ABSTRACT

Boundary Condition Dependence of Spectral Zeta Functions

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In this work we provide the analytic continuation of the spectral zeta function associated with the one-dimensional regular Sturm-Liouville problem and the two-dimensional Laplacian on the annulus. In the one-dimensional setting we consider general separated and coupled boundary conditions, and on the annulus we restrict our work to Dirichlet-Robin boundary conditions. In both cases, we use our results to calculate the coefficients of the asymptotic expansion of the associated heat kernel. In the one-dimensional case, we additionally use the analytically continued spectral zeta function to compute the determinant of the Sturm-Liouville operator.

Boundary Condition Dependence of Spectral Zeta Functions

by

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CHAPTER ONE

Introduction

In the mid-nineteenth century, the world was introduced to the Riemann zeta function,

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} , \tag{1.1}$$

where s is a complex variable [24]. Mathematicians since then have latched onto Riemann's seminal work in number theory, and the study of this function along with a variety of related functions has flourished. While the Riemann Hypothesis is most likely the first thing that comes to mind for a mathematician who sees (1.1), there are many other interesting paths one may take when researching various zeta functions with applications extending well beyond number theory to many areas of mathematics and physics [7, 20].

Of particular interest in this work is the spectral zeta function where the sum uses the spectrum of a differential operator as opposed to the positive integers. Our goal is to perform an analytic continuation of the spectral zeta function to the entire complex plane with the exception of simple poles, and use this analytic continuation in the calculation of the associated determinant and heat kernel coefficients.

This dissertation is organized as follows. In Chapter 2 we outline some of the fundamental definitions, methods and theorems used in our work. In particular, we more explicitly define the spectral zeta function and we discuss how the Cauchy Residue Theorem, along with WKB asymptotic analysis form the foundation for our analytic continuation. We also go into more detail on the connections between the spectral zeta function and the applications involving determinants and heat kernel coefficients.

In Chapter 3 we work with the general Sturm-Liouville differential operator on an interval on the real line with either separated or coupled boundary conditions [11]. The setting in Chapter 4 is the two-dimensional Laplacian acting on the annulus with Dirichlet-Robin boundary conditions. In both chapters, we use the methods outlined in Chapter 2 to perform our analytic continuation. We then apply our results to find the determinant of the Sturm-Liouville operator, along with the heat kernel coefficients for both the one and two-dimensional cases.

CHAPTER TWO

Preliminary Material

2.1 The Spectral Zeta Function

In this work we consider spectral functions for a self-adjoint Laplace-type operator, $\mathcal{L}$, on a compact Riemannian manifold. Our central focus will be on the spectral zeta function,

$$\zeta_{\mathcal{L}}(s) = \sum_n \lambda_n^{-s}, \tag{2.1}$$

where $\{\lambda_n\}_{n=0}^{\infty}$ is the spectrum of $\mathcal{L}$. In the setting of this paper, there is an infinite but countable number of eigenvalues, with no finite accumulation point [29]. The spectrum is real and bounded below, and as $n \to \infty$, we know that $\lambda_n \sim Cn^{2/D}$ where D is the dimension of the manifold being considered [29, 12]. This estimate allows us to say that $\zeta_{\mathcal{L}}(s)$ will be convergent on the half-plane $\mathrm{Re}(s) > D/2$. However, for the applications described below, it is necessary to evaluate the spectral zeta function to the left of this half-plane which can be accomplished through an analytic continuation of this function. To begin, we use the Cauchy Residue Theorem to give an integral representation of the spectral zeta function.

Theorem 2.1. *Cauchy's Residue Theorem [15]*

Suppose $f(z)$ is analytic on $[\gamma]$ except at a finite number of (isolated) singular points $a_1, a_2, \dots, a_n$ in (γ) (where γ is a curve). Then

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \mathit{Res}\, f(a_j). \tag{2.2}$$

Now, to see more clearly how the spectral zeta function may be related to a contour integral, suppose that the eigenvalues are implicitly defined as solutions to the equation

$$F(\lambda) = 0, \tag{2.3}$$

where $F(\lambda)$ is an analytic function. Then $\frac{F'(\lambda)}{F(\lambda)}$ has a simple pole at each eigenvalue with residue equal to the multiplicity of that eigenvalue. Thus, $\lambda^{-s}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln F(\lambda)$ has simple poles at each eigenvalue, λ_n, with residue equal to λ_n^{-s} times the multiplicity of the eigenvalue. Now, if we consider a contour, γ_N, enclosing the first N eigenvalues, then Cauchy's Residue Theorem gives us

$$\sum_{n=0}^{N} \lambda_n^{-s} = \frac{1}{2\pi i}\int_{\gamma_N} \mathrm{d}\lambda\, \lambda^{-s}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln F(\lambda). \tag{2.4}$$

Due to the behavior of the eigenvalues described above, if we consider taking $N \to \infty$, the contribution from γ_N at infinity tends to zero, so we have

$$\zeta_{\mathcal{L}}(s) = \frac{1}{2\pi i}\sum_{k}\int_{\gamma} \mathrm{d}\lambda\, \lambda^{-s}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln F_k(\lambda)\ , \tag{2.5}$$

where γ encloses the eigenvalues in a counterclockwise direction from $+\infty + i\varepsilon$, crossing the real axis at c, then continuing to $+\infty - i\varepsilon$, where $\varepsilon > 0$ and $0 < c < \lambda_0$ [2, 16, 19]. Note that here, and in the remainder of this dissertation, we assume the spectrum is positive. The reasons for this assumption are discussed more thoroughly in Section 3.1. The additional sum over k arises when the dimension, D, is larger than one. The relevant details regarding this sum will be discussed more thoroughly in Chapter 4.

This integral representation of the spectral zeta function is still only convergent on the half-plane $\mathrm{Re}(s) > D/2$. To extend this region to the left, we deform the contour of integration to the imaginary axis, and use the WKB method along with the appropriate boundary conditions to evaluate the asymptotic behavior of $\ln F_k(\lambda)$. This allows us to write,

$$\zeta_{\mathcal{L}}(s) = Z(s) + \sum_{i=-1}^{L} A_i(s)\ , \tag{2.6}$$

where the first $L+2$ terms of the appropriate asymptotic expansion of $\ln F_k(\lambda)$ have been subtracted from our integral, and then added back separately, shifting the region of convergence to the left with the exception of simple poles.

Aside from the work done in Chapters 3 and 4, the method for analytically continuing the spectral zeta function outlined above may be found in several published works. Just a few of the many examples of the contour integral representations for spectral zeta functions include [5, 6, 16, 17, 18, 19]. For further examples where WKB asymptotic analysis is used, see [2, 8, 3].

2.2 Applications

Before getting into specific applications, an important observation to make at this point is that (2.5) does not require knowledge of the exact eigenvalues. Only the characteristic equation (2.3) is needed. This is an important distinction due to the fact that there are times when information from spectral functions may be desired even though the spectrum is not explicitly known.

2.2.1 Functional Determinants

One such spectral function is the functional determinant. As a motivation for the formulas that follow, let us begin with the familiar case of an $N \times N$ Hermitian matrix, P. The determinant of P is the product of its eigenvalues, λ_n. Therefore, we may write,

$$\ln \det P = \sum_{n=1}^{N} \ln \lambda_n = -\frac{\mathrm{d}}{\mathrm{d}s} \sum_{n=1}^{N} \lambda_n^{-s} \bigg|_{s=0} = -\frac{\mathrm{d}}{\mathrm{d}s} \zeta_P(s) \bigg|_{s=0} . \tag{2.7}$$

Now, if we wish to extend the relationship to the differential operator $\mathcal{L}$ described at the beginning of Section 2.1, we run into problems with convergence of infinite products and sums. In fact, as we have seen earlier, the spectral zeta function is convergent in the half-plane $\mathrm{Re}(s) > D/2$, so to consider the derivative indicated in (2.7) requires an analytic continuation. As we will see in the chapters that follow, the analytically continued spectral zeta function is analytic in a neighborhood of $s = 0$, allowing us to compute the zeta-regularized functional determinant [16]. The definition,

$$\ln \det \mathcal{L} = -\zeta'_{\mathcal{L}}(0), \tag{2.8}$$

was first used by Ray and Singer in 1971 [23], and it has been used in many instances since then. Some examples of more recent results include [5, 17, 18, 19]. More specifically, in [5], the authors calculate the functional determinant on the generalized cone. In [17] and [18], the authors illustrate contour integral methods for computing the functional determinant with generalized Robin and non-separated boundary conditions imposed. They also consider the presence of a zero mode, and explain how to handle such situations which often arise in practice.

2.2.2 Heat Kernel Coefficients

The other application of interest to us in this work is the calculation of coefficients for the small-t expansion of the heat kernel. More specifically, the heat kernel,

$$K(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \ , \tag{2.9}$$

is related to the spectral zeta function by the equation

$$\zeta_{\mathcal{L}}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} K(t)\, \mathrm{d}t. \tag{2.10}$$

Furthermore, we have the small-t expansion of $K(t)$ is,

$$K(t) \sim \sum_{n=0,1/2,1,\ldots}^{\infty} a_n t^{n-D/2}, \tag{2.11}$$

where the coefficients a_n depend on both the differential operator and the boundary conditions being considered. In the case of the differential operator and boundary conditions described in Section 2.1, we have that

$$a_{D/2-s} = \Gamma(s) \mathrm{Res}\, \zeta_{\mathcal{L}}(s) \ , \tag{2.12}$$

for $s = D/2, (D-1)/2, \ldots, \frac{1}{2}, -\frac{2n+1}{2}$ with $n \in \mathbb{N}_0$, and

$$a_{D/2+s} = \frac{(-1)^s}{s!} \zeta_{\mathcal{L}}(-s) \ , \tag{2.13}$$

when $s \in \mathbb{N}_0$ [26].

Again, these results fundamentally rely on the analytically continued spectral zeta function. Some recent papers using the relations given above to calculate specific heat kernel coefficients include [5] where the authors give coefficients corresponding to an arbitrary elliptic operator on the D-dimensional ball with Dirichlet, Neumann, and Robin boundary conditions. In [6] the authors give coefficients on the generalized cone, again with Dirichlet, Neumann, and Robin boundary conditions. For many more examples, see [12, 13, 16].

CHAPTER THREE

The Sturm-Liouville Operator with Self-Adjoint Boundary Conditions

The content of this chapter comes from joint work with G. Fucci, and K. Kirsten [11].

3.1 Regular Sturm-Liouville Problems

In this chapter we will be concerned with the most general Sturm-Liouville differential operator. This symmetric second order differential operator, which we denote by $\mathcal{L}$, has the form [29]

$$\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + V(x)\ , \tag{3.1}$$

and acts on scalar functions defined on the interval $I = [0,1] \subset \mathbb{R}$. This choice of interval does not lack in generality since it is clear that the linear transformation $x' = a + (b-a)x$ with $a, b \in \mathbb{R}$ yields an operator of the same form as (3.1) but defined on the more general one-dimensional interval $[a, b]$. In addition we assume that $p(x) > 0$ for $x \in I$ and that both $p(x)$ and $V(x)$ belong to $\mathcal{C}^\infty(I)$.

For the symmetric operator $\mathcal{L}$ in (3.1) we consider the following differential equation

$$\mathcal{L}\,\varphi_\lambda(x) = \lambda^2\varphi_\lambda(x)\ , \tag{3.2}$$

where $\lambda \in \mathbb{C}$ and $\varphi_\lambda(x) \in C^2(I)$. Spectral functions associated with differential equations of the form (3.2) containing a differential operator of the type given in (3.1) can be conveniently studied by using the first-order formalism [17, 18]. By defining the vector

$$\mathrm{Y}_\lambda(x) = \begin{pmatrix} \varphi_\lambda(x) \\ p(x)\varphi'_\lambda(x) \end{pmatrix}\ , \tag{3.3}$$

with the prime denoting, here and in the rest of this chapter, differentiation with respect to the variable x, we can rewrite (3.2) as a system of first-order differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Y}_\lambda(x) = \mathrm{A}_\lambda(x)\mathrm{Y}_\lambda(x) \ , \tag{3.4}$$

with the 2×2 matrix $\mathrm{A}_\lambda(x)$ defined as

$$\mathrm{A}_\lambda(x) = \begin{pmatrix} 0 & p^{-1}(x) \\ V(x) - \lambda^2 & 0 \end{pmatrix} \ . \tag{3.5}$$

Amongst all the solutions to the differential equation (3.2), or equivalently to the system (3.4), we only select the ones that satisfy self-adjoint boundary conditions. With this restriction placed on the solutions, the numbers $\lambda \in \mathbb{R}$ become the eigenvalues and (3.2) coupled with self-adjoint boundary conditions represents a regular Sturm-Liouville problem. The self-adjoint boundary conditions that can be imposed on the solutions to (3.2) can be naturally divided into two classes [29].

Separated boundary conditions have the following general form

$$\begin{aligned} A_1\varphi_\lambda(0) - A_2 p(0)\varphi_\lambda'(0) &= 0 \ , \\ B_1\varphi_\lambda(1) + B_2 p(1)\varphi_\lambda'(1) &= 0 \ , \end{aligned} \tag{3.6}$$

with $A_1, A_2, B_1, B_2 \in \mathbb{R}$ and $(A_1, A_2) \neq (0,0)$, and $(B_1, B_2) \neq (0,0)$. In the first-order formalism these boundary conditions can be written in matrix form as

$$\begin{pmatrix} A_1 & -A_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_\lambda(0) \\ p(0)\varphi_\lambda'(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} \varphi_\lambda(1) \\ p(1)\varphi_\lambda'(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ . \tag{3.7}$$

We would like to point out that when $A_2 = B_2 = 0$ the equations (3.6) reduce to the Dirichlet boundary conditions, while for $A_2 \neq 0$, $B_2 \neq 0$, the system (3.6) represents Robin boundary conditions. Mixed or hybrid boundary conditions are obtained by setting $A_1 = B_2 = 0$ or $A_2 = B_1 = 0$.

Coupled boundary conditions can be expressed in general as

$$\begin{pmatrix} \varphi_\lambda(1) \\ p(1)\varphi_\lambda'(1) \end{pmatrix} = e^{i\gamma}\mathrm{K} \begin{pmatrix} \varphi_\lambda(0) \\ p(0)\varphi_\lambda'(0) \end{pmatrix} \ , \tag{3.8}$$

where $-\pi < \gamma \leq 0$ or $0 \leq \gamma < \pi$ and $\mathrm{K} \in \mathrm{SL}_2(\mathbb{R})$. Note that when $\gamma = 0$ and $\mathrm{K} = I_2$, with I_2 representing the 2×2 identity matrix, we obtain the familiar periodic boundary conditions.

Due to the fact that the functions $p(x)$ and $V(x)$ are arbitrary elements of $\mathcal{C}^\infty(I)$, the eigenvalues and corresponding eigenfunctions of (3.2) endowed with either separated or coupled boundary conditions cannot be found explicitly. However, by imposing either of the previously described self-adjoint boundary conditions to a general solution of the equation (3.2) we will be able to obtain an expression whose zeroes implicitly determine the eigenvalues.

