ABSTRACT

Multiplicity of Positive Solutions of Even-Order Nonhomogeneous
Boundary Value Problems

Britney Hopkins, Ph.D.

Advisor: Johnny Henderson, Ph.D.

In this work, we discuss multiplicity results for nonhomogeneous even-order boundary value problems on both discrete and continuous domains. We develop a method for establishing existence of positive solutions by transforming even-order problems into a series of second order problems satisfying homogeneous boundary conditions. We then construct a sequence of lemmas which give contraction and expansion relationships within a cone. This allows us to apply the Guo-Krasnosel'skii Fixed Point Theorem which, in turn, guarantees several positive solutions.

Multiplicity of Positive Solutions of Even-Order Nonhomogeneous
Boundary Value Problems

by

Britney Hopkins, B.S.,B.A.,M.S.

A Dissertation

Approved by the Department of Mathematics

Lance L. Littlejohn, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

Approved by the Dissertation Committee

Johnny Henderson, Ph.D., Chairperson

John M. Davis, Ph.D.

Lance L. Littlejohn, Ph.D.

Frank H. Mathis, Ph.D.

Dean M. Young, Ph.D.

Accepted by the Graduate School
May 2009

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

This work could not be accomplished without the guidance and support of many individuals. I would like to begin by thanking Baylor University and the Mathematics faculty for giving me this opportunity. A special thanks goes to my fellow graduate students. Making it through these last few years without them would be unimaginable. I would also like to thank my committee members for their time and commitment to this work. I especially thank my advisor, Dr. Johnny Henderson, for taking me under his wing. It is both an honor and privilege to work with such a man. To the man who introduced me to the world of mathematical research, I thank Dr. Nickolai Kosmatov for both believing in and encouraging me to continue my graduate studies. I thank Marc Patenaude for both motivating and helping me through the daunting writing process. Finally, I thank my family, especially my grandmother for teaching me and pushing me to strive for my full potential. She has never doubted me, even when I have.

CHAPTER ONE

Introduction

This work focuses on the multiplicity of positive solutions for various types of even order boundary value problems. By expounding upon the techniques used by Marcos, Lorca, and Ubilla in [26], we develop methods of reducing higher even order boundary value problems into a series of second order boundary value problems. We then construct a series of lemmas which lead to contraction and expansion estimates within a cone. This culminates in allowing applications of the Guo-Krasnosel'skii Fixed Point Theorem a total of three times, thus guaranteeing at least three positive solutions.

In Chapter 2, we consider even order boundary value problems of the form

$$u^{(2n)} = \lambda h\left(t, u, u'' \ldots, u^{(2(n-1))}\right),\ t \in (0,1),\ n \geq 2, \tag{1.1}$$

$$u^{(2k)}(0) = 0,\ \ k = 0, \ldots, n-1, \tag{1.2}$$

$$u^{(2k)}(1) = (-1)^k a_k,\ \ k = 0, \ldots, n-1. \tag{1.3}$$

We begin by giving a brief overview of the fourth order case studied in [26]. We then prove the result for the case where $n = 3$, highlighting the similarities and differences between the fourth order case and the sixth order case. This gives a better understanding of how to ultimately move to the generalized $2n$th order case. We conclude the chapter by establishing a generalized existence result. In Chapter 3, we expand the transformation technique to include even order problems of the form (1.1) satisfying the right focal boundary conditions,

$$u^{(2k)}(0) = 0,\ \ k = 0, \ldots, n-1,$$

$$u^{(2k+1)}(1) = (-1)^k a_k,\ \ k = 0, \ldots, n-1.$$

Again, we first develop an existence result for the fourth order problem, and then use it as a stepping stone for finding positive solutions for the $2n$th order case. Chapter 4 is devoted to modifying the transformation technique to include problems on discrete domains. We do so by considering a fourth order difference equation analogous to the one studied in Chapter 2; that is the fourth order problem,

$$\Delta^4 u(t-2) = \lambda h(t, u(t), \Delta^2 u(t-1)),\ t \in (0, N+2)_{\mathbb{Z}}, \tag{1.4}$$

under the conjugate boundary conditions,

$$u(0) = 0,\ \Delta^2 u(-1) = 0, \tag{1.5}$$

$$u(N+2) = a,\ \Delta^2 u(N+1) = -b. \tag{1.6}$$

Although the transformation technique is altered, we still assure solutions by first transforming the problem, and then constructing expansion and contraction estimates so that we may apply the desired fixed point theorem.

1.1 History

The study of multiple positive solutions for boundary value problems is a vastly researched field that has applications in modeling real world phenomena. Some examples of recent works concerning such results include [2], [3], [18], [6], [7], [9], [11], [13], [14], [15],[16], [19], [22], [24], [25], [26], [28], [29], [30], [31], [33], [36], [37] and the references therein. Many of these authors specifically focus on existence results for even order problems. For example, in [1], Agarwal gives an existence and uniqueness result for the fourth order problem $x^{(4)} = f(t, x, x', x'', x^{(3)})$, arising from beam analysis. Some other notable works on even order boundary value problems include, but are not limited to [2], [3], [4], [18], [10], [12], [16], [21], [22], [23], [26], [29], [32], [34].

Many of the even order boundary problems considered in this work revolve around the ability to transform a higher order problem into a system of second order differential equations of the form

$$-u''(t) = f(u(t)),\ t \in (0,1), \tag{1.7}$$

satisfying either the conjugate homogenous boundary conditions,

$$u(0) = u''(1) = 0, \tag{1.8}$$

or the right focal homogenous boundary conditions,

$$u(0) = u'(1) = 0. \tag{1.9}$$

It is well known that solutions for the differential equation (1.7), satisfying either of these boundary conditions are merely fixed points of the operator

$$Tu = \int_0^1 G(t,s)f(u(s))ds, \tag{1.10}$$

where $G(t, s)$ is the Green's function corresponding to the respective boundary conditions. Thus many authors arrive at existence results by applying various fixed point theorems. Some of the more commonly used fixed point theorems include the Leggett Williams Fixed Point Theorem, as seen in [8], [11], [21], [28], [29], [31], [37], and the Guo-Krasnosel'skii Fixed Point Theorem, as used in [5], [15], [20], [22], [25], [26].

In [3], Avery, Henderson, and O'Regan generalized the Five Functionals Fixed Point Theorem, a generalization of the Leggett Williams Fixed Point Theorem, and then applied it to a right focal boundary value problem of the form (1.7), (1.9) to show the existence of at least three positive solutions. In [26], Marcos, Lorca, and Ubilla considered a fourth order conjugate boundary value problem and found solutions by transforming the problem into two second order differential equations of the form (1.7), (1.8). They appealed to the Guo-Krasnosel'skii Fixed Point Theorem to ascertain at least three positive solutions. This method is discussed in further detail in Chapter 2 and is the motivation for this work.

Notice, equation (1.7) is for a differential equation, however, existence results are not limited to a continuous domain. Several authors have studied positive solutions for various types of boundary value problems on both discrete domains and on time scales, as seen in [2], [5], [6], [8], [11], [16], [27].

1.2 Definitions and Theorems

This section is devoted to presenting various definitions and results that are referred to throughout this work. As previously stated, our goal is to assure the existence of positive solutions for various boundary value problems by applying the Guo-Krasnosel'skii Fixed Point Theorem. As a result of this theorem, the solutions we find are contained within a cone. Thus, we give the following definition:

Definition 1.1. Let $(X, \|\cdot\|)$ be a real Banach Space. A cone, C, in X is a nonempty, closed, convex subset of X satisfying both of the following properties:

(1) If $x \in C$, and $\lambda > 0$, then $\lambda x \in C$.

(2) If $x \in C$ and $-x \in C$, then $x = 0$.

Now that we have defined a cone, we are ready to state the fixed point theorem that leads to our existence results.

Theorem 1.1. *[Guo-Krasnosel'skii Fixed Point Theorem] Let $(X, \|\cdot\|)$ be a Banach Space and $C \subset X$ be a cone. Suppose Ω_1, Ω_2 are open subsets of X satisfying $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ is a completely continuous operator such that either*

i $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_2$ or

ii $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_2$,

then T has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Although the cone we use on each type of boundary value problem differs, they each carry some sort of concavity condition. In Chapters 2 and 3, we repeatedly take advantage of this condition, combined with the result below.

Lemma 1.1. *Let* $u(t)$ *be a nonnegative concave function which is continuous on* $[0,1]$*. Then for all* $\alpha, \beta \in (0,1)$*, with* $\alpha < \beta$*, we have*

$$\inf_{t\in[\alpha,\beta]} u(t) \geq \alpha(1-\beta)\|u\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} u(t)$

Proof. If $u(t) \equiv 0$, the result clearly holds. Assume $u(t) \not\equiv 0$ and let $t_0 \in [0,1]$ be such that $u(t_0) = \|u\|_\infty$. Since $u(t)$ is concave, either $u(\alpha) = \inf_{t\in[\alpha,\beta]} u(t)$ or $u(\beta) = \inf_{t\in[\alpha,\beta]} u(t)$. First suppose $u(\alpha) = \inf_{t\in[\alpha,\beta]} u(t)$. Then $0 < \alpha \leq t_0 \leq 1$, and

$$\frac{u(\alpha)}{\alpha} \geq \frac{\|u\|_\infty}{t_0},$$

due to concavity. Furthermore, as $t_0^{-1} \geq 1$ and $(1-\beta) < 1$,

$$\frac{\|u\|_\infty}{t_0} \geq \|u\|_\infty > (1-\beta)\|u\|_\infty.$$

Therefore,

$$\inf_{t\in[\alpha,\beta]} u(t) = u(\alpha) > \alpha(1-\beta)\|u\|_\infty.$$

Now suppose $u(\beta) = \inf_{t\in[\alpha,\beta]} u(t)$. Then, by concavity, $0 \leq t_0 \leq \beta < 1$. Thus,

$$\frac{\|u\|_\infty}{1-t_0} \geq \|u\|_\infty > \alpha\|u\|_\infty.$$

Concavity also gives that

$$\frac{u(\beta)}{\beta-1} \leq \frac{\|u\|_\infty}{t_0-1}.$$

So,

$$\frac{u(\beta)}{1-\beta} \geq \frac{\|u\|_\infty}{1-t_0} > \alpha\|u\|_\infty,$$

yielding

$$\inf_{t\in[\alpha,\beta]} u(t) = u(\beta) > \alpha(1-\beta)\|u\|_\infty.$$

□

CHAPTER TWO

Differential Equations with Conjugate Boundary Conditions

The purpose of this chapter is to establish an existence result for even order ordinary differential equations of the form

$$u^{(2n)} = \lambda h\left(t, u, u'', \dots, u^{(2(n-1))}\right), \; t \in (0,1),$$

for $n \geq 2$, with the nonhomogeneous boundary conditions,

$$\begin{aligned} &u^{(2k)}(0) = 0, \;\; k = 0, \dots, n-1, \\ &u^{(2k)}(1) = (-1)^k a_k, \;\; k = 0, \dots, n-1, \end{aligned}$$

where $\lambda, a_k \geq 0, \; k = 0, \dots, n-1$, and $\sum_{k=0}^{n-1} a_k > 0$. We start, in the first section, by exploring the work of Marcos, Lorca, and Ubilla, who showed the existence of multiple solutions for the case where $n = 2$. In Section 2.2, we extend their result to the $n = 3$ case, paying close attention to how to accommodate the jump in order. This gives us a better understanding of the general even order case, which we present in the final section.

2.1 The Fourth Order Problem

In [26] Marcos, Lorca, and Ubilla considered the fourth order nonhomogeneous boundary value problem

$$u^{(4)} = \lambda h(t, u, u''), \;\; t \in (0,1), \tag{2.1}$$

$$u(0) = u''(0) = 0, \tag{2.2}$$

$$u(1) = a, \; u''(1) = -b, \tag{2.3}$$

where λ, a, b are nonnegative and $a + b > 0$. Ultimately, they showed the existence of three positive solutions via the Guo-Krasnosel'skii Fixed Point Theorem. Rather than working directly with (2.1), (2.2), and (2.3), they made a series of substitutions and then transformed the original problem into the system of second order ordinary differential equations,

$$-u''(t) = g(t, u + ta, v + tb), \;\; t \in (0, 1), \tag{2.4}$$

$$-v''(t) = \lambda f(t, u + ta, v + tb), \;\; t \in (0, 1), \tag{2.5}$$

having the homogeneous boundary conditions,

$$u(0) = v(0) = 0, \tag{2.6}$$

$$u(1) = v(1) = 0. \tag{2.7}$$

Their transformation technique plays a key role in establishing the main result and is a technique which is carried throughout this work. Our primary interest in using this method is in the interplay between the sign of the substitutions and how it effects the sign of the final solutions. Their goal, similar to ours, is to find positive solutions to the given boundary value problem, and since the Guo-Krasnosel'skii Fixed Point Theorem is being used, these solutions must reside in a cone. One of the requirements the authors place on their cone is that each of the elements is concave, making the second derivative of each element nonpositive. As $\lambda f = -v'' \geq 0$ and $g = -u'' \geq 0$, we see that sign constraints must be placed on both f and g. Acknowledging that this logic is easily reversed serves as motivation for making f and g nonnegative in the first hypothesis. It also tells us that h in (2.1) must be nonnegative.

After making the above transformation and stating their preliminary work, the authors set out to prove the main result, that is assume $f(t, u, v)$ and $g(t, u, v)$ satisfy the following hypotheses:

(H0) $f, g : [0, 1] \times [0, \infty)^2 \rightarrow [0, \infty)$ are continuous functions that are non-decreasing in the last two variables.

(H1) There exists an $\alpha_1, \beta_1 \in (0,1)$, with $\alpha_1 < \beta_1$, such that given $(u,v) > (0,0)$, there is a $c_1 = c_1(u,v) > 0$ so that

$$f(t,u,v) > c_1, \;\; t \in [\alpha_1, \beta_1],$$

and there exists and $\alpha_2, \beta_2 \in (0,1)$, with $\alpha_2 < \beta_2$, such that, given $v > 0$, there exists a $c_2 = c_2(v) > 0$ so that

$$g(t,0,v) > c_2, \;\; t \in [\alpha_2, \beta_2].$$

(H2) $\lim\limits_{u+v \to 0^+} = \dfrac{f(t,u,v)}{u+v} = 0$ uniformly for $t \in [0,1]$.

(H3) $\lim\limits_{u+v \to \infty} = \dfrac{f(t,u,v)}{u+v} = 0$ uniformly for $t \in [0,1]$.

(H4) There exists a $0 < \delta_2 < 8$ and $\overline{\rho} > 0$ such that, for all $(u,v) \in [0,\infty)^2$ with $0 < u+v < \overline{\rho}$, we have $g(t,u,v) \leq \delta_2(u+v)$, for each $t \in [0,1]$.

(H5) There exists a $0 < \delta_1 < 8$ and $\overline{R} > 0$ such that, for all $(u,v) \in [0,\infty)^2$ with $u+v > \overline{R}$, we have $g(t,u,v) \leq \delta_1(u+v)$, for each $t \in [0,1]$.

Then there is a Λ so that, given $\lambda > \Lambda$, there is a $\delta > 0$ such that the system

$$\begin{aligned}
&-u'' = g(t,u,v), \; t \in (0,1), \\
&-v'' = \lambda f(t,u,v), \; t \in (0,1), \\
&u(0) = v(0) = 0, \\
&u(1) = a, \;\; v(1) = b,
\end{aligned}$$

has at least three positive solutions for all $(a,b) \in [0,\infty)^2$ with $0 < a+b < \delta$.