In more detail, let $\varphi_\lambda(x)$ denote a general solution to the equation (3.2). For each $\lambda \in \mathbb{C}$ we choose $\varphi_\lambda(x)$ and $p(x)\varphi'_\lambda(x)$ such that they satisfy the following *initial conditions*

$$\varphi_\lambda(0) = A_2 \,, \quad \text{and} \quad p(0)\varphi'_\lambda(0) = A_1 \,. \tag{3.9}$$

With this particular choice of initial conditions, the first equation of the boundary conditions (3.6) is automatically satisfied while the second equation, describing the boundary condition at the endpoint $x = 1$, provides an implicit equation for the eigenvalues λ, namely

$$\Omega(\lambda) = B_1\varphi_\lambda(1) + B_2 p(1)\varphi'_\lambda(1) = 0 \,. \tag{3.10}$$

Since the boundary conditions (3.6) are self-adjoint, all the zeroes λ_n^2 of the equation (3.10) are real, but maybe positive or negative [29]. In what follows, for the purpose of presentation, we will assume eigenvalues to be non-negative. Negative eigenvalues will force one to use a non-standard branch-cut for the definition of the logarithm, the logarithm being needed to define λ_n^{-2s}. As a consequence, the zeta function becomes dependent on the choice of the branch-cut, but the functional determinant and heat kernel coefficients turn out to be independent of this choice such that final answers presented here for these quantities remain valid; for details regarding the purely technical nuances see [18].

For coupled boundary conditions, instead, we write the general solution $\varphi_\lambda(x)$ to (3.2) as a linear combination of two linearly independent solutions $u_\lambda(x)$ and $v_\lambda(x)$

$$\varphi_\lambda(x) = \alpha u_\lambda(x) + \beta v_\lambda(x) \;, \tag{3.11}$$

with arbitrary coefficients α and β. For each $\lambda \in \mathbb{C}$ the functions $u_\lambda(x)$ and $v_\lambda(x)$ are uniquely determined as solutions to (3.2) satisfying the initial conditions [1]

$$u_\lambda(0) = 0 \;, \quad \text{and} \quad p(0)u'_\lambda(0) = 1 \;, \tag{3.12}$$

for $u_\lambda(x)$, and

$$v_\lambda(0) = 1 \;, \quad \text{and} \quad p(0)v'_\lambda(0) = 0 \;, \tag{3.13}$$

for $v_\lambda(x)$. The coupled boundary conditions in (3.8) together with the formula for $\varphi_\lambda(x)$ in (3.11) lead to the following homogeneous linear system in the unknowns α and β

$$\begin{aligned} \alpha \left[u_\lambda(1) - e^{i\gamma}k_{12}\right] + \beta \left[v_\lambda(1) - e^{i\gamma}k_{11}\right] &= 0 \;, \\ \alpha \left[p(1)u'_\lambda(1) - e^{i\gamma}k_{22}\right] + \beta \left[p(1)v'_\lambda(1) - e^{i\gamma}k_{21}\right] &= 0 \;, \end{aligned} \tag{3.14}$$

where we have denoted with k_{ij} the entries of the matrix K and we have used the initial conditions $\varphi_\lambda(0) = \beta$ and $p(0)\varphi'_\lambda(0) = \alpha$ which follow directly from the relations (3.12) and (3.13). In order for the system (3.14) to have a non-trivial solution the determinant of its coefficient matrix must vanish. This condition provides the following implicit equation for the eigenvalues λ

$$\begin{aligned} \Delta(\lambda) &= \left(u_\lambda(1) - e^{i\gamma}k_{12}\right)\left(p(1)v'_\lambda(1) - e^{i\gamma}k_{21}\right) \\ &- \left(v_\lambda(1) - e^{i\gamma}k_{11}\right)\left(p(1)u'_\lambda(1) - e^{i\gamma}k_{22}\right) = 0 \;. \end{aligned} \tag{3.15}$$

The above implicit equation, however, can be simplified further. In fact, by performing the products and by using the relation detK $= 1$, which follows from the fact that K $\in \mathrm{SL}_2(\mathbb{R})$, we obtain

$$\begin{aligned}\Delta(\lambda) &= u_\lambda(1)p(1)v'_\lambda(1) - v_\lambda(1)p(1)u'_\lambda(1) - e^{2i\gamma} \\ &+ e^{i\gamma}\left[k_{22}v_\lambda(1) - k_{21}u_\lambda(1) + k_{11}p(1)u'_\lambda(1) - k_{12}p(1)v'_\lambda(1)\right] = 0\ . \end{aligned} \quad (3.16)$$

The first two terms on the right hand side of (3.16) constitute the Wronskian $W[u_\lambda(x), v_\lambda(x)]$ of the two linearly independent solutions $u_\lambda(x)$ and $v_\lambda(x)$ computed at $x = 1$. Since $W'[u_\lambda(x), v_\lambda(x)] = 0$ we can conclude that $W[u_\lambda(x), v_\lambda(x)] = C$. To find the constant C we set $x = 0$ and use the initial conditions (3.12) and (3.13) to obtain $C = W[u_\lambda(0), v_\lambda(0)] = -\det I_2 = -1$. The last remark proves that

$$u_\lambda(1)p(1)v'_\lambda(1) - v_\lambda(1)p(1)u'_\lambda(1) = -1\ , \quad (3.17)$$

and, consequently, the equation (3.16) which implicitly determines the eigenvalues λ becomes [1]

$$\Delta(\lambda) = 2\cos\gamma - \left[k_{22}v_\lambda(1) - k_{21}u_\lambda(1) + k_{11}p(1)u'_\lambda(1) - k_{12}p(1)v'_\lambda(1)\right] = 0\ . \quad (3.18)$$

The spectral zeta function associated with the eigenvalue equation (3.2) endowed with separated or coupled boundary conditions is

$$\zeta(s) = \sum_\lambda \lambda^{-2s}\ , \quad (3.19)$$

which is convergent, due to Weyl's estimate [12], in the semi-plane $\mathrm{Re}(s) > 1/2$. In order to perform the analytic continuation we represent $\zeta(s)$ in (3.19) in terms of a contour integral by exploiting Cauchy's residue theorem. In the case of separated boundary conditions the eigenvalues are implicitly given by (3.10) and, hence, the corresponding spectral zeta function can be expressed as [16]

$$\zeta^{\mathrm{S}}(s) = \frac{1}{2\pi i}\int_{\mathcal{C}} \mathrm{d}\lambda\, \lambda^{-2s} \frac{\partial}{\partial\lambda} \ln\Omega(\lambda)\ . \quad (3.20)$$

Similarly, for coupled boundary conditions the spectral zeta function can be represented as

$$\zeta^{\mathrm{C}}(s) = \frac{1}{2\pi i}\int_{\mathcal{D}} \mathrm{d}\lambda\, \lambda^{-2s}\frac{\partial}{\partial\lambda}\ln\Delta(\lambda)\ , \tag{3.21}$$

with the function $\Delta(\lambda)$ given by (3.18). In the above integral representations $\mathcal{C}$ and $\mathcal{D}$ are contours in the complex plane that encircle in the counterclockwise direction all the roots of $\Omega(\lambda)$ and $\Delta(\lambda)$, respectively. As indicated, these roots are assumed to be positive. The complex integrals in (3.20) and (3.21) are well defined, by construction, in the region $\mathrm{Re}(s) > 1/2$.

By deforming the contour of integration $\mathcal{C}$ in (3.20) to the imaginary axis and by using the property that $\Omega(i\lambda) = \Omega(-i\lambda)$, which can be proved from (3.10) by noticing that $\varphi_{i\lambda}(x) = \varphi_{-i\lambda}(x)$ and $\varphi'_{i\lambda}(x) = \varphi'_{-i\lambda}(x)$, we obtain, by changing variables $\lambda \to iz$, the representation

$$\zeta^{\mathrm{S}}(s) = \frac{\sin\pi s}{\pi}\int_0^\infty \mathrm{d}z\, z^{-2s}\frac{\partial}{\partial z}\ln\Omega(iz)\ . \tag{3.22}$$

By following a procedure analogous to the one used to obtain (3.22) from (3.20), one can prove that for coupled boundary conditions the spectral zeta function takes the form

$$\zeta^{\mathrm{C}}(s) = \frac{\sin\pi s}{\pi}\int_0^\infty \mathrm{d}z\, z^{-2s}\frac{\partial}{\partial z}\ln\Delta(iz)\ . \tag{3.23}$$

From the differential equation (3.2) one can show that the leading $z \to \infty$ behavior of its solutions is exponential growth, see (3.34). This remark and the explicit expressions (3.10) and (3.18) allow us to conclude that the functions $\Omega(iz)$ and $\Delta(iz)$ have a similar exponential behavior as $z \to \infty$ and, hence, the integral representations (3.22) and (3.23) converge at the upper limit of integration for $\mathrm{Re}(s) > 1/2$. Furthermore, since the solutions of the differential equation (3.2) are analytic with respect to the parameter λ, the functions $\Omega(iz)$ and $\Delta(iz)$ are analytic in the variable z. In particular this implies that in a neighborhood of $z = 0$ we have that $\Omega(iz) - \Omega(0) \sim z^2$ and linear terms in z are not present since $\Omega(iz) = \Omega(-iz)$. By assuming that $z = 0$ is not an eigenvalue of our problem, we can conclude that for $z \to 0$ we have $\ln \Omega(iz) = \ln \Omega(0) + \mathcal{O}(z^2)$. This last estimate implies that the integral representation (3.22) converges at the lower limit of integration for $\mathrm{Re}(s) < 1$. A completely similar analysis can be performed for the function $\Delta(iz)$ leading to the conclusion that the integral representation (3.23) is also convergent at the lower limit of integration for $\mathrm{Re}(s) < 1$. We can therefore state that the integral representations (3.22) and (3.23) converge in the region $1/2 < \mathrm{Re}(s) < 1$.

3.2 The WKB Analysis of the Sturm-Liouville Problem

To perform the analytic continuation of $\zeta^{\mathrm{S}}(s)$ and $\zeta^{\mathrm{C}}(s)$ to values lying outside of the strip $1/2 < \mathrm{Re}(s) < 1$ we will rely on the asymptotic expansion for large z of the eigenfunctions of the Sturm-Liouville problem (3.2) endowed with separated or coupled boundary conditions. This asymptotic expansion is found using the well established WKB expansion [4, 21].

To start the analysis we consider the differential equation (3.2), with $\lambda = iz$,

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + V(x)\right]\varphi_{iz}(x) = -z^2\varphi_{iz}(x)\ . \tag{3.24}$$

By utilizing the following ansatz for the solution

$$\varphi_{iz}(x) = \exp\left\{\int_0^x \mathcal{S}(t,z)\mathrm{d}t\right\}\ , \tag{3.25}$$

and by substituting it into the differential equation (3.24), we obtain a first order nonlinear differential equation for $\mathcal{S}(x,z)$

$$[p(x)\mathcal{S}(x,z)]' = V(x) + z^2 - p(x)\mathcal{S}^2(x,z)\ . \tag{3.26}$$

The last differential equation, although nonlinear, is very suitable for the application of WKB techniques to obtain the large-z behavior of $\mathcal{S}(x,z)$ and consequently, through the relation (3.25), of $\varphi_{iz}(x)$ [4].

For $z \to \infty$ we assume that $\mathcal{S}(x,z)$ has an asymptotic expansion of the form

$$\mathcal{S}(x,z) \sim zS_{-1}(x) + \sum_{i=0}^{\infty}\frac{S_i(x)}{z^i}\ . \tag{3.27}$$

By substituting the above expansion into the differential equation (3.26) we have

$$\sum_{i=-1}^{\infty} \frac{[p(x)S_i(x)]'}{z^i} = V(x) + z^2 - p(x)\left(zS_{-1}(x) + \sum_{i=0}^{\infty} \frac{S_i(x)}{z^i}\right)^2 . \tag{3.28}$$

The last term on the right hand side of (3.28) may be written as

$$p(x)\left(z^2 S_{-1}^2(x) + 2S_{-1}(x)\sum_{i=-1}^{\infty} \frac{S_{i+1}(x)}{z^i} + \sum_{j=0}^{\infty} \frac{C_j(x)}{z^j}\right), \tag{3.29}$$

where the coefficients, $C_j(x)$, are

$$\sum_{m=0}^{j} S_m(x) S_{j-m}(x). \tag{3.30}$$

By equating like powers of z in (3.28) with the help of (3.29) and (3.30), we obtain

$$S_{-1}^{\pm}(x) = \pm\frac{1}{\sqrt{p(x)}} , \qquad S_0^{\pm}(x) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(p(x)S_{-1}^{\pm}(x)\right) , \tag{3.31}$$

for the leading and the first subleading term of the asymptotic expansion. For the term with $i = 1$ we have

$$S_1^{\pm}(x) = \frac{1}{2p(x)S_{-1}^{\pm}(x)}\left[V(x) - p(x)\left(S_0^{\pm}\right)^2(x) - \left(p(x)S_0^{\pm}(x)\right)'\right] , \tag{3.32}$$

while for the higher asymptotic orders with $i \geq 1$ we obtain the recurrence relation

$$S_{i+1}^{\pm}(x) = -\frac{1}{2p(x)S_{-1}^{\pm}(x)}\left[\left(p(x)S_i^{\pm}(x)\right)' + p(x)\sum_{m=0}^{i} S_m^{\pm}(x)S_{i-m}^{\pm}(x)\right] . \tag{3.33}$$

The $\pm$ represents the different choice of sign in the leading term S_{-1} of the asymptotic expansion (3.27). The different signs correspond to the two solutions $\mathcal{S}^+(z,x)$ and $\mathcal{S}^-(z,x)$ to the differential equation (3.26). These solutions provide, through the relation (3.25), the exponentially growing and decaying parts of the function $\varphi_{iz}(x)$. The correct asymptotic behavior of $\varphi_{iz}(x)$ for $z \to \infty$ is obtained as a linear combination of the asymptotically increasing and decaying terms as follows

$$\varphi_{iz}(x) = A\exp\left\{\int_0^x \mathcal{S}^+(t,z)\mathrm{d}t\right\} + B\exp\left\{\int_0^x \mathcal{S}^-(t,z)\mathrm{d}t\right\} , \tag{3.34}$$

where the terms A and B are uniquely determined once the initial conditions are imposed.