Before proving this result, the authors first state and prove four lemmas which give expansion and compression relationships in terms of their defined operator. These culminate to not only establish the existence of a Λ and a δ, but also to satisfy the (a) portion of the Guo-Krasnosel'skii Fixed Point Theorem once, and the (b) portion twice, thus leading to a simple existence proof of not one, but at least three positive solutions.

2.2 *Sixth Order Problem*

This section centers on the multiplicity of solutions for sixth order boundary value problems of the form

$$u^{(6)} = \lambda h\left(t, u, u'', u^{(4)}\right), \ \ t \in (0,1), \tag{2.8}$$

$$u(0) = u''(0) = u^{(4)}(0) = 0, \tag{2.9}$$

$$u(1) = a, \ u''(1) = -b, \ u^{(4)}(1) = c, \tag{2.10}$$

where $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to (-\infty, 0]$ is continuous, $a, b, c, \lambda \geq 0$, and $a+b+c > 0$. Although we prove an existence theorem similar to that of the fourth order result, the purpose of this section is not only to illustrate the result for the case where $n = 3$, but also to establish the commonalities and differences between the fourth order and the sixth order problems. This helps pave the way towards the overall goal of this chapter by providing a blue-print of how to state and prove a main result for the even order generalization.

2.2.1 *Preliminaries*

We start by laying the ground work for an existence result similar to the one mentioned in the previous section. By setting $v = -u''$, $w = -v'' = u^{(4)}$, $g_1(t,u,v,w) = v$, $g_2(t,u,v,w) = w$, and $-h(t,u,v,w) = f(t,u,-v,w)$, we observe that solutions of (2.8) with boundary conditions (2.9), (2.10) are in one to one correspondence with solutions of the system of second order boundary value problems,

$$-w'' = \lambda f(t,u,v,w), \tag{2.11}$$

$$-v'' = g_2(t,u,v,w), \tag{2.12}$$

$$-u'' = g_1(t,u,v,w), \tag{2.13}$$

$$u(0) = v(0) = w(0) = 0, \tag{2.14}$$

$$u(1) = a, \ v(1) = b, \ w(1) = c, \tag{2.15}$$

where $a, b, c, \lambda \geq 0$ with $a+b+c > 0$, and $t \in (0,1)$.

Appealing to the technique used in [26], note that in order to obtain positive solutions of (2.11)-(2.15), certain constraints must be placed on f, or h if you are looking at the original problem, which lead to constraints on g_1 and g_2. Using the reversible reasoning mentioned in Section 2.1, we start by looking at the conditions a positive solution, u, of (2.8)-(2.10) must satisfy. Since $u''(1) = -b < 0$, $u(t)$ is concave at $t = 1$. In fact, the cone used to eventually find solutions carries a concavity condition. This, in turn, tells us that $u'', -u^{(4)}, u^{(6)} \leq 0$ on all $[0, 1]$, falling in line with the signs of our original boundary conditions. Notice, the sign of $u^{(2k)}$ for $k = 0, 1, 2, 3$, oscillates from nonnegative to nonpositive depending on the choice of k. This observation is important in selecting the substitutions used above and plays a role in the generalization process. Since we are taking $g_1 = v = -u'' \geq 0$ and $g_2 = w = u^{(4)} \geq 0$, we see that g_1 and g_2 are nonnegative. Similarly, $f = -w'' = -u^{(6)} \geq 0$, giving that f is nonnegative. Combining this with the substitution tells that h must be nonpositive. Noting the reversibility of this argument, we place the following assumptions on the functions f:

(H0) $f : [0, 1] \times [0, \infty)^3 \to [0, \infty)$ is a continuous function which is nondecreasing in the last three variables.

(H1) There exists $\alpha_1, \beta_1 \in (0, 1)$ such that, given $(u, v, w) \in [0, \infty)^3$ with $u + v + w \neq 0$, there is a $k > 0$ such that

$$f(t, u, v, w) > k$$

for $t \in (\alpha_1, \beta_1)$.

(H2) Let $z = u + v + w$. Then $\displaystyle\lim_{z \to 0^+} \frac{f(t, u, v, w)}{(u + v + w)} = 0$ uniformly for $t \in [0, 1]$.

(H3) Let $z = u + v + w$. Then $\displaystyle\lim_{z \to \infty} \frac{f(t, u, v, w)}{(u + v + w)} = 0$ uniformly for $t \in [0, 1]$.

For simplicity, we transform the system (2.11)-(2.15) into the system of second order differential equations,

$$-u'' = g_1(t, u + ta, v + tb, w + tc),\ t \in (0, 1), \tag{2.16}$$

$$-v'' = g_2(t, u + ta, v + tb, w + tc),\ t \in (0, 1), \tag{2.17}$$

$$-w'' = \lambda f(t, u + ta, v + tb, w + tc),\ t \in (0, 1), \tag{2.18}$$

under the homogeneous conjugate boundary conditions,

$$u(0) = u(1) = 0, \tag{2.19}$$

$$v(0) = v(1) = 0, \tag{2.20}$$

$$w(0) = w(1) = 0. \tag{2.21}$$

The main result given in the concluding portion of this section actually gives an existence result for the boundary value problem above. However, since these systems are virtually equivalent, if solutions to the above boundary value problem exist, then they also exist for (2.8)-(2.10). We note that by integrating and then taking the boundary conditions into consideration, solutions of system (2.18)-(2.21) are of the form

$$\begin{aligned} u(t) &= \int_0^1 G(t,s) g_1(s, u(s) + sa, v(s) + sb, w(s) + sc)\, ds, \\ v(t) &= \int_0^1 G(t,s) g_2(s, u(s) + sa, v(s) + sb, w(s) + sc)\, ds, \\ w(t) &= \lambda \int_0^1 G(t,s) f(s, u(s) + sa, v(s) + sb, w(s) + sc)\, ds, \end{aligned}$$

where $G(t, s)$ denotes the Green's function

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

By simply glancing at $G(t, s)$, it is easy to see that it is a positive function. Furthermore, since f, g_1, and g_2 are positive as well, the solutions u, v, and w are positive, if they exist. Some additional useful properties concerning the Green's function are that

$$\max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{8}, \tag{2.22}$$

and

$$\max_{t\in[0,1]} \int_0^1 \left| \frac{\partial G(t,s)}{\partial t} \right| ds = \frac{1}{2}, \tag{2.23}$$

both of which appear in the complete continuity argument at the end of this section. Equation (2.22) also plays a role in acquiring the estimates given in the next section and is an underlying factor in the bound on $\delta_1 + \delta_2$ found in the proof of Lemma 2.4, as well as the bound on η in used in Lemma 2.5.

Recall that in order to apply the Guo-Krosnosel'skii Fixed Point Theorem, we need a Banach space containing a cone. With this in mind, let $(X, \|\cdot\|)$ denote the Banach space $\prod_{j=1}^3 C([0,1];\mathbb{R})$ endowed with the norm

$$\|(u,v,w)\| = \|u\|_\infty + \|v\|_\infty + \|w\|_\infty,$$

where $\|u\|_\infty = \sup\limits_{t\in[0,1]} |u(t)|$, and let $C \subset X$ denote the set

$$C = \{(u,v,w) \in X | (u,v,w)(0) = (u,v,w)(1) = (0,0,0) \text{ and } u,v,w \text{ are concave}\}.$$

The fact that C is a cone follows directly from the definition. Moreover, take Ω_ρ to be the open set $\Omega_\rho = \{(u,v,w) \in X : \|(u,v,w)\| < \rho\}$. Finally, we define the operator $T : X \to X$ by $T(u,v,w)(t) = (A_1(u,v,w), A_2(u,v,w), A_3(u,v,w))(t)$, where

$$A_i(u,v,w)(t) = \int_0^1 G(t,s) g_i(s, u(s)+as, v(s)+sb, w(s)+sc)\, ds, \;\; i = 1,2,$$

$$A_3(u,v,w)(t) = \lambda \int_0^1 G(t,s) f(s, u(s)+as, v(s)+sb, w(s)+sc) ds.$$

Note that solutions of (2.18)-(2.21) are fixed points of T.

We conclude the preliminary section by giving a lemma yielding two properties of the operator, T.

Lemma 2.1. *T is a completely continuous operator and $T : C \to C$.*

Proof. We start by showing that T preserves the cone. To this end, let $(u, v, w) \in C$ and note that for $t \in [0, 1]$,

$$A_i''(u, v, w)(t) = -g_i(t, u(t) + ta, v(t) + tb, w(t) + tc) \leq 0, \;\; i = 1, 2,$$

and

$$A_3''(u, v, w)(t) = -\lambda f(t, u(t) + ta, v(t) + tb, w(t) + tc) \leq 0.$$

Hence $A_i(u, v, w)(t)$ is concave for $i = 1, 2, 3$. Since $G(0, s) = 0 = G(1, s)$, we have

$$\begin{aligned}(A_1(u, v, w), A_2(u, v, w), A_3(u, v, w))(0) &= (A_1(u, v, w), A_2(u, v, w), A_3(u, v, w))(1)\\ &= (0, 0, 0),\end{aligned}$$

giving that $T(u, v, w) \in C$.

Next we show that T is a completely continuous operator. It suffices to show that each $A_i : X \to C([0, 1]; \mathbb{R})$, $i = 1, 2, 3$ is completely continuous. Let's start by looking at A_1. Set $B_r := \{(u, v, w) \in X : \|u\|_\infty + \|v\|_\infty + \|w\|_\infty \leq r\}$. Since g_1 is continuous, there is a $k = k(r) \in \mathbb{Z}$ such that $|g_1(t, u(t)+ta, v(t)+tb, w(t)+tc)| < k$ for every $(u, v, w) \in B_r$ and $t \in [0, 1]$. Let $\{(u_n, v_n, w_n)\}$ be a sequence in B_r. Then, by (2.22),

$$\begin{aligned}|A_1(u_n, v_n, w_n)(t)| &\leq \int_0^1 |G(t, s)g_1(s, u_n(s) + sa, v_n(s) + sb, w_n(s) + sc)|ds\\ &\leq k\int_0^1 G(t, s)ds\\ &\leq \frac{k}{8}\end{aligned}$$

for $t \in [0, 1]$, giving $\|A_1(u_n, v_n, w_n)\|_\infty \leq k/8$, i.e., $\{A_1(u_n, v_n, w_n)\}$ is uniformly bounded. Furthermore, by (2.23), we see that

$$\begin{aligned}\left|\frac{d}{dt}A_1(u_n, v_n, w_n)(t)\right| &= \left|\int_0^1 \frac{\partial}{\partial t}G(t, s)g_1(s, u_n(s) + sa, v_n(s) + sb, w_n(s) + sc)ds\right|\\ &\leq \frac{k}{2}\end{aligned}$$

for $t \in [0,1]$. Let $\epsilon > 0$ and pick $\delta > \frac{2\epsilon}{k}$. Then for any $t_1 < t_2$ in $(0,1)$ with $|t_1 - t_2| < \delta$, we have by the Mean Value Theorem, that there is a $\tau \in (t_1, t_2)$ such that

$$\begin{aligned} |A_1(u_n, v_n, w_n)(t_1) - A_1(u_n, v_n, w_n)(t_2)| &\leq \left| \frac{d}{dt} A_1(u_n, v_n, w_n)(\tau) \right| |t_1 - t_2| \\ &\leq \frac{k\delta}{2} \\ &< \epsilon. \end{aligned}$$

Hence, $\{A_1(u_n, v_n, w_n)\}$ is a uniformly equicontinuous sequence of functions. Therefore, by the Arzela-Ascoli Theorem, it has a convergent subsequence, giving that A_1 is a completely continuous operator. Since the proof is the same for A_2 and A_3 it follows that T is completely continuous. □

2.2.2 Lemmas

Now that the preliminary work is in place, we present four lemmas, each yielding a relationship between $\|T(u,v,w)\|$ and $\|(u,v,w)\|$ at certain points within the cone C. The proof of the main result follows directly by applying these four lemmas in conjunction with the Guo-Krasnosel'skii Fixed Point Theorem.

The first two lemmas give sets in C for which we have the relationship

$$\|T(u,v,w)\| \geq \|(u,v,w)\|,$$

based only on assumptions (H0) and (H1). Since we are looking for sets having lower bounds on T, it suffices to show that either A_1, A_2, or A_3 is bounded below. With this in mind, we take advantage of assumptions (H0) and (H1) to find lower bounds on A_3.

Lemma 2.2. *Suppose* (H0) *and* (H1) *hold and let* $\rho^* > 0$. *Then there exists a* Λ *such that for every* $\lambda \geq \Lambda$ *and* $(a,b,c) \in [0,\infty)^3$,

$$\|T(u,v,w)\| \geq \|(u,v,w)\|$$

for each $(u,v,w) \in C \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and $(u,v,w) \in C \cap \partial\Omega_{\rho^*}$. Set $r = \alpha_1(1-\beta_1)$ and $\Lambda \geq \left[rM\int_{\alpha_1}^{\beta_1} G(1/2,s)ds\right]^{-1}$, where

$$M = \inf\left\{\frac{f(t,ra_1,ra_2,ra_3)}{r(a_1+a_2+a_3)} : t \in [\alpha_1,\beta_1],\ a_1,a_2,a_3 > 0,\ (a_1+a_2+a_3) = \rho^*\right\}.$$

It follows from (H1) that M exists and is nonzero. By Lemma 1.1, $u(t)+ta \geq u(t) \geq r\|u\|_\infty$ for $t \in [\alpha_1,\beta_1]$. Similar inequalities hold for v and w. This combined with the nondecreasing property placed on f in (H0) gives

$$\begin{aligned}
\|T(u,v,w)\| &\geq \|A_3(u,v,w)\|_\infty \\
&\geq \lambda\int_0^1 G(1/2,s)f(s,u(s)+sa,v(s)+sb,w(s)+sc)ds \\
&\geq \lambda\int_{\alpha_1}^{\beta_1} G(1/2,s)f(s,r\|u\|_\infty,r\|v\|_\infty,r\|w\|_\infty)ds \\
&\geq \lambda r\|(u,v,w)\|\int_{\alpha_1}^{\beta_1} G(1/2,s)\frac{f(s,r\|u\|_\infty,r\|v\|_\infty,r\|w\|_\infty)}{r\|(u,v,w)\|}ds \\
&\geq \Lambda rM\|(u,v,w)\|\int_{\alpha_1}^{\beta_1} G(1/2,s)ds \\
&\geq \|(u,v,w)\|,
\end{aligned}$$

for $\lambda \geq \Lambda$, completing the proof. □

Lemma 2.3. *Fix* $\Lambda > 0$ *and suppose* (H0) *and* (H1) *hold. Then for every* $\lambda \geq \Lambda$ *and* $(a,b,c) \in [0,\infty)^3$ *with* $a+b+c \geq 0$, *there is a* $\rho_1 = \rho_1(\Lambda,a,b,c)$ *such that for every* $\rho \leq \rho_1$, *we have*

$$\|T(u,v,w)\| \geq \|(u,v,w)\|$$

for $(u,v,w) \in C \cap \partial\Omega_\rho$.