3.2.1 Separated Boundary Conditions

For separated boundary conditions we impose the initial conditions (3.9) on the function $\varphi_{iz}(x)$ in (3.34) to obtain

$$A = -\frac{A_2 p(0)\mathcal{S}^-(0,z) - A_1}{p(0)\left[\mathcal{S}^+(0,z) - \mathcal{S}^-(0,z)\right]}\ , \qquad B = \frac{A_2 p(0)\mathcal{S}^+(0,z) - A_1}{p(0)\left[\mathcal{S}^+(0,z) - \mathcal{S}^-(0,z)\right]}\ . \tag{3.35}$$

By substituting the expressions (3.35) into (3.34) and then by using the resulting expression for $\varphi_{iz}(x)$ in the formula for $\Omega(iz)$ in (3.10) we obtain

$$\begin{aligned}\Omega(iz) &= -\frac{\left[A_2 p(0)\mathcal{S}^-(0,z) - A_1\right]\left[B_2 p(1)\mathcal{S}^+(1,z) + B_1\right]}{p(0)\left[\mathcal{S}^+(0,z) - \mathcal{S}^-(0,z)\right]} \\ &\times\ \exp\left\{\int_0^1 \mathcal{S}^+(t,z)\mathrm{d}t\right\}(1+\mathcal{E}(z))\ ,\end{aligned} \tag{3.36}$$

where we have denoted with $\mathcal{E}(z)$ exponentially small contributions in z. The integral representation of the spectral zeta function $\zeta^{\mathrm{S}}(s)$ in (3.22) contains the term $\ln\Omega(iz)$ and, therefore, its asymptotic expansion is needed for the process of analytic continuation of $\zeta^{\mathrm{S}}(s)$. From the expression (3.36) we have

$$\begin{aligned}\ln\Omega(iz) &= \ln\left[-A_2 p(0)\mathcal{S}^-(0,z) + A_1\right] + \ln\left[B_2 p(1)\mathcal{S}^+(1,z) + B_1\right] \\ &- \ln\left[p(0)\left(\mathcal{S}^+(0,z) - \mathcal{S}^-(0,z)\right)\right] + \int_0^1 \mathcal{S}^+(t,z)\mathrm{d}t + \tilde{\mathcal{E}}(z)\ ,\end{aligned} \tag{3.37}$$

where $\tilde{\mathcal{E}}(z)$ denotes exponentially decaying terms. At this point, it is necessary to further expand for $z \to \infty$ each term appearing in (3.37). From the results (3.31)-(3.33) one obtains

$$\mathcal{S}^+(0,z) - \mathcal{S}^-(0,z) = \frac{2z}{\sqrt{p(0)}}\left[1 + \sum_{i=1}^{\infty}\frac{\omega_i(0)}{z^{i+1}}\right]\ , \tag{3.38}$$

where for $i \in \mathbb{N}^+$

$$\omega_i(0) = \frac{\sqrt{p(0)}}{2}\left[S_i^+(0) - S_i^-(0)\right]\ . \tag{3.39}$$

To further simplify (3.39), we may use the recurrence relation (3.33) to show that S_i^+ and S_i^- satisfy the relation

$$S_i^-(x) = (-1)^i S_i^+(x)\ . \tag{3.40}$$

To prove (3.40), it is easy to check the base case by inspection. Then, if we assume the relation holds up to $S_i^-(x)$, we have

$$\begin{aligned} S_{i+1}^-(x) &= -\frac{1}{2p(x)S_{-1}^-(x)}\left[\left(p(x)S_i^-(x)\right)' + p(x)\sum_{m=0}^{i} S_m^-(x)S_{i-m}^-(x)\right] \\ &= \frac{1}{2p(x)S_{-1}^+(x)}\left[(-1)^i\left(p(x)S_i^+(x)\right)'\right. \\ &+ \left. p(x)\sum_{m=0}^{i}(-1)^m S_m^+(x)(-1)^{i-m}S_{i-m}^+(x)\right] \\ &= \frac{(-1)^i}{2p(x)S_{-1}^+(x)}\left[\left(p(x)S_i^+(x)\right)' + p(x)\sum_{m=0}^{i} S_m^+(x)S_{i-m}^+(x)\right] \\ &= (-1)^{i+1}S_{i+1}^+(x). \end{aligned} \tag{3.41}$$

Therefore, from (3.39) we have that

$$\omega_{2i}(0) = 0\ , \qquad \omega_{2i+1}(0) = \sqrt{p(0)}S_{2i+1}^+(0)\ . \tag{3.42}$$

We use the results (3.38), (3.39) and (3.42) to obtain

$$\ln\left[p(0)\left(\mathcal{S}^+(0,z) - \mathcal{S}^-(0,z)\right)\right] = \frac{1}{2}\ln p(0) + \ln 2z + \sum_{i=1}^{\infty}\frac{D_{2i-1}(0)}{z^{2i}}\ , \tag{3.43}$$

where the terms $D_{2i-1}(0)$ are determined through the cumulant expansion

$$\ln\left[1 + \sum_{k=0}^{\infty}\frac{\omega_{2k+1}(0)}{z^{2k+2}}\right] \simeq \sum_{i=1}^{\infty}\frac{D_{2i-1}(0)}{z^{2i}}\ . \tag{3.44}$$

More care is needed in the expansion of the first two terms on the right hand side of (3.37). For these terms the cases $A_2 = B_2 = 0$, and $A_2 = 0$ or $B_2 = 0$ need to be distinguished from all the other cases. This distinction is necessary since the large z asymptotic behavior of $\ln\Omega(iz)$ critically depends on whether A_2 or B_2, or both, vanish. In the case $A_2 = B_2 = 0$, which can be easily recognized to represent Dirichlet boundary conditions, the arguments of $\ln[-A_2p(0)\mathcal{S}^-(0,z) + A_1]$ and $\ln[B_2p(1)\mathcal{S}^+(1,z) + B_1]$ reduce to just constants and, hence, no further analysis is necessary. In order to avoid analyzing each case separately, we introduce the function

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} , \tag{3.45}$$

and replace the first term on the right hand side of (3.37) by

$$[1 - \delta(A_2)] \ln \left[-A_2 p(0) \mathcal{S}^-(0, z) + A_1\right] + \delta(A_2) \ln A_1 , \tag{3.46}$$

and the second term by

$$[1 - \delta(B_2)] \ln \left[B_2 p(1) \mathcal{S}^+(1, z) + B_1\right] + \delta(B_2) \ln B_1 . \tag{3.47}$$

From the asymptotic expansion (3.27), we obtain

$$-A_2 p(0) \mathcal{S}^-(0, z) + A_1 = A_2 \sqrt{p(0)}\, z + \sum_{i=0}^{\infty} \frac{\sigma_i^-(0)}{z^i} , \tag{3.48}$$

and

$$B_2 p(1) \mathcal{S}^+(1, z) + B_1 = B_2 \sqrt{p(1)}\, z + \sum_{i=0}^{\infty} \frac{\sigma_i^+(1)}{z^i} , \tag{3.49}$$

where, by recalling (3.40),

$$\sigma_0^-(0) = -A_2 p(0) S_0^+(0) + A_1 , \qquad \sigma_i^-(0) = (-1)^{i+1} A_2 p(0) S_i^+(0) , \quad i \geq 1 \tag{3.50}$$

and

$$\sigma_0^+(1) = B_2 p(1) S_0^+(1) + B_1 , \qquad \sigma_i^+(1) = B_2 p(1) S_i^+(1) , \quad i \geq 1 . \tag{3.51}$$

The results obtained in (3.48) and (3.49) lead to the expansions

$$\ln \left[-A_2 p(0) \mathcal{S}^-(0, z) + A_1\right] = \ln \left(A_2 \sqrt{p(0)}\right) + \ln z + \sum_{i=1}^{\infty} \frac{\mathcal{Z}_i^-(0)}{z^i} , \tag{3.52}$$

and

$$\ln \left[B_2 p(1) \mathcal{S}^+(1, z) + B_1\right] = \ln \left(B_2 \sqrt{p(1)}\right) + \ln z + \sum_{i=1}^{\infty} \frac{\mathcal{Z}_i^+(1)}{z^i} , \tag{3.53}$$

where $\mathcal{Z}_i^-(0)$ and $\mathcal{Z}_i^+(1)$ are found through the relations

$$\ln \left[1 + \frac{1}{A_2 \sqrt{p(0)}} \sum_{i=1}^{\infty} \frac{\sigma_{i-1}^-(0)}{z^i}\right] \simeq \sum_{k=1}^{\infty} \frac{\mathcal{Z}_k^-(0)}{z^k} , \tag{3.54}$$

$$\ln\left[1+\frac{1}{B_2\sqrt{p(1)}}\sum_{i=1}^{\infty}\frac{\sigma_{i-1}^{+}(1)}{z^i}\right] \simeq \sum_{k=1}^{\infty}\frac{\mathcal{Z}_k^{+}(1)}{z^k} . \tag{3.55}$$

By using the expansions (3.27) and (3.43) and by recalling that for a unified treatment of all separated boundary conditions we need to replace the first two terms of the right hand side of (3.37) by (3.46) and (3.47), whose expansions can be obtained from (3.48) and (3.49), we find

$$\begin{aligned} \ln\Omega(iz) &= -\frac{1}{4}\ln\left(p(0)p(1)\right)+[1-\delta(A_2)]\ln\left(A_2\sqrt{p(0)}\right) \\ &+ [1-\delta(B_2)]\ln\left(B_2\sqrt{p(1)}\right)+\delta(A_2)\ln A_1+\delta(B_2)\ln B_1-\ln 2z \\ &+ [2-\delta(A_2)-\delta(B_2)]\ln z+z\int_0^1 S_{-1}^{+}(t)\mathrm{d}t+\sum_{i=1}^{\infty}\frac{\mathcal{M}_i}{z^i} , \end{aligned} \tag{3.56}$$

where we have discarded exponentially decreasing terms and we have used the relation

$$\int_0^1 S_0^{+}(t)\mathrm{d}t = -\frac{1}{4}\ln\frac{p(1)}{p(0)} . \tag{3.57}$$

Since, for $i\in\mathbb{N}^+$, $D_{2i}(0)=0$, we have that, when $i=2m+1$ with $m\in\mathbb{N}_0$,

$$\mathcal{M}_{2m+1} = \int_0^1 S_{2m+1}^{+}(t)\mathrm{d}t+[1-\delta(A_2)]\,\mathcal{Z}_{2m+1}^{-}(0)+[1-\delta(B_2)]\,\mathcal{Z}_{2m+1}^{+}(1) , \tag{3.58}$$

while, for $i=2m$ with $m\in\mathbb{N}^+$,

$$\mathcal{M}_{2m} = \int_0^1 S_{2m}^{+}(t)\mathrm{d}t-D_{2m-1}(0)+[1-\delta(A_2)]\,\mathcal{Z}_{2m}^{-}(0)+[1-\delta(B_2)]\,\mathcal{Z}_{2m}^{+}(1) . \tag{3.59}$$

3.2.2 Coupled Boundary Conditions

The implicit equation for the eigenvalues in the case of coupled boundary conditions, namely $\Delta(iz)=0$ in (3.18), contains two linearly independent solutions $u_{iz}(x)$ and $v_{iz}(x)$ to (3.24) which are found by imposing the initial conditions (3.12) and (3.13) to a general solution of (3.24). For $z\to\infty$ both functions $u_{iz}(x)$ and $v_{iz}(x)$ can be expressed as a linear combination of an exponentially increasing and an exponentially decreasing term in the same way as indicated in (3.34). By imposing the initial conditions (3.12) to the general solution in (3.34) we obtain for A and B the relations

$$A+B=0\,,\quad A=\frac{1}{p(0)\left[\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)\right]}\,, \tag{3.60}$$

which lead to the following result

$$\begin{aligned} u_{iz}(x) &= \frac{1}{p(0)\left[\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)\right]} \\ &\times \left[\exp\left\{\int_0^x \mathcal{S}^{+}(t,z)\mathrm{d}t\right\}-\exp\left\{\int_0^x \mathcal{S}^{-}(t,z)\mathrm{d}t\right\}\right]\,. \end{aligned} \tag{3.61}$$

For the function $v_{iz}(x)$ we impose, instead, the initial conditions (3.13) to (3.34) to obtain

$$A+B=1\,,\quad A=-\frac{\mathcal{S}^{-}(0,z)}{\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)}\,, \tag{3.62}$$

which provide the following expression

$$\begin{aligned} v_{iz}(x) &= \frac{1}{\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)}\left[-\mathcal{S}^{-}(0,z)\exp\left\{\int_0^x \mathcal{S}^{+}(t,z)\mathrm{d}t\right\}\right. \\ &+ \left.\mathcal{S}^{+}(0,z)\exp\left\{\int_0^x \mathcal{S}^{-}(t,z)\mathrm{d}t\right\}\right]\,. \end{aligned} \tag{3.63}$$

Obviously, expressions for the functions $u'_{iz}(x)$ and $v'_{iz}(x)$, which appear in $\Delta(iz)$, are obtained by simply differentiating (3.61) and (3.63). The last remark and the explicit expressions (3.61) and (3.63) can be used in the formula for $\Delta(iz)$ in (3.18) to obtain

$$\begin{aligned} \Delta(iz) &= \left[\frac{k_{22}\mathcal{S}^{-}(0,z)}{\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)}+\frac{k_{21}}{p(0)\left[\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)\right]}\right. \\ &- \left.\frac{k_{11}p(1)\mathcal{S}^{+}(1,z)}{p(0)\left[\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)\right]}-\frac{k_{12}p(1)\mathcal{S}^{-}(0,z)\mathcal{S}^{+}(1,z)}{\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)}\right] \\ &\times \exp\left\{\int_0^1 \mathcal{S}^{+}(t,z)\mathrm{d}t\right\}+2\cos\gamma+\hat{\varepsilon}(z) \\ &= \frac{\left[k_{21}+k_{22}p(0)\mathcal{S}^{-}(0,z)-k_{11}p(1)\mathcal{S}^{+}(1,z)-k_{12}p(1)p(0)\mathcal{S}^{-}(0,z)\mathcal{S}^{+}(1,z)\right]}{p(0)\left[\mathcal{S}^{+}(0,z)-\mathcal{S}^{-}(0,z)\right]} \\ &\times \exp\left\{\int_0^1 \mathcal{S}^{+}(t,z)\mathrm{d}t\right\}(1+\varepsilon(z))\,, \end{aligned} \tag{3.64}$$

where exponentially small contributions have been collectively denoted by $\hat{\varepsilon}(z)$ and $\varepsilon(z)$. From this expression it is not very difficult to obtain

$$\begin{aligned}\ln\Delta(iz) &= -\ln\left[p(0)\left(\mathcal{S}^+(0,z)-\mathcal{S}^-(0,z)\right)\right]+\int_0^1 \mathcal{S}^+(t,z)\mathrm{d}t \\ &+ \ln\left[k_{21}+k_{22}p(0)\mathcal{S}^-(0,z)-k_{11}p(1)\mathcal{S}^+(1,z)\right. \\ &- \left.k_{12}p(1)p(0)\mathcal{S}^-(0,z)\mathcal{S}^+(1,z)\right]+\tilde{\varepsilon}(z)\ ,\end{aligned} \tag{3.65}$$

with $\tilde{\varepsilon}(z)$ being exponentially decreasing terms as $z\to\infty$. The expression (3.65) is the starting point for the computation of the large-z asymptotic expansion of $\ln\Delta(iz)$. The asymptotic expansion of the first two terms on the right hand side of (3.65) has already been found in the case of separated boundary conditions and, hence, will not be repeated here. Instead, we will concentrate only on the last logarithmic term in (3.65). From the asymptotic expansion of $\mathcal{S}^+$ and $\mathcal{S}^-$ we obtain, for the terms linear in $\mathcal{S}^+$ and $\mathcal{S}^-$

$$\begin{aligned}k_{21} &+k_{22}p(0)\mathcal{S}^-(0,z)-k_{11}p(1)\mathcal{S}^+(1,z) = -zk_{22}\sqrt{p(0)}+k_{21}+k_{22}p(0)S_0^+(0) \\ &+ \sum_{i=1}^{\infty}\frac{(-1)^i k_{22}p(0)S_i^+(0)}{z^i}-zk_{11}\sqrt{p(1)}-k_{11}p(1)S_0^+(1)-\sum_{i=1}^{\infty}\frac{k_{11}p(1)S_i^+(1)}{z^i} \\ &= -z\left(k_{22}\sqrt{p(0)}+k_{11}\sqrt{p(1)}\right)-\sum_{i=0}^{\infty}\frac{\Phi_i}{z^i}\ ,\end{aligned} \tag{3.66}$$

with

$$\begin{aligned}\Phi_0 &= -k_{21}-k_{22}p(0)S_0^+(0)+k_{11}p(1)S_0^+(1)\ , \\ \Phi_i &= (-1)^{i+1}k_{22}p(0)S_i^+(0)+k_{11}p(1)S_i^+(1)\ ,\quad i\geq 1\ ,\end{aligned} \tag{3.67}$$

where in the previous relations we have exploited (3.40). In addition, for the term proportional to $\mathcal{S}^-\mathcal{S}^+$ we have

$$\begin{aligned}k_{12}\,p(1)p(0)\mathcal{S}^-(0,z)\mathcal{S}^+(1,z) &= z^2k_{12}p(1)p(0)\sum_{i=0}^{\infty}\frac{S_{i-1}^-(0)}{z^i}\sum_{i=0}^{\infty}\frac{S_{i-1}^+(1)}{z^i} \\ &= z^2k_{12}p(1)p(0)\left(-\frac{1}{\sqrt{p(0)p(1)}}+\sum_{i=1}^{\infty}z^{-i}\left[\sum_{m=0}^{i}(-1)^{m-1}S_{m-1}^+(0)S_{i-m-1}^+(1)\right]\right)\end{aligned}$$

$$= \quad -z^2 k_{12}\sqrt{p(0)p(1)}\left[1+\sum_{i=1}^{\infty} z^{-i}\left(\sum_{m=0}^{i}(-1)^m \bar{S}_m(0)\bar{S}_{i-m}(1)\right)\right] , \tag{3.68}$$

with $\bar{S}_0(x) = 1$ and, for $i \in \mathbb{N}^+$,

$$\bar{S}_i(x) = \sqrt{p(x)}S^+_{i-1}(x) . \tag{3.69}$$

In (3.68) we have used, once again, the relation (3.40) in order to express $S^-_i(x)$ in terms of $S^+_i(x)$.

From the relations (3.66) and (3.68) one can notice that the leading term of the asymptotic expansion of the argument of the last logarithmic term in (3.65) strictly depends on whether or not the coefficient k_{12} of the matrix K vanishes. Because of this different leading behavior as $z \to \infty$ we will need to distinguish between two cases: $k_{12} \neq 0$, and $k_{12} = 0$. In order to describe both cases simultaneously we proceed as in the previous subsection and replace the third logarithmic term in (3.65) by

$$\begin{aligned}&[1-\delta(k_{12})]\ln\big[k_{21}+k_{22}p(0)\mathcal{S}^-(0,z)-k_{11}p(1)\mathcal{S}^+(1,z)\\&-k_{12}p(1)p(0)\mathcal{S}^-(0,z)\mathcal{S}^+(1,z)\big]+\delta(k_{12})\ln\big[k_{21}+k_{22}p(0)\mathcal{S}^-(0,z)\\&-k_{11}p(1)\mathcal{S}^+(1,z)\big] .\end{aligned} \tag{3.70}$$

For the first terms in (3.70) the results obtained in (3.66) and (3.68) lead to the expression

$$\begin{aligned}&\ln\big[k_{21}+k_{22}p(0)\mathcal{S}^-(0,z)-k_{11}p(1)\mathcal{S}^+(1,z)-k_{12}p(1)p(0)\mathcal{S}^-(0,z)\mathcal{S}^+(1,z)\big]\\=\;&\ln k_{12}\sqrt{p(0)p(1)}+2\ln z+\sum_{i=1}^{\infty}\frac{\Lambda_i}{z^i} ,\end{aligned} \tag{3.71}$$

where the coefficients Λ_i are obtained through the cumulant expansion

$$\ln\left[1+\sum_{k=1}^{\infty}\frac{\Psi_k}{z^k}\right] \simeq \sum_{i=1}^{\infty}\frac{\Lambda_i}{z^i} , \tag{3.72}$$

with the definitions

$$\Psi_1 = \bar{S}_1(1) - \bar{S}_1(0) - \frac{k_{22}\sqrt{p(0)}+k_{11}\sqrt{p(1)}}{k_{12}\sqrt{p(0)p(1)}} , \tag{3.73}$$

and, when $i \geq 2$,

$$\Psi_i = \sum_{m=0}^{i} (-1)^m \bar{S}_m(0)\bar{S}_{i-m}(1) - \frac{\Phi_{i-2}}{k_{12}\sqrt{p(0)p(1)}} \ . \tag{3.74}$$

The asymptotic expansion of the second term in (3.70) is derived from (3.66) and reads

$$\begin{aligned} &\ln\left[-k_{21} - k_{22}p(0)\mathcal{S}^{-}(0,z) + k_{11}p(1)\mathcal{S}^{+}(1,z)\right] \\ = \ &\ln\left(k_{22}\sqrt{p(0)} + k_{11}\sqrt{p(1)}\right) + \ln z + \sum_{i=1}^{\infty}\frac{\Pi_i}{z^i} \ , \end{aligned} \tag{3.75}$$

where the coefficients Π_i can be found from the relation

$$\ln\left[1 + \frac{1}{k_{22}\sqrt{p(0)} + k_{11}\sqrt{p(1)}}\sum_{k=1}^{\infty}\frac{\Phi_{k-1}}{z^k}\right] \simeq \sum_{i=1}^{\infty}\frac{\Pi_i}{z^i} \ . \tag{3.76}$$

By using (3.43), the expansion (3.27), and the result (3.71) and (3.75) we can conclude that

$$\begin{aligned} \ln\Delta(iz) &= -\frac{1}{4}\ln p(0)p(1) + [1-\delta(k_{12})]\ln k_{12}\sqrt{p(0)p(1)} \\ &+\delta(k_{12})\ln\left(k_{22}\sqrt{p(0)} + k_{11}\sqrt{p(1)}\right) \\ &+[2-\delta(k_{12})]\ln z - \ln 2z + z\int_0^1 S_{-1}^{+}(t)\mathrm{d}t + \sum_{i=1}^{\infty}\frac{\mathcal{N}_i}{z^i} \ , \end{aligned} \tag{3.77}$$

where we have replaced the third logarithmic term in (3.65) with the modified expression in (3.70). The functions $\mathcal{N}_i$ introduced in (3.77) have the expression, for $i = 2m+1$, $m \in \mathbb{N}_0$,

$$\mathcal{N}_{2m+1} = \int_0^1 S_{2m+1}^{+}(t)\mathrm{d}t + [1-\delta(k_{12})]\,\Lambda_{2m+1} + \delta(k_{12})\Pi_{2m+1} \ , \tag{3.78}$$

and for $i = 2m$, $m \in \mathbb{N}^{+}$,

$$\mathcal{N}_{2m} = \int_0^1 S_{2m}^{+}(t)\mathrm{d}t - D_{2m-1}(0) + [1-\delta(k_{12})]\,\Lambda_{2m} + \delta(k_{12})\Pi_{2m} \ . \tag{3.79}$$

The expansions (3.56) and (3.77) represent the only information needed in order to perform the analytic continuation of the spectral zeta function associated with the Sturm-Liouville problem (3.2) with separated or coupled boundary conditions.

3.3 Analytic Continuation of the Spectral Zeta Function

To perform the analytic continuation of the spectral zeta function associated with self-adjoint Sturm-Liouville problems we will need the results about the asymptotic expansions obtained in the previous section and the integral representations (3.22) and (3.23). In the case of separated boundary conditions the representation (3.22) can be conveniently rewritten as a sum of two terms

$$\zeta^{\mathrm{S}}(s)=\frac{\sin\pi s}{\pi}\int_0^1 \mathrm{d}z\, z^{-2s}\frac{\partial}{\partial z}\ln\Omega(iz)+\frac{\sin\pi s}{\pi}\int_1^\infty \mathrm{d}z\, z^{-2s}\frac{\partial}{\partial z}\ln\Omega(iz)\ . \qquad (3.80)$$

The first integral represents an analytic function for $\mathrm{Re}(s)<1$ while the second one defines an analytic function in the region $\mathrm{Re}(s)>1/2$. In order to analytically continue the spectral zeta function to a region extending to the left of the strip $1/2<\mathrm{Re}(s)<1$ we need to subtract and add a suitable number of terms of the asymptotic expansion of $\ln\Omega(iz)$ from the second integral [16]. By using the first $L+2$ terms of the expansion (3.56) the spectral zeta function can be written as

$$\zeta^{\mathrm{S}}(s)=Z^{\mathrm{S}}(s)+\sum_{i=-1}^{L}A_i^{\mathrm{S}}(s)\ . \qquad (3.81)$$

The function $Z^{\mathrm{S}}(s)$ is analytic in the region $\mathrm{Re}(s)>-(1+L)/2$ and has the form

$$\begin{aligned}Z^{\mathrm{S}}(s) &= \frac{\sin\pi s}{\pi}\int_0^\infty \mathrm{d}z\, z^{-2s}\frac{\partial}{\partial z}\Bigg\{\ln\Omega(iz)-H(z-1)\Bigg[-\frac{1}{4}\ln p(0)p(1)\\ &+ [1-\delta(A_2)]\ln A_2\sqrt{p(0)}+[1-\delta(B_2)]\ln B_2\sqrt{p(1)}+\delta(A_2)\ln A_1\\ &+ \delta(B_2)\ln B_1+[2-\delta(A_2)-\delta(B_2)]\ln z-\ln 2z+z\int_0^1 S_{-1}^{+}(t)\mathrm{d}t\\ &+ \sum_{i=1}^{L}\frac{\mathcal{M}_i}{z^i}\Bigg]\Bigg\}\ ,\end{aligned} \qquad (3.82)$$

with $H(z-1)$ denoting the Heaviside step function. The functions $A_i^{\mathrm{S}}(s)$ are meromorphic for $s\in\mathbb{C}$ and their expressions are

$$A_{-1}^{\mathrm{S}}(s)=\frac{\sin\pi s}{\pi}\int_1^\infty \mathrm{d}z\, z^{-2s}\frac{\partial}{\partial z}\left[z\int_0^1 S_{-1}^{+}(t)\mathrm{d}t\right]\ , \qquad (3.83)$$

$$A_0^{\mathrm{S}}(s)=\frac{\sin\pi s}{\pi}\int_1^\infty \mathrm{d}z\, z^{-2s}\frac{\partial}{\partial z}\left\{[2-\delta(A_2)-\delta(B_2)]\ln z-\ln 2z\right\}\ , \qquad (3.84)$$

while, for $i \geq 1$,

$$A_i^{\mathrm{S}}(s) = \frac{\sin \pi s}{\pi} \int_1^{\infty} \mathrm{d}z\, z^{-2s} \frac{\partial}{\partial z} \left[\frac{\mathcal{M}_i}{z^i} \right] . \tag{3.85}$$

By using the expression (3.81) and by performing the elementary integrations in (3.84)-(3.85) we obtain the following analytically continued expression for the spectral zeta function

$$\zeta^{\mathrm{S}}(s) = Z^{\mathrm{S}}(s) + \frac{\sin \pi s}{\pi} \left[\frac{1 - \delta(A_2) - \delta(B_2)}{2s} + \frac{1}{2s-1} \int_0^1 S_{-1}^{+}(t) \mathrm{d}t - \sum_{i=1}^{L} i \frac{\mathcal{M}_i}{2s+i} \right] . \tag{3.86}$$

The last expression clearly shows that $\zeta^{\mathrm{S}}(s)$ is a meromorphic function of s with a simple pole at the points $s = 1/2$ and $s = -(2k+1)/2$, $k \in \mathbb{N}_0$.

For coupled boundary conditions the spectral zeta function can be analogously written as a sum of two terms

$$\zeta^{\mathrm{C}}(s) = \frac{\sin \pi s}{\pi} \int_0^1 \mathrm{d}z\, z^{-2s} \frac{\partial}{\partial z} \ln \Delta(iz) + \frac{\sin \pi s}{\pi} \int_1^{\infty} \mathrm{d}z\, z^{-2s} \frac{\partial}{\partial z} \ln \Delta(iz) , \tag{3.87}$$

where the first integral converges for $\mathrm{Re}(s) < 1$ and the second for $\mathrm{Re}(s) > 1/2$. To obtain the analytically continued expression of $\zeta^{\mathrm{C}}(s)$ we proceed as for the case of separated boundary conditions by subtracting and adding from the second integral in (3.87) $L+2$ terms of the asymptotic expansion in (3.77). This leads to the expression

$$\zeta^{\mathrm{C}}(s) = Z^{\mathrm{C}}(s) + \sum_{i=-1}^{L} A_i^{\mathrm{C}}(s) . \tag{3.88}$$

The function $Z^{\mathrm{C}}(s)$ is analytic for $\mathrm{Re}(s) > -(1+L)/2$ and has the expression

$$\begin{aligned} Z^{\mathrm{C}}(s) &= \frac{\sin \pi s}{\pi} \int_0^{\infty} \mathrm{d}z\, z^{-2s} \frac{\partial}{\partial z} \Bigg\{ \ln \Delta(iz) - H(z-1) \Bigg[-\frac{1}{4} \ln p(0)p(1) \\ &+ [1 - \delta(k_{12})] \ln k_{12} \sqrt{p(0)p(1)} + \delta(k_{12}) \ln \left(k_{22}\sqrt{p(0)} + k_{11}\sqrt{p(1)} \right) \\ &+ [2 - \delta(k_{12})] \ln z - \ln 2z + z \int_0^1 S_{-1}^{+}(t) \mathrm{d}t + \sum_{i=1}^{L} \frac{\mathcal{N}_i}{z^i} \Bigg] \Bigg\} , \end{aligned} \tag{3.89}$$

while the functions $A_i^{\mathrm{C}}(s)$ are meromorphic in s and have the form

$$A_{-1}^{\mathrm{C}}(s) = A_{-1}^{\mathrm{S}}(s) \; , \quad A_0^{\mathrm{C}}(s) = \frac{\sin \pi s}{\pi} \int_1^{\infty} \mathrm{d}z \, z^{-2s} \frac{\partial}{\partial z} \left\{ [2 - \delta(k_{12})] \ln z - \ln 2z \right\} \; , \tag{3.90}$$

and, for $i \geq 1$,

$$A_i^{\mathrm{C}}(s) = \frac{\sin \pi s}{\pi} \int_1^{\infty} \mathrm{d}z \, z^{-2s} \frac{\partial}{\partial z} \left[\frac{\mathcal{N}_i}{z^i} \right] \; . \tag{3.91}$$

By utilizing (3.90) and (3.91) in (3.88) we get

$$\zeta^{\mathrm{C}}(s) = Z^{\mathrm{C}}(s) + \frac{\sin \pi s}{\pi} \left[\frac{1 - \delta(k_{12})}{2s} + \frac{1}{2s-1} \int_0^1 S_{-1}^{+}(t) \mathrm{d}t - \sum_{i=1}^{L} i \frac{\mathcal{N}_i}{2s+i} \right] \; . \tag{3.92}$$

The expressions (3.86) and (3.92) represent the desired analytically continued expressions for the spectral zeta function and the starting point for the computation of the functional determinant of the Sturm-Liouville operator and of the coefficients of the heat kernel asymptotic expansion.