Proof. By (H1) and the nondecreasing properties of f, there is a $k > 0$ such that,

$$f(t,u(t)+ta,v(t)+tb,w(t)+tc) \geq f(t,\alpha_1 a,\alpha_1 b,\alpha_1 c) > k,$$

for $t \in [\alpha_1, \beta_1]$. Take $\rho_1 = \Lambda k \int_{\alpha_1}^{\beta_1} G(1/2, s)ds$. Then, for all $(u, v, w) \in C \cap \Omega_\rho$ where $\rho \leq \rho_1$,

$$\begin{aligned}\|T(u,v,w)\| &\geq \lambda \int_{\alpha_1}^{\beta_1} G(1/2,s) f(s, \alpha_1 a, \alpha_1 b, \alpha_1 c) ds \\ &\geq \lambda k \|(u,v,w)\| \int_{\alpha_1}^{\beta_1} \frac{G(1/2,s)}{\|(u,v,w)\|} ds \\ &\geq \|(u,v,w)\|.\end{aligned}$$

□

We emphasize that although each of the above lemmas produce lower bounds on T, they do so for separate sets. In fact, the open set introduced in Lemma 2.3, Ω_{ρ_1}, is contained in the open set Ω_{ρ^*} from Lemma 2.2. We will see that this inclusion is proper by Lemma 2.4. Also notice that both Lemma 2.2 and Lemma 2.3 rely solely on assumptions (H0) and (H1). These assumptions are quite similar to the first two hypotheses made in the fourth order case, the only difference being that f has one more variable. By making slight modifications on assumptions (H0) and (H1) which take the extra terms into consideration, Lemmas 2.2 and 2.3 can easily be extended to the generalized case.

The last two lemmas give sets for which we have the opposite relationship between $\|T(u, v, w)\|$ and $\|(u, v, w)\|$. Unlike the first two lemmas, these lemmas make use of the fact that both g_1 and g_2 are nondecreasing in the last n variables. Furthermore, Lemma 2.4 takes advantage of the growth conditions placed on f in (H2) as well as the ability to find bounds on g_1 and g_2 when $u, v,$ and w have a sufficiently small sum. Lemma 2.5 gives a similar result, however, it uses assumption (H3) and bounds on g_1 and g_2, which depend on the sum of $u, v,$ and w being sufficiently large.

Lemma 2.4. *Suppose* (H0) *and* (H2) *hold and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$*, there is a* $\rho_2 \in (0, \rho^*)$ *and a* $\delta > 0$ *such that for every* $(a, b, c) \in [0, \infty)^3$ *with*

$0 < a + b + c < \delta,$

$$\|T(u,v,w)\| \le \|(u,v,w)\|,$$

for $(u,v,w) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Given $\lambda > 0$, pick an $\epsilon > 0$ so that $\lambda\epsilon < 4$. Then, by (H2), we can find a $\rho_2 \in (0, \rho^*)$ such that for $u + v + w = \rho_2$ and $a + b + c \le \rho_2$, we have

$$f(t, u+a, v+b, w+c) \le \epsilon[(u+a) + (v+b) + (w+c)]$$

for $t \in [0,1]$. Take $(u,v,w) \in C \cap \partial\Omega_{\rho_2}$ and suppose $a + b + c \le \rho_2$. Then, for $t \in [0,1]$,

$$\begin{aligned} A_3(u,v,w)(t) &= \lambda \int_0^1 G(t,s) f(s, u(s) + sa, v(s) + sb, w(s) + sc) ds \\ &\le \lambda \int_0^1 G(t,s) f(s, \|u\|_\infty + a, \|v\|_\infty + b, \|w\|_\infty + c) ds \\ &\le \lambda\epsilon \left[\|(u,v,w)\| + (a+b+c)\right] \int_0^1 G(t,s) ds \\ &\le 2\lambda\epsilon \|(u,v,w)\| \int_0^1 G(t,s) ds \\ &\le \frac{\lambda\epsilon}{4} \|(u,v,w)\|, \end{aligned}$$

by (2.23), giving

$$\|A_3(u,v,w)\|_\infty \le \frac{1}{4}\lambda\epsilon\|(u,v,w)\|.$$

Due to their projective nature, both g_1 and g_2 are continuous functions which are nondecreasing in the last n variables. Furthermore, there are $\delta_1, \delta_2 > 0$, satisfying

$$\delta_1 + \delta_2 < 8,$$

and a $q > 0$ such that for $(u,v,w) \in [0,\infty)^3$ with $u + v + w < q$, we have

$$g_i(t,u,v,w) \le \delta_i(u + v + w),$$

for $i = 1, 2$ and $t \in [0,1]$. Now pick ρ_2 so that $\rho_2 < \frac{1}{2}q$. Then $[(u+a) + (v+b) + (w+c)] < 2\rho_2 < q$. Hence, for $i = 1, 2$,

$$g_i(t, u+a, v+b, w+c) \le \delta_i[(u+a) + (v+b) + (w+c)].$$

Let $\delta' < 1$ and set $\delta = \delta'\rho_2$. Taking $a+b+c < \delta$ and $(u,v,w) \in C \cap \partial\Omega_{\rho_2}$, we have

$$\begin{aligned} A_i(u,v,w)(t) &= \int_0^1 G(t,s) g_i(s, u(s)+sa, v(s)+sb, w(s)+sc) ds \\ &\le \delta_i[\|(u,v,w)\| + (a+b+c)] \int_0^1 G(t,s) ds \\ &\le \delta_i(1+\delta')\|(u,v,w)\| \int_0^1 G(t,s) ds \\ &\le \frac{\delta_i(1+\delta')}{8}\|(u,v,w)\|, \end{aligned}$$

for $i = 1,2$ and $t \in [0,1]$. Thus

$$\|A_i(u,v,w)\|_\infty \le \frac{\delta_i(1+\delta')}{8}\|(u,v,w)\|,$$

for $i = 1,2$. Therefore, we see that

$$\|T(u,v,w)\| \le \left[\frac{(\delta_1+\delta_2)(1+\delta')}{8} + \frac{\epsilon\lambda}{4}\right]\|(u,v,w)\|,$$

for $(u,v,w) \in C \cap \Omega_{\rho_2}$ and $(a,b,c) \in [0,\infty)^3$, with $0 < a+b+c < \delta$. Since we can pick ϵ and δ' small enough so that $\|T(u,v,w)\| \le \|(u,v,w)\|$, we have our desired result. □

Lemma 2.5. *Suppose* $0 < a+b+c < \delta$, *where* $\delta > 0$ *is given, and assumptions* (H0) *and* (H3) *hold. Then for every* $\lambda > 0$, *there is a* $\rho_3 = \rho_3(\delta,\lambda)$ *such that for every* $\rho \ge \rho_3$,

$$\|T(u,v,w)\| \le \|(u,v,w)\|,$$

where $(u,v,w) \in C \cap \partial\Omega_\rho$.