3.4 Functional Determinant of Sturm-Liouville Operators

The Sturm-Liouville operator (3.1) is elliptic, self-adjoint and acts on suitable scalar functions defined on a one-dimensional finite interval. For such operators the zeta regularized functional determinant is defined as follows [23, 25]

$$\det(\mathcal{L}) = \exp\{-\zeta'(0)\} \; , \tag{3.93}$$

where the derivative of the zeta function at the point $s = 0$ is computed after $\zeta(s)$ has been analytically continued to a neighborhood of $s = 0$. For the case of separated and coupled boundary conditions the analytically continued expressions of $\zeta^{\mathrm{S}}(s)$ and $\zeta^{\mathrm{C}}(s)$ are provided in Section 3.3 and represent the starting point for the computation of the functional determinant.

By differentiating (3.86) with respect to the variable s we obtain

$$(\zeta^{\mathrm{S}})'(s) \quad = \quad (Z^{\mathrm{S}})'(s) + \cos(\pi s) \left[\frac{1 - \delta(A_2) - \delta(B_2)}{2s} + \frac{1}{2s-1} \int_0^1 S_{-1}^{+}(t) \mathrm{d}t \right.$$

$$- \sum_{i=1}^{L} i\frac{\mathcal{M}_i}{2s+i}\Bigg] - 2\frac{\sin\pi s}{\pi}\left[\frac{1-\delta(A_2)-\delta(B_2)}{4s^2} + \frac{1}{(2s-1)^2}\int_0^1 S_{-1}^{+}(t)\mathrm{d}t - \sum_{i=1}^{L} i\frac{\mathcal{M}_i}{(2s+i)^2}\right] . \tag{3.94}$$

In the limit $s \to 0$ the last expression simplifies to

$$(\zeta^{\mathrm{S}})'(0) = (Z^{\mathrm{S}})'(0) - \int_0^1 S_{-1}^{+}(t)\mathrm{d}t - \sum_{i=1}^{L}\mathcal{M}_i \ , \tag{3.95}$$

which can be obtained by noting that since the function $Z^{\mathrm{S}}(s)$ is analytic in the region $\mathrm{Re}(s) > -(1+L)/2$, the value $s=0$ can simply be set into the expression for $(Z^{\mathrm{S}})'(s)$. The explicit expression for $(Z^{\mathrm{S}})'(0)$ is found by differentiating (3.82) and by setting $s=0$, namely

$$\begin{aligned}(Z^{\mathrm{S}})'(0) &= -\ln 2\Omega(0) - \frac{1}{4}\ln p(0)p(1) + [1-\delta(A_2)]\ln A_2\sqrt{p(0)} \\ &+ [1-\delta(B_2)]\ln B_2\sqrt{p(1)} + \delta(A_2)\ln A_1 + \delta(B_2)\ln B_1 \\ &+ \int_0^1 S_{-1}^{+}(t)\mathrm{d}t + \sum_{i=1}^{L}\mathcal{M}_i \ .\end{aligned} \tag{3.96}$$

The substitution of the result (3.96) in the relation (3.95) and the use of the definition (3.93) leads to the following expression for the functional determinant of the Sturm-Liouville operator $\mathcal{L}^{\mathrm{S}}$ endowed with separated boundary conditions

$$\det\left(\mathcal{L}^{\mathrm{S}}\right) = 2[p(0)p(1)]^{\frac{1}{4}}\Omega(0)\frac{\left(A_2\sqrt{p(0)}\right)^{\delta(A_2)-1}\left(B_2\sqrt{p(1)}\right)^{\delta(B_2)-1}}{A_1^{\delta(A_2)}B_1^{\delta(B_2)}} \ . \tag{3.97}$$

In the case of coupled boundary conditions we differentiate the expression in (3.92) with respect to the variable s. Since $Z^{\mathrm{C}}(s)$ is analytic for $\mathrm{Re}(s) > -(1+L)/2$ and the remaining terms are meromorphic functions of s having no pole at the origin we set $s=0$ in the differentiated expression of (3.92) to obtain

$$(\zeta^{\mathrm{C}})'(0) = (Z^{\mathrm{C}})'(0) - \int_0^1 S_{-1}^{+}(t)\mathrm{d}t - \sum_{i=1}^{L}\mathcal{N}_i \ . \tag{3.98}$$

After differentiating (3.89) and setting $s=0$ it is not very difficult to obtain the following expression

$$(Z^{\mathrm{C}})'(0) = -\ln 2\Delta(0) - \frac{1}{4}\ln p(0)p(1) + [1-\delta(k_{12})]\ln k_{12}\sqrt{p(0)p(1)}$$

$$+ \quad \delta(k_{12}) \ln\Big(k_{22}\sqrt{p(0)} + k_{11}\sqrt{p(1)}\Big) + \int_0^1 S_{-1}^{+}(t)\mathrm{d}t + \sum_{i=1}^{L} \mathcal{N}_i \,. \tag{3.99}$$

From (3.99), (3.98) and the definition (3.93) we find, for the functional determinant of the Sturm-Liouville operator (3.1) with coupled boundary conditions, the result

$$\det\left(\mathcal{L}^{\mathrm{C}}\right) = 2[p(0)p(1)]^{\frac{1}{4}}\Delta(0)\frac{\Big(k_{12}\sqrt{p(0)p(1)}\Big)^{\delta(k_{12})-1}}{\Big(k_{22}\sqrt{p(0)} + k_{11}\sqrt{p(1)}\Big)^{\delta(k_{12})}} \,. \tag{3.100}$$

The terms $\Omega(0)$ and $\Delta(0)$ in the expressions (3.97) and (3.100) for the functional determinant are obtained by setting $\lambda = 0$ in (3.10) and (3.18), respectively. The function φ_0 which enters in the expression for $\Omega(0)$ represents a non-trivial solution to the homogeneous differential equation $\mathcal{L}\varphi_0 = 0$ satisfying the initial conditions (3.9) and can be found either numerically or analytically. In the case of $\Delta(0)$, instead, a suitable solution to the homogeneous differential equation satisfies the initial conditions given in (3.12) and (3.13). Also in this case the solution can either be found numerically or analytically.

The above expressions for the functional determinant have been obtained under the assumption that zero is not part of the spectrum of the Sturm-Liouville operator (3.1) with separated or coupled boundary conditions. When a zero mode is present, however, it has to be excluded from the definition of the functional determinant. The case with the zero mode extracted is detailed in [11].

3.5 Coefficients of the Heat Kernel Asymptotic Expansion

In this section we will use the analytic continuation of the spectral zeta function of the Sturm-Liouville operator (3.1), endowed with either separated or coupled boundary conditions presented in Section 3.3, to compute the coefficients of the small-t asymptotic expansion of the heat kernel trace $\theta(t) = \mathrm{Tr}_{\mathscr{L}^2} e^{-t\mathcal{L}}$. For the Sturm-Liouville operator (3.1) the small-t asymptotic expansion of $\theta(t)$ has the following general form [12, 14, 22, 28]

$$\theta(t) = \frac{1}{\sqrt{4\pi t}} \sum_{n=0}^{\infty} a_{\frac{n}{2}} t^{\frac{n}{2}} \;, \tag{3.101}$$

where the terms $a_{n/2}$ denote the heat kernel coefficients. Thanks to the relation that exists between the trace of the heat kernel and the spectral zeta function given by the Mellin transform [26] one can prove, in the one-dimensional case, that the following relations hold

$$a_{\frac{1}{2}-s} = \Gamma(s) \mathrm{Res}\, \zeta(s) \;, \tag{3.102}$$

when $s = 1/2$ and $s = -(2n+1)/2$ with $n \in \mathbb{N}_0$, and furthermore

$$a_{\frac{1}{2}+n} = \frac{(-1)^n}{n!} \zeta(-n) \;. \tag{3.103}$$

In the expression (3.102), Res denotes the residue of the spectral zeta function. Since the relations (3.102) and (3.103) express $a_{n/2}$ in terms of either the residue or the value of the spectral zeta function at a specific point of the real line, they will be used in this section for the computation of the coefficients of the small-t expansion of $\theta(t)$ as that information can be extracted from the analytically continued expression of $\zeta(s)$ obtained in Section 3.3.

For separated boundary conditions the analytically continued expression for the spectral zeta function is given by (3.86). By choosing $L = 2n+1$ with $n \in \mathbb{N}_0$ in (3.86) the function $Z^{\mathrm{S}}(s)$ becomes analytic for $\mathrm{Re}(s) > -(n+1)$ and, therefore, does not contribute to the residue of $\zeta^{\mathrm{S}}(s)$ at $s = -(2n+1)/2$. In particular from (3.86) we have

$$\mathrm{Res}\, \zeta^{\mathrm{S}} \left(\frac{1}{2} \right) = \frac{1}{2\pi} \int_0^1 \frac{\mathrm{d}t}{\sqrt{p(t)}} \;, \tag{3.104}$$

where we have used the expression in (3.31), and for $n \in \mathbb{N}_0$,

$$\mathrm{Res}\, \zeta^{\mathrm{S}} \left(-\frac{2n+1}{2} \right) = \frac{(-1)^n}{2\pi} (2n+1) \mathcal{M}_{2n+1} \;. \tag{3.105}$$

In addition, due to the prefactor $\sin(\pi s)/\pi$ in the expression (3.82), we have that $Z^{\mathrm{S}}(-n)=0$. This implies that

$$\zeta^{\mathrm{S}}(0)=\frac{1-\delta(A_2)-\delta(B_2)}{2}\ , \quad \text{and} \quad \zeta^{\mathrm{S}}(-n)=(-1)^{n+1}n\,\mathcal{M}_{2n}\ , \quad n\in\mathbb{N}^{+}\ . \quad (3.106)$$

By using the results obtained in (3.104) through (3.106) together with the relations (3.102) and (3.103), we find the following expression for the heat kernel coefficients when separated boundary conditions are imposed

$$a_0^{\mathrm{S}}=\frac{1}{2\sqrt{\pi}}\int_0^1\frac{\mathrm{d}t}{\sqrt{p(t)}}\ , \qquad (3.107)$$

$$a_{n+1}^{\mathrm{S}}=-\frac{2^{2n}n!}{\sqrt{\pi}(2n)!}\mathcal{M}_{2n+1}\ , \qquad (3.108)$$

which has been obtained by utilizing the formula [27]

$$\Gamma\left(-\frac{2n+1}{2}\right)=\frac{\sqrt{\pi}(-1)^{n+1}2^{2n+1}n!}{(2n+1)!}\ , \quad n\in\mathbb{N}_0\ . \qquad (3.109)$$

For the coefficients with half-integer index we find, instead,

$$a_{\frac{1}{2}}^{\mathrm{S}}=\frac{1-\delta(A_2)-\delta(B_2)}{2}\ , \quad a_{\frac{2n+1}{2}}^{\mathrm{S}}=-\frac{1}{(n-1)!}\mathcal{M}_{2n}\ , \quad n\in\mathbb{N}^{+}\ . \qquad (3.110)$$

By using the explicit expressions for $\mathcal{M}_i$ displayed in the Appendix and the relations (3.108) and (3.110), we find

$$\begin{aligned} a_1^{\mathrm{S}} &= -\frac{1}{32\sqrt{\pi}}\int_0^1\frac{1}{\sqrt{p(t)}}\left[16V(t)-\frac{p'(t)^2}{p(t)}+4p''(t)\right]\mathrm{d}t \\ &- \frac{1-\delta(A_2)}{4A_2\sqrt{\pi p(0)}}\left(A_2p'(0)+4A_1\right)+\frac{1-\delta(B_2)}{4B_2\sqrt{\pi p(1)}}\left(B_2p'(1)-4B_1\right)\ , \end{aligned} \qquad (3.111)$$

and

$$\begin{aligned} a_{\frac{3}{2}}^{\mathrm{S}} &= \frac{1}{4}\left[V(0)+V(1)\right]-\frac{1}{64}\left[\frac{p'(0)^2}{p(0)}+\frac{p'(1)^2}{p(1)}\right]+\frac{1}{16}\left[p''(0)+p''(1)\right] \\ &+ \left[1-\delta(A_2)\right]\left[\frac{A_1^2}{2A_2^2p(0)}-\frac{V(0)}{2}+\frac{A_1p'(0)}{4A_2p(0)}+\frac{p'(0)^2}{16p(0)}-\frac{p''(0)}{8}\right] \\ &+ \left[1-\delta(B_2)\right]\left[\frac{B_1^2}{2B_2^2p(1)}-\frac{V(1)}{2}-\frac{B_1p'(1)}{4B_2p(1)}+\frac{p'(1)^2}{16p(1)}-\frac{p''(1)}{8}\right]\ . \end{aligned} \qquad (3.112)$$

Note, that in the above formulas and in the following, the understanding is that the terms proportional to $1-\delta(A_2)$ and $1-\delta(B_2)$ do not contribute when $A_2 = 0$, respectively $B_2 = 0$, despite the presence of A_2 and B_2 in denominators.

These leading heat kernel coefficients can be compared with known results [12, 16, 28]. In order to do so, the operator (3.1) and boundary conditions (3.6) have to be written using geometric invariants. To this end we want to identify the Sturm-Liouville operator (3.1) with a Laplacian on the interval $[0,1]$. The second derivative term is matched if we consider $g(x) = p^{-1}(x)$ to be the metric on the interval. With this metric, the Laplacian for scalars on the interval is

$$\Delta_I = -\sqrt{p(x)}\frac{d}{dx}\left(\sqrt{p(x)}\frac{d}{dx}\right) = -p(x)\frac{d^2}{dx^2} - \frac{1}{2}p'(x)\frac{d}{dx} \ .$$

The fact that the first order derivative term does not match the one in $\mathcal{L}$,

$$\mathcal{L} = -p(x)\frac{d^2}{dx^2} - p'(x)\frac{d}{dx} + V(x) \ ,$$

is incorporated by using a connection one-form $\omega(x) = p'(x)/(4p(x))$. It is easily seen that a suitable rewriting of $\mathcal{L}$ then is

$$\mathcal{L} = -\sqrt{p(x)}\left(\frac{d}{dx} + \omega(x)\right)\sqrt{p(x)}\left(\frac{d}{dx} + \omega(x)\right) - E \ ,$$

where

$$E = -\left(V(x) + \frac{1}{4}p''(x) - \frac{1}{16}\frac{(p'(x))^2}{p(x)}\right)$$

is the relevant "*invariant potential*" to use in order to write down the heat kernel coefficients.

For Dirichlet boundary conditions, $A_2 = B_2 = 0$, this is all that is needed to verify the above coefficients from known results; note, the Riemannian volume element is $p^{-1/2}(x)$ and Riemann tensor as well as extrinsic curvature vanish on an interval.

For the cases involving the derivatives in (3.6) we have to identify the relevant parameter as it occurs in Robin boundary conditions. In general this condition is written as

$$[\varphi_{;m}(x) - R_1\varphi(x)]\,|_{x=0} = 0\;, \qquad [\varphi_{;m}(x) - R_2\varphi(x)]|\big|_{x=1} = 0\;,$$

where $\varphi_{;m}$ denotes the covariant derivative with respect to the *exterior* normal. More explicitly for our situation,

$$\left[-\sqrt{p(0)}\left(\frac{d}{dx}+\omega(0)\right)-R_1\right]\varphi(x)\bigg|_{x=0} = 0\;,$$

and

$$\left[\sqrt{p(1)}\left(\frac{d}{dx}+\omega(1)\right)-R_2\right]\varphi(x)\bigg|_{x=1} = 0\;.$$

This establishes the relations

$$A_1 = -\frac{1}{4}\frac{p'(0)}{\sqrt{p(0)}} - R_1\;,\quad A_2 = \frac{1}{\sqrt{p(0)}}\;,\quad B_1 = \frac{1}{4}\frac{p'(1)}{\sqrt{p(1)}} - R_2\;,\quad B_2 = \frac{1}{\sqrt{p(1)}}\;,$$

and using these identifications, the known results for the leading coefficients are obtained; note, that the standard notation for R_i, $i = 1, 2$, is S [12, 16, 28].

In addition to making sure that the leading coefficients agree with known results one might wonder how our computations do reproduce the fact that half-integer coefficients only contain boundary contributions. This means that only values of $p(x)$ and $V(x)$ and their derivatives at $x = 0$ and $x = 1$ appear in the expressions for the heat kernel coefficients with half-integer index. This can be shown as follows: In (3.37) it is seen that the only possible volume contributions are contained in the term $\int_0^1 \mathcal{S}^+(t, z)dt$, and, there, the even powers in the large-z expansion contribute to half-integer coefficients. These terms with even powers of z integrate to boundary terms only. To prove this statement, first note that

$$\mathcal{S}^{\pm}(x, z) = \pm\mathcal{S}_{odd}(x, z) + \mathcal{S}_{even}(x, z)\;,$$

where

$$\mathcal{S}_{odd}(x, z) = \sum_{i=-1}^{\infty}\frac{S^+_{2i+1}(x)}{z^{2i+1}}\;,\quad \mathcal{S}_{even} = \sum_{i=0}^{\infty}\frac{S^+_{2i}(x)}{z^{2i}}\;.$$

By construction both, $\mathcal{S}^+(x,z)$ and $\mathcal{S}^-(x,z)$ satisfy the differential equation (3.26), from which one easily obtains

$$\left(p(x)\mathcal{S}^+(x,z)\right)' + p(x)\left(\mathcal{S}^+(x,z)\right)^2 = \left(p(x)\mathcal{S}^-(x,z)\right)' + p(x)\left(\mathcal{S}^-(x,z)\right)^2 \ ,$$

or, in terms of $\mathcal{S}_{odd}$ and $\mathcal{S}_{even}$,

$$2\left(p(x)\mathcal{S}_{odd}(x,z)\right)' + 4p(x)\mathcal{S}_{odd}(x,z)\mathcal{S}_{even}(x,z) = 0 \ .$$

This implies

$$\mathcal{S}_{even}(x,z) = -\frac{1}{2}\frac{d}{dx}\ln\left(p(x)\mathcal{S}_{odd}(x,z)\right) \ , \tag{3.113}$$

from which the wanted assertion follows.

Let us next focus on the case of coupled boundary conditions. In this case, the relevant analytically continued expression of the spectral zeta function is (3.92). By choosing once again $L = 2n+1$, $n \in \mathbb{N}_0$, the function $Z^{\mathrm{C}}(s)$ becomes analytic in the region $\mathrm{Re}(s) > -(n+1)$ and, in addition, vanishes for $s = -n$. From the last remark, by using the expression (3.92) in (3.102) and (3.103), we obtain for the coefficients with integer index

$$a_0^{\mathrm{C}} = a_0^{\mathrm{S}} \ , \quad a_{n+1}^{\mathrm{C}} = -\frac{2^{2n} n!}{\sqrt{\pi}(2n)!}\mathcal{N}_{2n+1} \ , \tag{3.114}$$

with $n \in \mathbb{N}_0$, and for those with half-integer index

$$a_{\frac{1}{2}}^{\mathrm{C}} = \frac{1-\delta(k_{12})}{2} \ , \quad a_{\frac{2n+1}{2}}^{\mathrm{C}} = -\frac{1}{(n-1)!}\mathcal{N}_{2n} \ , \tag{3.115}$$

with $n \in \mathbb{N}^+$. In particular, for $n=0$ and $n=1$ we have

$$\begin{aligned} a_1^{\mathrm{C}} &= -\frac{1}{32\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{p(t)}}\left[16V(t) - \frac{p'(t)^2}{p(t)} + 4p''(t)\right]\mathrm{d}t \\ &+ \frac{\delta(k_{12})}{4\sqrt{\pi}}\left[\frac{k_{11}p'(1) - k_{22}p'(0) + 4k_{21}}{\sqrt{p(1)}k_{11} + \sqrt{p(0)}k_{22}}\right] \\ &+ \frac{1-\delta(k_{12})}{\sqrt{\pi}}\left[\frac{k_{22}\sqrt{p(0)} + k_{11}\sqrt{p(1)}}{k_{12}\sqrt{p(0)p(1)}} - \frac{p'(0)}{4\sqrt{p(0)}} + \frac{p'(1)}{4\sqrt{p(1)}}\right] \ , \end{aligned} \tag{3.116}$$

and

$$
\begin{aligned}
a^{\mathrm{C}}_{\frac{3}{2}} &= \frac{1}{4}\left[V(0)+V(1)\right]-\frac{1}{64}\left[\frac{p'(0)^2}{p(0)}+\frac{p'(1)^2}{p(1)}\right]+\frac{1}{16}\left[p''(0)+p''(1)\right] \\
&+ \frac{[1-\delta(k_{12})]}{2}\Bigg\{\left(\frac{k_{22}\sqrt{p(0)}+k_{11}\sqrt{p(1)}}{k_{12}\sqrt{p(0)p(1)}}\right)^2-(V(0)+V(1)) \\
&+ \frac{1}{8}\left(\frac{p'(0)^2}{p(0)}-\frac{p'(1)^2}{p(1)}\right)-\frac{1}{4}(p''(0)+p''(1))-\frac{2k_{21}}{k_{12}\sqrt{p(0)p(1)}} \\
&- \frac{1}{2k_{12}}\left[\frac{k_{11}p'(0)}{p(0)}-\frac{k_{22}p'(1)}{p(1)}\right]\Bigg\}+\frac{\delta(k_{12})}{32}\Bigg\{\left(\frac{k_{11}p'(1)-k_{22}p'(0)+4k_{21}}{\sqrt{p(1)}k_{11}+\sqrt{p(0)}k_{22}}\right)^2 \\
&- \frac{1}{\sqrt{p(1)}k_{11}+\sqrt{p(0)}k_{22}}\Bigg(4\sqrt{p(0)}k_{22}\left(4V(0)+p''(0)\right) \\
&+ 4\sqrt{p(1)}k_{11}\left(4V(1)+p''(1)\right)-\frac{k_{22}p'(0)^2}{\sqrt{p(0)}}-\frac{k_{11}p'(1)^2}{\sqrt{p(1)}}\Bigg)\Bigg\}. \qquad (3.117)
\end{aligned}
$$

The set of boundary conditions characterized by $k_{12}=0$ contains, as a particular case, periodic boundary conditions. These are obtained by imposing, in addition to $k_{12}=0$, the constraints $k_{21}=0$, $k_{11}=k_{22}=1$ and, for $n\in\mathbb{N}_0$, $\lim_{t\to 0}p^{(n)}(t)=\lim_{t\to 1}p^{(n)}(t)\neq 0$. From the results obtained in this Section it is not very difficult to verify that for periodic boundary conditions $a^{\mathrm{C}}_{1/2}=a^{\mathrm{C}}_{3/2}=0$. Indeed, one can show that all half-integer coefficients vanish. This follows from (3.65) by noting that under the given assumptions we also have $\mathcal{S}^{\pm}(0,z)=\mathcal{S}^{\pm}(1,z)$. Substituting the given values of k_{ij} and using (3.113), one obtains

$$
\ln\Delta(iz)=\int_0^1 \mathcal{S}_{odd}(t,z)dt\ ,
$$

which implies that half-integer coefficients vanish as no even powers in $1/z$ occur.

We would like to point out that higher order heat kernel coefficients for each of the cases presented here may be found from their general formulas (3.108), (3.110), (3.114), and (3.115) with the help of an algebraic computer program.

CHAPTER FOUR

The Laplacian on the Annulus with Dirichlet-Robin Boundary Conditions

4.1 The Zeta Function

Moving from the one dimensional case to two dimensions, we consider the second order differential operator P acting on the plane given by

$$P = -\Delta + V(r), \tag{4.1}$$

where $V(r)$ is radially smooth. Our goal is to analyze spectral functions associated with the eigenvalue problem

$$P\mu = \lambda^2\mu, \tag{4.2}$$

but for the sake of the computations that follow, we begin by considering the operator

$$P_m = -\Delta + V(r) + m^2, \tag{4.3}$$

where the mass, $m > 0$, will later be sent to zero. The eigenvalue problem associated with P_m is

$$P_m\mu = \nu^2\mu, \tag{4.4}$$

where $\nu^2 = \lambda^2 + m^2$. We will consider (4.2) in an annular region, so in polar coordinates we have

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mu}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2\mu}{\partial\theta^2} + V(r)\mu = \lambda^2\mu, \quad 0 < a < r < b, \ -\pi < \theta \leq \pi. \tag{4.5}$$

We will consider Dirichlet boundary conditions at $r = a$, and at $r = b$ we apply Robin boundary conditions,

$$\left.\frac{\partial}{\partial r}\mu(r,\theta)\right|_{r=b} + c\mu(b,\theta) = 0, \quad -\pi < \theta \leq \pi, \tag{4.6}$$

where $c \in \mathbb{R}$. By using separation of variables with $\mu(r,\theta) = R(r)\Theta(\theta)$, we have

$$-\frac{r^2}{R(r)}R''(r) - \frac{r}{R(r)}R'(r) + r^2(V(r) - \lambda^2) - \frac{1}{\Theta(\theta)}\Theta''(\theta) = 0, \tag{4.7}$$

which gives the system of differential equations

$$-\frac{r^2}{R(r)}R''(r) - \frac{r}{R(r)}R'(r) + r^2(V(r) - \lambda^2) = -k^2, \tag{4.8}$$

$$\frac{1}{\Theta(\theta)}\Theta''(\theta) = -k^2. \tag{4.9}$$

The inherent periodicity in (4.9) results in having $k \in \mathbb{Z}$, so for each k we may now use the initial value problem,

$$R''(r)+\frac{1}{r}R'(r)-R(r)\left(V(r) - \lambda^2 + \frac{k^2}{r^2}\right) = 0, \quad R_k(a,\lambda) = 0,\ R'_k(a,\lambda) = 1, \tag{4.10}$$

in our analytic continuation of the spectral zeta function,

$$\zeta(s) = \sum_{k=-\infty}^{\infty} \sum_{\lambda_k} (\lambda_k^2 + m^2)^{-s}. \tag{4.11}$$

Imposing the Robin boundary conditions to a solution of (4.10), the eigenvalues λ are implicitly defined by

$$\Omega_k(\lambda) = R'_k(b,\lambda) + cR_k(b,\lambda) = 0, \tag{4.12}$$

so we may use the Cauchy Residue Theorem to express the corresponding spectral zeta function as

$$\zeta(s) = \frac{1}{2\pi i}\sum_{k=-\infty}^{\infty}\int_{\mathcal{C}} \mathrm{d}\lambda\, (\lambda^2 + m^2)^{-s}\frac{\partial}{\partial\lambda}\ln\Omega_k(\lambda)\ , \tag{4.13}$$

where $\mathcal{C}$ is a contour in the complex plane that encloses all the roots of $\Omega_k(\lambda)$ in a counterclockwise direction. We assume that the roots of (4.12) are on the positive real axis, allowing us to deform our contour $\mathcal{C}$ to the imaginary axis. As in the one dimensional case, we have that $\Omega_k(i\lambda) = \Omega_k(-i\lambda)$, so by changing variables $\lambda \to iz$ we obtain

$$\zeta(s) = \frac{\sin\pi s}{\pi}\sum_{k=-\infty}^{\infty}\int_m^{\infty} \mathrm{d}z\, (z^2 - m^2)^{-s}\frac{\partial}{\partial z}\ln\Omega_k(iz)\ . \tag{4.14}$$

4.2 The WKB Analysis

In order to perform the analytic continuation of $\zeta(s)$, we begin by considering the differential equation

$$\phi''(r) + \frac{1}{r}\phi'(r) - \phi(r)\left(V(r) + z^2 + \frac{k^2}{r^2}\right) = 0. \tag{4.15}$$

We must analyze the asymptotic behavior of the solutions $\phi_k(iz)$ both for $z \to \infty$ and $|k| \to \infty$, so we will use a uniform asymptotic expansion by setting $z = ku$. However, we must consider a separate case for large z when $k = 0$.