Proof. It follows from the definition and their projective natures that both g_1 and g_2 are nondecreasing in the last n variables and that there are $\eta_1, \eta_2 > 0$, satisfying that

$$0 < \eta < 4,$$

where $\eta = \max\{\eta_1, \eta_2\}$, and a $p > 0$ such that for $(u,v,w) \in [0,\infty)^3$ with $u+v+w > p$,

$$g_i(t,u,v,w) \le \eta_i(u+v+w),$$

for $i = 1, 2$ and $t \in [0, 1]$. Thus, given any $q_1 \geq p$, where p is as above, we have

$$g_i(t, u + ta, v + tb, w + tc) \leq \eta_i[(u + a) + (v + b) + (w + c)],$$

for $u + v + w \geq q_1$ and $i = 1, 2$. Let $\epsilon > 0$ and pick q_1 large enough so that $\epsilon > \frac{\eta\delta}{q_1}$. Then,

$$\begin{aligned} g_i(t, u + ta, v + tb, w + tc) &\leq \eta(u + v + w) + \eta(a + b + c) \\ &\leq \eta(u + v + w) + \epsilon(u + v + w) \\ &= (\eta + \epsilon)(u + v + w). \end{aligned}$$

So, for any $(u, v, w) \in C \cap \partial\Omega_{q_1}$ and $i = 1, 2$, we have

$$\begin{aligned} A_i(u, v, w)(t) &= \int_0^1 G(t, s) g_i(s, u(s) + as, v(s) + bs, w(s) + cs) ds \\ &\leq (\eta + \epsilon)\|(u, v, w)\| \int_0^1 G(t, s) ds \\ &\leq \frac{(\eta + \epsilon)}{8}\|(u, v, w)\|, \end{aligned}$$

giving

$$\|A_i(u, v, w)\|_\infty < \frac{\eta + \epsilon}{8}\|(u, v, w)\|,$$

for $i = 1, 2$.

Now we examine $\|A_3(u, v, w)\|_\infty$. Let $\delta' > 0$. Then, by (H0) and (H3), there is a $q_2 > 0$ such that for every $(u, v, w) \in [0, \infty)^3$ with $u + v + w \geq q_2$, we have

$$f(t, u + ta, v + tb, w + tc) < \delta'[(u + a) + (v + b) + (w + c)].$$

Let $q_3 = \max\{\delta, q_2\}$. By the above and the fact that $a + b + c < \delta$, for every $(u, v, w) \in [0, \infty)^3$ with $u + v + w \geq q_3$, we have

$$\begin{aligned} f(t, u + ta, v + tb, w + tc) &\leq \delta'(u + v + w) + \delta'\delta \\ &\leq 2\delta'(u + v + w). \end{aligned}$$

It then follows that for $(u, v, w) \in C \cap \Omega_{q_3}$,

$$A_3(u, v, w) \leq 2\delta' \lambda \|(u, v, w)\| \int_0^1 G(t, s) ds \leq \frac{1}{4} \delta' \lambda \|(u, v, w)\|,$$

giving that $\|A_3(u, v, w)\|_\infty \leq \frac{1}{4}\delta'\lambda\|(u, v, w)\|$. Take $\rho_3 = \max\{q_1, q_3\}$ and let $\rho \geq \rho_3$. Then given $(u, v, w) \in C \cap \partial\Omega_\rho$, we have

$$\|T(u, v, w)\| \leq \left(\frac{\epsilon + \eta + \lambda\delta'}{4}\right) \|(u, v, w)\|.$$

Picking ϵ and δ' small enough so that $\epsilon + \lambda\delta' \leq 4 - \eta$ yields

$$\|T(u, v, w)\| \leq \|(u, v, w)\|.$$

□

As with the first two lemmas, Lemmas 2.4 and 2.5 each give the same relationship between $\|T(u, v, w)\|$ and $\|(u, v, w)\|$ for different sets, $C \cap \partial\Omega_{\rho_2}$ and $C \cap \partial\Omega_{\rho_3}$. In fact, when the four lemmas are combined, we see that

$$\Omega_{\rho_1} \subsetneq \Omega_{\rho_2} \subsetneq \Omega_{\rho^*} \subsetneq \Omega_{\rho_3},$$

which helps to satisfy one of the requirements necessary to apply the Guo-Krasnosel'skii Fixed Point Theorem. Moreover, not unlike Lemmas 2.2 and 2.3, the assumptions used in Lemmas 2.4 and 2.5 to find an upper bound on A_3 rely solely on f and can be just as easily amended to ensure the same results for the $2n$th order problem.

Of course, finding a bound on A_3 is only part of what is accomplished in Lemmas 2.4 and 2.5. Upper bounds on A_1 and A_2 are given using the inherent bounds and nondecreasing property of both g_1 and g_2. In Lemma 2.4, we eventually arrive at the inequality

$$\|A_1(u, v, w)\|_\infty + \|A_2(u, v, w)\|_\infty < m\|(u, v, w)\|$$

for some m. Now, it is essential that $m < 1$, and the interplay between the bound involving the integral of the Green's function, (2.22), and the bound placed on $\delta_1+\delta_2$ in the proof of Lemma 2.4 is a key factor in securing this inequality. Lemma 2.5 has a similar working relationship between (2.22) and the bound on η. In the fourth order case, only one operator utilized the bound on η. Hence, it was sufficient to require $\eta < 8$. However, we are concerned with finding bounds for the sum of two operators and as a result, need $\eta < 4$. This provides some insight into how the proofs of the final two lemmas can be altered to produce results similar to Lemmas 2.4 and 2.5 for the generalized case.

2.2.3 An Existence Result

We are now ready to assemble the pieces above by producing an existence result for sixth order boundary value problems of the form (2.18)-(2.21). Luckily, the bulk of the work needed to prove the theorem is contained in Section 2.2.2. After constructing the statement of the main result, the only work that remains is piecing the lemmas together properly to allow a triple application of the Guo-Krasnosel'skii Fixed Point Theorem.

Theorem 2.1. *Let f satisfy assumptions* (H0)-(H3)*. Then there exists a $\Lambda > 0$ such that given any $\lambda \geq \Lambda$, there is a $\delta > 0$ such that for every $a, b, c \geq 0$ satisfying $0 < a + b + c < \delta$, the system* (2.18)-(2.21) *has at least three positive solutions.*

Proof. Suppose f satisfies hypotheses (H0)-(H3) and let $\rho^* > 0$ be fixed. By Lemma 2.2, there is a $\Lambda > 0$ such that for every $\lambda \geq \Lambda$ and $a, b, c \geq 0$,

$$\|T(u, v, w)\| \geq \|(u, v, w)\|, \text{ for } (u, v, w) \in C \cap \partial\Omega_{\rho^*}.$$

Now, fix $\lambda \geq \Lambda$. Lemmas 2.3-2.5 give that there is a $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$, satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$, such that for $(a, b, c) \in [0, \infty)^3$ with $0 < a + b + c < \delta$, we have

$$\|T(u,v,w)\| \geq \|(u,v,w)\|, \text{ for } (u,v,w) \in C \cap \partial\Omega_{\rho_1},$$

$$\|T(u,v,w)\| \leq \|(u,v,w)\|, \text{ for } (u,v,w) \in C \cap \partial\Omega_{\rho_2},$$

$$\|T(u,v,w)\| \leq \|(u,v,w)\|, \text{ for } (u,v,w) \in C \cap \partial\Omega_{\rho_3}.$$

Applying the Guo-Krasnosel'skii Fixed Point Theorem, we have the existence of three positive solutions, $(u_1,v_1,w_1),(u_2,v_2,w_2),(u_3,v_3,w_3) \in C$, such that

$$\rho_1 < \|(u_1,v_1,w_1)\| < \rho_2 < \|(u_2,v_2,w_2)\| < \rho^* < \|(u_3,v_3,w_3)\| < \rho_3.$$

□

Note that the above theorem is very close in nature to the main result given by Marcos, Lorca, and Ubilla, the only difference being the inclusion of g_2 and c in the construction of the problem, to accommodate the jump in order. It is natural to assume that the generalized main result will also be in a similar vein. This is further supported by the fact that the assumptions and lemmas from the $n = 3$ case can be just as easily amended to include any even order problem as they were in the move from the fourth order to the sixth order problem. In the next section, we will see that this is, in fact, the case.