4.2.1 Zero Case

For the case $k = 0$, we begin by using the ansatz for the solution to (4.15),

$$\phi_0(r, iz) = \exp\left\{\int_a^r \tilde{\mathcal{S}}(t, z)\, \mathrm{d}t\right\}. \tag{4.16}$$

Plugging this into the differential equation (4.15), we get the first order differential equation,

$$\tilde{\mathcal{S}}'(r, z) = V(r) + z^2 - \tilde{\mathcal{S}}^2(r, z) - \frac{1}{r}\tilde{\mathcal{S}}(r, z), \tag{4.17}$$

which is where we begin the WKB analysis. We use the asymptotic expansion

$$\tilde{\mathcal{S}}(r, z) \simeq \sum_{i=-1}^{\infty} \tilde{S}_i(r) z^{-i} \tag{4.18}$$

for large z so we may then obtain the behavior of $\phi_0(r, iz)$ as $z \to \infty$ through the relation (4.16). Substituting this expansion into the differential equation (4.17), we may compare like powers of z on each side of the equation to recursively define the $\tilde{S}_i(r)$ terms as follows:

$$\tilde{S}_{-1}^{\pm}(r) = \pm 1, \qquad \tilde{S}_0^{\pm}(r) = -\frac{1}{2r}, \tag{4.19}$$

and

$$\tilde{S}_1^{\pm}(r) = \pm\frac{1}{2}\left(V(r) - (\tilde{S}_0^{\pm})'(r) - \frac{1}{r}\tilde{S}_0^{\pm}(r) - (\tilde{S}_0^{\pm})^2(r)\right) \tag{4.20}$$

are the first terms, and for $i \geq 1$ we have

$$\tilde{S}^{\pm}_{i+1}(r) = \pm\frac{1}{2}\left(-(\tilde{S}^{\pm}_i)'(r) - \frac{1}{r}\tilde{S}^{\pm}_i(r) - \sum_{m=0}^{i}\tilde{S}^{\pm}_m(r)\tilde{S}^{\pm}_{i-m}(r)\right), \tag{4.21}$$

where the $\pm$ indicates the choice of sign for the leading $\tilde{S}_{-1}$ term in the asymptotic expansion (4.18).

4.2.2 Non-zero Case

For $k \neq 0$, we plug

$$\phi_k(r, iku) = \exp\left\{\int_a^r \hat{\mathcal{S}}(t,k)\,\mathrm{d}t\right\} \tag{4.22}$$

into the differential equation

$$\phi''(r) + \frac{1}{r}\phi'(r) - \phi(r)\left(V(r) + k^2\left(u^2 + \frac{1}{r^2}\right)\right) = 0 \tag{4.23}$$

to get the first order differential equation,

$$\hat{\mathcal{S}}'(r,k) = k^2\left(u^2 + \frac{1}{r^2}\right) + V(r) - \hat{\mathcal{S}}^2(r,k) - \frac{1}{r}\hat{\mathcal{S}}(r,k). \tag{4.24}$$

As before, we now consider the large k behavior of $\hat{\mathcal{S}}(r,k)$ through the asymptotic expansion

$$\hat{\mathcal{S}}(r,k) \simeq \sum_{i=-1}^{\infty}\hat{S}_i(r)k^{-i}. \tag{4.25}$$

Plugging the expansion (4.25) into our first order differential equation (4.24) allows us to find the following expressions for the $\hat{S}_i(r)$ terms:

$$\hat{S}^{\pm}_{-1}(r) = \pm\sqrt{u^2 + \frac{1}{r^2}}, \qquad \hat{S}^{\pm}_0(r) = -\frac{1}{2r} - \frac{(\hat{S}^{\pm}_{-1})'(r)}{2\hat{S}^{\pm}_{-1}(r)}, \tag{4.26}$$

and

$$\hat{S}^{\pm}_1(r) = \frac{1}{2\hat{S}^{\pm}_{-1}(r)}\left(V(r) - (\hat{S}^{\pm}_0)'(r) - \frac{1}{r}\hat{S}^{\pm}_0(r) - (\hat{S}^{\pm}_0)^2(r)\right) \tag{4.27}$$

are the first terms, and for $i \geq 1$ we have

$$\hat{S}^{\pm}_{i+1}(r) = \frac{1}{2\hat{S}^{\pm}_{-1}(r)}\left(-(\hat{S}^{\pm}_i)'(r) - \frac{1}{r}\hat{S}^{\pm}_i(r) - \sum_{m=0}^{i}\hat{S}^{\pm}_m(r)\hat{S}^{\pm}_{i-m}(r)\right), \tag{4.28}$$

where the $\pm$ indicates the choice of sign for the leading $\hat{S}_{-1}$ term in the asymptotic expansion (4.25).

Now, returning to the differential equation (4.10) with $\lambda = iz$, we may use the analysis from the previous section to describe the asymptotic behavior for $z \to \infty$ and $|k| \to \infty$ of $R_k(r, iz)$ through the linear combination

$$R_k(r, iz) = A^+ \exp\left(\int_a^r \mathcal{S}^+(t, z)\,\mathrm{d}t\right) + A^- \exp\left(\int_a^r \mathcal{S}^-(t, z)\,\mathrm{d}t\right), \tag{4.29}$$

where

$$\mathcal{S}^\pm(t, z) = \begin{cases} \tilde{\mathcal{S}}^\pm(t, z) & \text{if } k = 0, \\ \hat{\mathcal{S}}^\pm(t, k) & \text{if } k \neq 0, \end{cases} \tag{4.30}$$

and we will similarly use the convention that $S_i^\pm$ refers to either $\tilde{S}_i^\pm$ or $\hat{S}_i^\pm$, depending on the case being considered. Applying the initial conditions given in (4.10), we find that

$$A^+ = -A^-, \qquad A^+ = \frac{1}{\mathcal{S}^+(a, z) - \mathcal{S}^-(a, z)}, \tag{4.31}$$

and now we may utilize (4.12) along with (4.29) to express $\Omega_k(iz)$ as

$$\Omega_k(iz) = \frac{c + \mathcal{S}^+(b, z)}{\mathcal{S}^+(a, z) - \mathcal{S}^-(a, z)} \exp\left(\int_a^b \mathcal{S}^+(t, z)\,\mathrm{d}t\right)(1 + \mathcal{E}_k(z)), \tag{4.32}$$

where $\mathcal{E}_k(z)$ represents exponentially damped terms for large z and large k. We may now begin to analyze the asymptotic behavior of $\ln \Omega_k(iz)$ which is the foundation for our analytic continuation of the spectral zeta function (4.14). From (4.32) we write

$$\ln \Omega_k(iz) = \ln[c + \mathcal{S}^+(b, z)] - \ln[\mathcal{S}^+(a, z) - \mathcal{S}^-(a, z)] + \int_a^b \mathcal{S}^+(t, z)\,\mathrm{d}t + \tilde{\mathcal{E}}_k(z), \tag{4.33}$$

where $\tilde{\mathcal{E}}_k(z)$ is exponentially damped. We now continue by considering the expansion of each term in (4.33). Beginning with $c + \mathcal{S}^+(b, z)$, we have

$$c + \mathcal{S}^+(b, z) = \begin{cases} z + \displaystyle\sum_{i=0}^{\infty} \frac{\sigma_i(b)}{z^i} & \text{if } k = 0, \\ \sqrt{u^2 + \dfrac{1}{b^2}}\, k + \displaystyle\sum_{i=0}^{\infty} \frac{\sigma_i(b)}{k^i} & \text{if } k \neq 0, \end{cases} \tag{4.34}$$

with

$$\sigma_0(b) = S_0^+(b) + c, \qquad \sigma_i(b) = S_i^+(b),\ i \geq 1. \tag{4.35}$$

Now, we use (4.34) to get

$$\ln[c + \mathcal{S}^+(b,z)] = \begin{cases} \ln z + \sum_{j=1}^{\infty} \frac{\tilde{\mathcal{Z}}_j(b)}{z^j} & \text{if } k = 0, \\ \frac{1}{2}\ln\left[u^2 + \frac{1}{b^2}\right] + \ln k + \sum_{j=1}^{\infty} \frac{\hat{\mathcal{Z}}_j(b)}{k^j} & \text{if } k \neq 0, \end{cases} \tag{4.36}$$

where $\tilde{\mathcal{Z}}_j(b)$ and $\hat{\mathcal{Z}}_j(b)$ are given through the relations

$$\ln\left[1 + \sum_{i=1}^{\infty} \frac{\sigma_{i-1}(b)}{z^i}\right] \simeq \sum_{j=1}^{\infty} \frac{\tilde{\mathcal{Z}}_j(b)}{z^j}\ , \tag{4.37}$$

$$\ln\left[1 + \frac{1}{\sqrt{u^2 + 1/b^2}} \sum_{i=1}^{\infty} \frac{\sigma_{i-1}(b)}{k^i}\right] \simeq \sum_{j=1}^{\infty} \frac{\hat{\mathcal{Z}}_j(b)}{k^j}\ . \tag{4.38}$$

Turning our focus to $\ln[\mathcal{S}^+(a,z) - \mathcal{S}^-(a,z)]$ in (4.33), we begin by pointing out that (4.21) and (4.28) gives us the relation

$$S_i^-(r) = (-1)^i S_i^+(r). \tag{4.39}$$

Using (4.39) allows us to write

$$\mathcal{S}^+(a,z) - \mathcal{S}^-(a,z) = 2z\left[1 + \sum_{i=1}^{\infty} \frac{S_{2i-1}^+(a)}{z^{2i}}\right] \tag{4.40}$$

for $k = 0$, and when $k \neq 0$ we get

$$\mathcal{S}^+(a,z) - \mathcal{S}^-(a,z) = 2\sqrt{u^2 + \frac{1}{a^2}}\ k\left[1 + \frac{1}{\sqrt{u^2 + 1/a^2}} \sum_{i=1}^{\infty} \frac{S_{2i-1}^+(a)}{k^{2i}}\right]. \tag{4.41}$$

We may now use these two equations to show that

$$\ln[\mathcal{S}^+(a,z) - \mathcal{S}^-(a,z)] = \begin{cases} \ln 2z + \sum_{j=1}^{\infty} \frac{\tilde{D}_{2j-1}(a)}{z^{2j}} & \text{if } k = 0 \\ \frac{1}{2}\ln\left[u^2 + \frac{1}{a^2}\right] + \ln 2k + \sum_{j=1}^{\infty} \frac{\hat{D}_{2j-1}(a)}{k^{2j}} & \text{if } k \neq 0 \end{cases}, \tag{4.42}$$

where $\tilde{D}_{2j-1}(a)$ and $\hat{D}_{2j-1}(a)$ are given through the relations

$$\ln\left[1+\sum_{i=1}^{\infty}\frac{S_{2i-1}^{+}(a)}{z^{2i}}\right] \simeq \sum_{j=1}^{\infty}\frac{\tilde{D}_{2j-1}(a)}{z^{2j}}\,, \tag{4.43}$$

$$\ln\left[1+\frac{1}{\sqrt{u^2+1/a^2}}\sum_{i=1}^{\infty}\frac{S_{2i-1}^{+}(a)}{k^{2i}}\right] \simeq \sum_{j=1}^{\infty}\frac{\hat{D}_{2j-1}(a)}{k^{2j}}\,. \tag{4.44}$$

Finally, we use (4.36) and (4.42) to write

$$\ln\Omega_0(iz) = (b-a)z - \ln 2 + \int_a^b S_0^{+}(t)\mathrm{d}t + \sum_{j=1}^{\infty}\frac{\widetilde{\mathcal{M}}_j}{z^j}, \tag{4.45}$$

and for $k \neq 0$, we have

$$\begin{aligned}\ln\Omega_k(iz) &= k\int_a^b S_{-1}^{+}(t)\mathrm{d}t + \frac{1}{2}\left(\ln\left[u^2+\frac{1}{b^2}\right] - \ln\left[u^2+\frac{1}{a^2}\right]\right)\\ &- \ln 2 + \int_a^b S_0^{+}(t)\mathrm{d}t + \sum_{j=1}^{\infty}\frac{\widehat{\mathcal{M}}_j}{k^j},\end{aligned} \tag{4.46}$$

where for $j = 2m+1$ with $m \in \mathbb{N}_0$,

$$\mathcal{M}_{2m+1} = \int_a^b S_{2m+1}^{+}(t)\mathrm{d}t + \mathcal{Z}_{2m+1}(b), \tag{4.47}$$

and for $j = 2m$ with $m \in \mathbb{N}^{+}$,

$$\mathcal{M}_{2m} = \int_a^b S_{2m}^{+}(t)\mathrm{d}t + \mathcal{Z}_{2m}(b) - D_{2m-1}(a). \tag{4.48}$$

Again, we omit the "tilde" and "hat" notation with the understanding that the coefficients depend on whether $k = 0$ or $k \neq 0$.

4.4 Analytic Continuation

In order to extend the region of convergence to the left of the half-plane $\mathrm{Re}(s) > 1$, we will subtract and add a suitable number of terms from the asymptotic expansion of $\ln\Omega_k(iz)$. From (4.14), we write

$$\zeta(s) = Z(s) + \sum_{j=-1}^{L} A_j(s), \tag{4.49}$$

where $Z(s) = \tilde{Z}(s) + \hat{Z}(s)$ and $A_j(s) = \tilde{A}_j(s) + \hat{A}_j(s)$, with

$$\begin{aligned}\tilde{Z}(s) &= \frac{\sin(\pi s)}{\pi}\int_m^\infty \mathrm{d}z\,(z^2-m^2)^{-s} \\ &\times \frac{\partial}{\partial z}\left[\ln\Omega_0(iz) - (b-a)z + \ln 2 - \int_a^b S_0^+(t)\mathrm{d}t - \sum_{j=1}^{L}\frac{\mathcal{M}_j}{z^j}\right], \quad (4.50)\end{aligned}$$

and

$$\begin{aligned}\hat{Z}(s) &= \frac{2\sin(\pi s)}{\pi}\sum_{k=1}^{\infty}\int_{m/k}^\infty \mathrm{d}u\,((uk)^2-m^2)^{-s}\frac{\partial}{\partial u}\Bigg[\ln\Omega_k(iuk) \\ &- k\int_a^b S_{-1}^+(t)\mathrm{d}t - \frac{1}{2}\left(\ln\left[u^2+\frac{1}{b^2}\right] - \ln\left[u^2+\frac{1}{a^2}\right]\right) \\ &+ \ln 2 - \int_a^b S_0^+(t)\mathrm{d}t - \sum_{j=1}^{L}\frac{\mathcal{M}_j}{k^j}\Bigg]. \quad (4.51)\end{aligned}$$

From (4.50) we can see that $\tilde{Z}(s)$ is analytic in the region $\mathrm{Re}(s) > -\frac{L+1}{2}$. Also, as can be seen from $\widehat{\mathcal{M}}_1$ and $\widehat{\mathcal{M}}_2$ given in the appendix, the leading large u behavior of $\widehat{\mathcal{M}}_j$ is $\frac{1}{u^j}$, so we have that the u-integral in (4.51) is convergent in this same region. However, with (4.51) we must also consider the series convergence. This restricts the region where $\hat{Z}(s)$ is analytic to $\mathrm{Re}(s) > -\frac{L}{2}$.

The terms $\tilde{A}_j(s)$ are given by

$$\tilde{A}_{-1}(s) = \frac{\sin(\pi s)}{\pi}\int_m^\infty \mathrm{d}z\,(z^2-m^2)^{-s}\frac{\partial}{\partial z}(b-a)z, \quad (4.52)$$

$$\tilde{A}_0(s) = 0, \quad (4.53)$$

and

$$\tilde{A}_i(s) = \frac{\sin(\pi s)}{\pi}\int_m^\infty \mathrm{d}z\,(z^2-m^2)^{-s}\frac{\partial}{\partial z}\frac{\mathcal{M}_i}{z^i}, \quad (4.54)$$

for $i \geq 1$, while $\hat{A}_j(s)$ is

$$\hat{A}_{-1}(s) = \frac{2\sin(\pi s)}{\pi}\sum_{k=1}^{\infty}\int_{m/k}^\infty \mathrm{d}u\,((uk)^2-m^2)^{-s}\frac{\partial}{\partial u}k\int_a^b S_{-1}^+(t)\mathrm{d}t, \quad (4.55)$$

$$\hat{A}_0(s) = \frac{2\sin(\pi s)}{\pi}\sum_{k=1}^{\infty}\int_{m/k}^\infty \mathrm{d}u\,((uk)^2-m^2)^{-s}$$

$$\times \quad \frac{\partial}{\partial u}\left(\frac{1}{2}\left(\ln\left[u^2+\frac{1}{b^2}\right]-\ln\left[u^2+\frac{1}{a^2}\right]\right)+\int_a^b S_0^+(t)\mathrm{d}t\right), \quad (4.56)$$

and

$$\hat{A}_i(s) = \frac{2\sin(\pi s)}{\pi}\sum_{k=1}^{\infty}\int_{m/k}^{\infty}\mathrm{d}u\,((uk)^2-m^2)^{-s}\frac{\partial}{\partial u}\frac{\mathcal{M}_i}{k^i}, \quad (4.57)$$

for $i \geq 1$. These terms will be calculated explicitly as needed for determining the heat kernel coefficients in the next section.

4.5 Coefficients of the Heat Kernel

We now may proceed as described in Section 3.5 to find the heat kernel coefficients, except in the two-dimensional case these coefficient are given by the formulas [16],

$$a_{1-s} = \Gamma(s)\mathrm{Res}\,\zeta(s)\ , \quad (4.58)$$

for $s = 1, \frac{1}{2}, -\frac{2n+1}{2}$ with $n \in \mathbb{N}_0$, and

$$a_{1+s} = \frac{(-1)^s}{s!}\zeta(-s)\ , \quad (4.59)$$

when $s \in \mathbb{N}_0$. As indicated in the previous section, equations (4.52) to (4.57) will be used in finding these coefficients. The relevant computations for the coefficients given below are,

$$\tilde{A}_{-1}(s) = \frac{(b-a)\sin(\pi s)}{2\pi(s-1/2)}\ , \quad (4.60)$$

$$\tilde{A}_0(s) = 0\ , \quad (4.61)$$

and

$$\tilde{A}_1(s) = \frac{\sin(\pi s)}{2\pi(s+1/2)}\left(\frac{1}{2b}-c-\frac{1}{8}\left(\frac{1}{b}-\frac{1}{a}\right)-\frac{1}{2}\int_a^b V(t)\mathrm{d}t\right)\ , \quad (4.62)$$

for the case when $k = 0$. In this case, it was necessary to take m to zero before performing the z-integration in (4.52) and (4.54). We, therefore, considered the integral from zero to one, and from one to infinity. The integral from zero to one does not contribute to the heat kernel coefficients, and so (4.60) and (4.62) only include the results from integrating from one to infinity. When $k \neq 0$, we have

$$\hat{A}_{-1}(s) = \frac{(b^{2s} - a^{2s})\Gamma(s-1/2)}{2\sqrt{\pi}\,\Gamma(s+1)}\,\zeta_R(2s-1)\,, \tag{4.63}$$

$$\hat{A}_0(s) = \frac{(b^{2s} - a^{2s})}{2}\,\zeta_R(2s)\,, \tag{4.64}$$

$$\begin{aligned}\hat{A}_1(s) &= \Bigg[\Gamma\left(s+\frac{1}{2}\right)\left(\frac{1}{12}(5s+1)\left(b^{2s}-a^{2s}\right) - \frac{1}{2}\int_a^b t^{2s+1}\,V(t)\,dt\right.\\ &\left.-cb^{2s+1} + \frac{b^{2s}}{2}\right) - b^{2s}\,\Gamma\left(s+\frac{3}{2}\right)\Bigg]\frac{2\,\zeta_R(2s+1)}{\sqrt{\pi}\,\Gamma(s)}\,,\end{aligned} \tag{4.65}$$

and

$$\begin{aligned}\hat{A}_2(s) &= \Bigg[a^{2s+2}V(a) - \frac{a^{2s}}{16}(5s^2+3s) + b^{2s+2}(c^2 - V(b)) + b^{2s+1}cs + \frac{b^{2s}}{16}(7s^2+s)\\ &+ (1+s)\int_a^b t^{2s+1}V(t)\mathrm{d}t + \frac{1}{2}\int_a^b t^{2s+2}V'(t)\mathrm{d}t\Bigg]\,s\,\zeta_R(2s+2)\,,\end{aligned} \tag{4.66}$$

where we use ζ_R to indicate the zeta function of Riemann. In this case, we waited until after performing the u-integration to send m to zero. Now, to calculate a_0 using (4.58), we need to consider the residue of the spectral zeta function when $s = 1$. In this case, taking L to be -1 in (4.49) will make $Z(s)$ analytic around $s = 1$, and we see that the only contribution to the residue comes from $\hat{A}_{-1}(s)$ which gives us,

$$a_0 = \frac{b^2 - a^2}{4}\,. \tag{4.67}$$

Continuing in this manner, taking L to be 0 allows us to find

$$a_{1/2} = \frac{\pi(b-a)}{4}\,. \tag{4.68}$$

For a_1 we use (4.59) to find

$$a_1 = -bc - \frac{1}{2}\int_a^b tV(t)dt\,. \tag{4.69}$$

Finally, for $a_{3/2}$, we use equations (4.58) and (4.60)-(4.66) to get,

$$a_{3/2} = \frac{\sqrt{\pi}}{2}\left(\frac{1}{64a} + \frac{a}{2}V(a) + \frac{5}{64b} + bc^2 - \frac{c}{2} - \frac{b}{2}V(b)\right)\,. \tag{4.70}$$

As was mentioned in Section 3.5, these coefficients may be compared against known results [12, 16, 28], and higher order coefficients may be found using the formulas given in this text.

APPENDICES

APPENDIX A

The Coefficients $\mathcal{M}_i$ and $\mathcal{N}_i$

Here we will give the coefficients $\mathcal{M}_i$ and $\mathcal{N}_i$ up to $i = 2$ that we used in our analytic continuation in Chapter 3. Obviously, higher order coefficients are easily obtained with the help of a simple computer program. In order to simplify the notation, for any integrable function f on I we use

$$[f] = \int_0^1 f(t)\mathrm{d}t \ . \tag{A.1}$$

By using the relations (3.58) and (3.59), the cumulant expansions (3.44), (3.54), and (3.55), and the recurrence relation (3.33), one obtains

$$\begin{aligned}\mathcal{M}_1 &= \left[\frac{V}{2\sqrt{p}} - \frac{(p')^2}{32p^{3/2}} + \frac{p''}{8\sqrt{p}}\right] + \frac{1-\delta(A_2)}{4A_2\sqrt{p(0)}}\left(A_2 p'(0) + 4A_1\right) \\ &- \frac{1-\delta(B_2)}{4B_2\sqrt{p(1)}}\left(B_2 p'(1) - 4B_1\right) \ ,\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}\mathcal{M}_2 &= \left[-\frac{(p')^3}{64p^2} - \frac{V'}{4} + \frac{p'p''}{32p} - \frac{1}{16}p^{(3)}\right] - \frac{V(0)}{2} + \frac{p'(0)^2}{32p(0)} - \frac{p''(0)}{8} \\ &+ (1-\delta(A_2))\left(-\frac{A_1^2}{2A_2^2 p(0)} + \frac{V(0)}{2} - \frac{A_1 p'(0)}{4A_2 p(0)} - \frac{p'(0)^2}{16p(0)} + \frac{p''(0)}{8}\right) \\ &+ (1-\delta(B_2))\left(-\frac{B_1^2}{2B_2^2 p(1)} + \frac{V(1)}{2} + \frac{B_1 p'(1)}{4B_2 p(1)} - \frac{p'(1)^2}{16p(1)} + \frac{p''(1)}{8}\right) \ .\end{aligned} \tag{A.3}$$

For the coefficients $\mathcal{N}_i$ we use the definitions (3.78) and (3.79), and the expansion (3.72) through (3.76) to find

$$\begin{aligned}\mathcal{N}_1 &= \left[\frac{V}{2\sqrt{p}} - \frac{(p')^2}{32p^{3/2}} + \frac{p''}{8\sqrt{p}}\right] - \delta(k_{12})\left(\frac{k_{11}p'(1) - k_{22}p'(0) + 4k_{21}}{4\left(\sqrt{p(1)}k_{11} + \sqrt{p(0)}k_{22}\right)}\right) \\ &- (1-\delta(k_{12}))\left(\frac{k_{22}\sqrt{p(0)} + k_{11}\sqrt{p(1)}}{k_{12}\sqrt{p(0)p(1)}} - \frac{p'(0)}{4\sqrt{p(0)}} + \frac{p'(1)}{4\sqrt{p(1)}}\right) \ ,\end{aligned} \tag{A.4}$$

and

$$\mathcal{N}_2 = \left[-\frac{(p')^3}{64p^2} - \frac{V'}{4} + \frac{p'p''}{32p} - \frac{1}{16}p^{(3)}\right] - \frac{V(0)}{2} + \frac{p'(0)^2}{32p(0)} - \frac{p''(0)}{8}$$

$$
\begin{aligned}
- &\quad \frac{(1-\delta(k_{12}))}{2}\Bigg\{ \left(\frac{k_{22}\sqrt{p(0)}+k_{11}\sqrt{p(1)}}{k_{12}\sqrt{p(0)p(1)}}\right)^2 - (V(0)+V(1)) \\
+ &\quad \frac{1}{8}\left(\frac{p'(0)^2}{p(0)}+\frac{p'(1)^2}{p(1)}\right) - \frac{1}{4}(p''(0)+p''(1)) - \frac{2k_{21}}{k_{12}\sqrt{p(0)p(1)}} \\
- &\quad \frac{k_{11}p'(0)}{2p(0)k_{12}} + \frac{k_{22}p'(1)}{2p(1)k_{12}}\Bigg\} - \frac{\delta(k_{12})}{32}\Bigg\{ \left(\frac{k_{11}p'(1)-k_{22}p'(0)+4k_{21}}{\sqrt{p(1)}k_{11}+\sqrt{p(0)}k_{22}}\right)^2 \\
- &\quad \frac{1}{\sqrt{p(1)}k_{11}+\sqrt{p(0)}k_{22}}\Bigg(4\sqrt{p(0)}k_{22}\left(4V(0)+p''(0)\right) \\
+ &\quad 4\sqrt{p(1)}k_{11}\left(4V(1)+p''(1)\right) - \frac{k_{22}p'(0)^2}{\sqrt{p(0)}} - \frac{k_{11}p'(1)^2}{\sqrt{p(1)}}\Bigg)\Bigg\}\,. \qquad \text{(A.5)}
\end{aligned}
$$

APPENDIX B

The Coefficients $\widetilde{\mathcal{M}}_j$ and $\widehat{\mathcal{M}}_j$

Here we give the coefficients $\widetilde{\mathcal{M}}_j$ and $\widehat{\mathcal{M}}_j$ up to $j = 2$ that were used in our analytic continuation in Chapter 4. Both of these coefficients rely heavily on $S^{+}(r)$, so we begin by using (4.20), (4.21), (4.27), and (4.28) to find that

$$\tilde{S}_1^{+}(r) = \frac{1}{2}\left(V(r) - \frac{1}{4r^2}\right) , \tag{B.1}$$

$$\tilde{S}_2^{+}(r) = -\frac{1}{8r^3} - \frac{1}{4}V'(r) , \tag{B.2}$$

$$\hat{S}_1^{+}(r) = \frac{-r^2u^4 + 4\left(r^2u^2+1\right)^2 V(r) + 4u^2}{8\sqrt{\frac{1}{r^2}+u^2}\left(r^2u^2+1\right)^2} , \tag{B.3}$$

and

$$\hat{S}_2^{+}(r) = -\frac{r\left(2r\left(r^2u^2+1\right)^3 V'(r) + 4\left(r^2u^2+1\right)^2 V(r) + u^2\left(r^4u^4 - 10r^2u^2 + 4\right)\right)}{8\left(r^2u^2+1\right)^4} . \tag{B.4}$$

Finally, equations (4.47) and (4.48) may be used to show that

$$\widetilde{\mathcal{M}}_1 = \frac{1}{2}\int_a^b V(t) - \frac{1}{4t^2}\, dt - \frac{1}{2b} + c , \tag{B.5}$$

$$\widetilde{\mathcal{M}}_2 = \frac{1}{8a^2} + \int_a^b \left(-\frac{1}{8t^3} - \frac{1}{4}V'(t)\right) dt - \frac{V(a)}{2} + \frac{-2b^2c^2 + 2b^2V(b) + 2bc - 1}{4b^2} , \tag{B.6}$$

$$\begin{aligned} \widehat{\mathcal{M}}_1 \;&=\; \int_a^b \frac{1}{8\sqrt{\frac{1}{t^2}+u^2}\left(t^2u^2+1\right)^2}\left(-t^2u^4 + 4\left(t^2u^2+1\right)^2 V(t) + 4u^2\right) dt \\ &+\; \frac{1}{\sqrt{\frac{1}{b^2}+u^2}}\left(\frac{1}{2b^3u^2+2b} - \frac{1}{2b} + c\right) , \end{aligned} \tag{B.7}$$

and

$$\widehat{\mathcal{M}}_2 \;=\; -\frac{a^2\left(-a^2u^4 + 4\left(a^2u^2+1\right)^2 V(a) + 4u^2\right)}{8\left(a^2u^2+1\right)^3}$$

$$
\begin{aligned}
- & \frac{b^2\left(-2c\,(b^3u^4+bu^2)+2\,(b^2cu^2+c)^2-2\,(b^2u^2+1)^2\,V(b)+u^2\,(b^2u^2-2)\right)}{4\,(b^2u^2+1)^3} \\
- & \int_a^b \frac{t}{8\,(t^2u^2+1)^4}\Big(2t\,\left(t^2u^2+1\right)^3 V'(t)+4\,\left(t^2u^2+1\right)^2 V(t) \\
+ & u^2(t^4u^4-10t^2u^2+4)\Big]\, dt\,. \qquad \text{(B.8)}
\end{aligned}
$$

Again, higher order coefficients may easily be obtained with the help of a computer program.

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