2.3 2nth Order Problem

With the blue-print provided by the previous two sections in hand, we can now focus our attention on the multiplicity of positive solutions for even order differential equations of the form

$$u^{(2n)} = \lambda h\left(t,u,u'',\ldots,u^{(2(n-1))}\right), \;\; t \in (0,1), \;\; n \geq 2, \tag{2.24}$$

satisfying the boundary conditions:

$$u^{(2k)}(0) = 0, \;\; k = 0,\ldots,n-1, \tag{2.25}$$

$$u^{(2k)}(1) = (-1)^k a_k, \;\; k = 0,\ldots,n-1, \tag{2.26}$$

where $\lambda, a_k \geq 0$, with $\sum_{k=0}^{n-1} a_k > 0$, and $h : [0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,\infty) \to (-1)^n[0,\infty)$ is continuous. Although the generic order makes constructing an existence result slightly more involved than in either the $n = 2$ or $n = 3$ cases, the basic steps are somewhat similar. If we can generate a set of hypotheses close enough to those in the previous cases, we can make our way to a quartet of lemmas with results corresponding to the ones in Section 2.2.2. Once these results are in place, the sought-after existence theorem will essentially reveal itself.

2.3.1 Preliminaries

Our construction of a generalized existence result starts by assembling a sturdy foundation similar to the one constructed in the case where $n = 3$. Thus, we again turn to the transformation technique used in [26] and then expounded upon for the sixth order problem. Note, however, this process is more tedious for the $2n$th order problem than for previous problems. This is due to both the generic order and the oscillatory nature of signs briefly discussed in Section 2.2.1.

Working backwards, we start by assuming that u is a positive solution of (2.24)-(2.26). As before, the cone used to apply the Guo-Krasnosel'skii Fixed Point Theorem includes a concavity condition allowing us to assume that u is not only positive, but concave as well. Notice that this falls in line with the boundary condition $u''(1) = -a_2$, since $a_2 \geq 0$. Upon further inspection, a simple calculus argument tells us that if u'' is convex on $[0,1]$, then $u^{(4)}$ is nonnegative. In fact, we can keep reapplying the same argument through $k = n$, each time noting that $u^{(2k)}$ switches between being a nonnegative concave function and a nonpositive convex function. More specifically, $u^{(2k)}$ is nonnegative when k is even, and it is nonpositive when k is odd. With this in mind, set $u_k = (-1)^k u^{(2k)}$ for $k = 0, \ldots, n-1$, and observe that the $(-1)^k$ makes each u_k nonnegative. Therefore, if we set $u_{k+1} = g_k(t, u_0, u_1, \ldots, u_n)$, for $k = 0, \ldots, n-2$, and $t \in [0,1]$, each g_k is nonnegative. Furthermore, since

$$-u_k'' = (-1)\left[(-1)^k u^{(2k)}\right]'' = (-1)^{k+1} u^{(2k+2)} = u_{k+1},$$

we have $-u_k'' = g(t, u_0, u_1, \ldots, u_{n-1})$ for each $k = 0, \ldots, n-2$. Now, for the final substitution, set

$$f(t, u_0, u_1, \ldots, u_{n-1}) = (-1)^n h(t, u_0, -u_1, \ldots, (-1)^{n-1} u_{n-1}).$$

Since $(-1)^n \lambda u^{(2n)} \geq 0$, f is nonnegative as well. Rewriting (2.24)-(2.26) in terms of the above substitutions, we have the system of second order ordinary differential equations,

$$-u_{n-1}'' = \lambda f(t, u_0, u_1, \ldots, u_{n-1}), \;\; t \in (0,1), \tag{2.27}$$

$$-u_k'' = g_k(t, u_0, u_1, \ldots, u_{n-1}), \;\; k = 0, \ldots, n-2, \;\; t \in (0,1), \tag{2.28}$$

satisfying the boundary conditions;

$$u_k(0) = 0, \;\; k = 0, \ldots, n-1, \tag{2.29}$$

$$u_k(1) = a_k, \;\; k = 0, \ldots, n-1. \tag{2.30}$$

Note that solutions of (2.27)-(2.30) are in one-to-one correspondence with solutions of (2.24)-(2.26). Moreover, as the above sign argument is completely reversible, confining f to be nonnegative leads to solutions being positive, provided they exist. In fact, this is why we placed a constraint on the sign of h in the original problem. With this in mind, we place the the following hypotheses on f;

(H0) $f : [0,1] \times [0,\infty)^n \to [0,\infty)$, is a continuous function which is nondecreasing in the last n variables.

(H1) There exists $\alpha_1, \beta_1 \in (0,1)$, with $\alpha_1 < \beta_1$, such that, given $(u_0, \ldots, u_{n-1}) \in [0,\infty)^n$ satisfying $\sum_{k=0}^{n-1} u_k \neq 0$, there is a $c > 0$ such that

$$f(t, u_0, \ldots, u_{n-1}) > c, \;\; t \in [\alpha_1, \beta_1].$$

(H2) Let $z = \sum_{k=0}^{n-1} u_k$. Then $\lim_{z \to 0^+} \frac{f(t, u_0, \ldots, u_{n-1})}{z} = 0$, uniformly for $t \in [0, 1]$.

(H3) Let $z = \sum_{k=0}^{n-1} u_k$. Then $\lim_{z \to \infty} \frac{f(t, u_0, \ldots, u_{n-1})}{z} = 0$, uniformly for $t \in [0, 1]$.

The above assumptions are practically identical to the assumptions made in previous cases, the only exceptions being simple amendments to accommodate the generic order. For example, the number of variables in f and where it is nondecreasing has been altered to depend on the order of the problem. In addition, there is an increase in the number of g_k's in the construction of the problem, which is also dependent on n.

Following our blue-print, the next step is to transform the system, (2.27)-(2.30) into a system of second order differential equations of the form,

$$-u''_{n-1} = \lambda f(t, u_0 + ta_0, \ldots, u_{n-1} + ta_{n-1}), \tag{2.31}$$

$$-u''_k = g_k(t, u_0 + ta_0, \ldots, u_{n-1} + ta_{n-1}), \;\; k = 0, \ldots, n-2, \tag{2.32}$$

for $t \in (0, 1)$, under the homogeneous conjugate boundary conditions,

$$u_k(0) = u_k(1) = 0, \;\; k = 0, \ldots, n-1. \tag{2.33}$$

As was the case for the sixth order problem, this is the system we actually find solutions for. But, given the substitutions and transformations made above, if solutions of (2.31)-(2.33) exist, they must automatically exist for (2.24)-(2.26). Motivated by this, note that integration tells us that solutions of (2.31)-(2.33) are of the form,

$$u_{n-1}(t) = \lambda \int_0^1 G(t, s) f(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds,$$

$$u_k(t) = \int_0^1 G(t, s) g_k(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds, k = 0, \ldots, n-2,$$

where $G(t, s)$ denotes the Green's function,

$$G(t, s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

As a by-product of the transformation process, this is the same Green's function as the one used in the $n = 3$ case, which we noted as being positive. This juxtaposed with the sign constraints resulting from (H0) tells us that each of the above integrals is nonnegative, reinforcing the positivity of solutions. For convenience, recall the property,

$$\max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{8}. \tag{2.22}$$

This equation will once again crop up in finding the estimates given in the next section and is a driving force behind the appearance of the eight as a bound in the proofs of the last two lemmas.

We still need a Banach space containing a cone and a completely continuous operator in order to apply the Guo-Krasnosel'skii Fixed Point Theorem. Both the space and the cone used in the sixth order problem worked quite efficiently and are easily modified to meet our current needs. To this end, let $(X, \|\cdot\|)$ denote the Banach space $X = \prod_{j=1}^{n} C([0,1];\mathbb{R})$, endowed with the norm

$$\|(u_0, u_1, \ldots, u_{n-1})\| = \|u_0\|_\infty + \|u_1\|_\infty + \cdots + \|u_{n-1}\|_\infty,$$

where $\|u\|_\infty = \sup\limits_{t\in[0,1]} |u(t)|$. Furthermore, let $C \subset X$ be the set

$$C = \{(u_0, \ldots, u_{n-1}) \in X | (u_0, \ldots, u_{n-1})(0) = (u_0, \ldots, u_{n-1})(1) = (0, \ldots, 0) \text{ and}$$
$$u_0, \ldots, u_{n-1} \text{ are concave}\}.$$

Assuring that C satisfies the criteria in the definition of a cone is straight forward; hence C is a cone. Define the operator $T : X \to X$ by

$$T(u_0, \ldots, u_{n-1}) = (A_0(u_0, \ldots, u_{n-1}), \ldots, A_{n-1}(u_0, \ldots, u_{n-1}))\,,$$

where

$$A_{n-1}(u_0, \ldots, u_{n-1})(t) = \lambda \int_0^1 G(t,s) f(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1})ds,$$

and, for $k = 0, \ldots, n-2$,

$$A_k(u_0, \ldots, u_{n-1})(t) = \int_0^1 G(t,s) g_k(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds,$$

and note that solutions of (2.31)-(2.33) are fixed points of T.

The following lemma gives two properties of T that are needed to satisfy the assumptions of our desired fixed point theorem.

Lemma 2.6. *T is a completely continuous operator and $T : C \to C$.*

The proof of Lemma 2.6 is a standard application of the Arzela-Ascoli Theorem. In fact, it is practically identical to the proof given for Lemma 2.1.

2.3.2 Lemmas

We emphasize that the power of the previous result, Theorem 2.1, rests in the formulation of the lemmas in Section 2.2.2. Utilizing this, assumptions (H0)-(H3) have been reformulated to produce a series of four additional lemmas, given below, generating similar contraction and expansion relationships at particular points within the cone, C. These relationships, in turn, allow us to apply the Guo-Krasnosel'skii Fixed Point Theorem three times, thus yielding at least three positive solutions.

The first two lemmas, Lemmas 2.7 and 2.8, follow a similar pattern to the corresponding lemmas for the sixth order problem. They each use hypotheses (H0) and (H1), respectively, to find lower bounds for T. Since it is sufficient to show that A_{n-1} is bounded below, we only use properties pertaining to f. That is not to say that the additional nondecreasing properties naturally associated with each g_k are not needed. In fact, they play a crucial role in proving the last two lemmas. It is just not needed here.

Lemma 2.7. *Suppose* (H0) *and* (H1) *hold and let $\rho^* > 0$. Then there exists a $\Lambda > 0$*

such that, for every $\lambda \geq \Lambda$ *and* $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$,

$$\|T(u_0, \ldots, u_{n-1})\| \geq \|(u_0, \ldots, u_{n-1})\|$$

for $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho^*}$. Set $r = \alpha_1(1 - \beta_1)$ and $\Lambda \geq \left[rM \int_{\alpha_1}^{\beta_1} G(1/2, s)\, ds\right]^{-1}$, where

$$\begin{aligned} M &= \inf\left\{ \frac{f(t, rz_0, \ldots, rz_{n-1})}{r(z_0 + l \cdots + z_{n-1})} : t \in [\alpha_1, \beta_1],\ (z_0, \ldots, z_{n-1}) \in (0, \infty)^n \right. \\ &\qquad \left. \text{and } \sum_{i=0}^{n-1} z_i = \rho^* \right\}. \end{aligned}$$

The fact that M exists and is positive follows from (H1). Take $\lambda \geq \Lambda$. By the nondecreasing properties of f along with Lemma (1.1), we have

$$\begin{aligned} \|T(u_0, \ldots, u_{n-1})\| &\geq \|A_{n-1}(u_0, \ldots, u_{n-1})\|_\infty \\ &\geq \lambda \int_0^1 G(1/2, s) f(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds \\ &\geq \lambda \int_{\alpha_1}^{\beta_1} G(1/2, s) f(s, r\|u_0\|_\infty, \ldots, r\|u_{n-1}\|_\infty) ds \end{aligned}$$

$$\begin{aligned} &\geq \lambda r \|(u_0, \ldots, u_{n-1})\| \int_{\alpha_1}^{\beta_1} G(1/2, s) \frac{f(s, r\|u_0\|_\infty, \ldots, r\|u_{n-1}\|_\infty)}{r\|(u_0, \ldots, u_{n-1})\|} ds \\ &\geq \Lambda r M \|(u_0, \ldots, u_{n-1})\| \int_{\alpha_1}^{\beta_1} G(1/2, s) ds \\ &\geq \|(u_0, \ldots, u_{n-1})\|. \end{aligned}$$

□

Lemma 2.8. *Fix* $\Lambda > 0$ *and suppose* (H0) *and* (H1) *hold. Then for every* $\lambda \geq \Lambda$ *and* $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$, *with* $\sum_{i=0}^{n-1} a_i < 0$, *there is a* $\rho_1 = \rho_1(\Lambda, a_0, \ldots, a_{n-1})$ *such that, for every* $\rho \leq \rho_1$, *we have*

$$\|T(u_0, \ldots, u_{n-1})\| \geq \|(u_0, \ldots, u_{n-1})\|$$

for $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$.

Proof. By (H1) and the nondecreasing properties of f, we have for $t \in [\alpha_1, \beta_1]$,

$$f(t, u_0(t) + ta_0, \ldots, u_{n-1}(t) + ta_{n-1}) > f(t, \alpha_1 a_0, \alpha_1 a_1, \ldots, \alpha_1 a_{n-1}) > k,$$

Take $\rho_1 = \Lambda c \int_{\alpha_1}^{\beta_1} G(1/2, s)ds$. Then for all $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$, where $\rho \leq \rho_1$,

$$\begin{aligned} \|T(u_0, \ldots, u_{n-1})\| &\geq \lambda \int_{\alpha_1}^{\beta_1} G(1/2, s) f(s, \alpha_1 a_0, \ldots, \alpha_1 a_{n-1}) ds \\ &\geq \Lambda c \|(u_0, \ldots, u_{n-1})\| \int_{\alpha_1}^{\beta_1} \frac{G(1/2, s)}{\|(u_0, \ldots, u_{n-1})\|} ds \\ &\geq \|(u_0, \ldots, u_{n-1})\|. \end{aligned}$$

□

Although on the surface the above lemmas appear to be the same, their results are slightly different. Granted, they each establish the same relationship between $\|T(u_0, \ldots, u_{n-1})\|$ and $\|(u_0, \ldots, u_{n-1})\|$, but, as we mentioned before, this relationship holds for two different sets. Also notice that Lemma 2.7 guarantees the existence of a Λ, whereas Lemma 2.8 relies on this Λ to find a $\rho_1 < \rho^*$. This gives an important containment relationship between the two open sets generated by Lemmas 2.7 and 2.8, namely $\Omega_{\rho_1} \subset \Omega_{\rho^*}$.

The final two lemmas, Lemma 2.9 and Lemma 2.10, each give reverse inequalities between $\|T(u_0, \ldots, u_{n-1})\|$ and $\|(u_0, \ldots, u_{n-1})\|$ by first showing that each A_k, $k = 0, \ldots, n-1$, has an upper bound. Then, by taking advantage of the defined norm, those bounds are summed to give an upper estimate of the form

$$\|T(u_0, \ldots, u_{n-1})\| \leq m\|(u_0, \ldots, u_{n-1})\|,$$

for some constant m. As was the case with the sixth order problem, it is essential for $m \leq 1$ if we wish to apply the Guo-Krasnosel'skii Fixed Point Theorem. Hence, we need to find points in C where the upper bound on each A_k, $k = 0, \ldots, n-1$, is sufficiently small. In both lemmas, the growth conditions placed on f in (H2) and (H3) are respectively used in conjunction with the nondecreasing restraint on f from (H0) to acquire points where $\|A_{n-1}(u_0,\ldots,n-1)\|_\infty$ is small. In order to find points where the upper bound on each remaining A_k is sufficiently small, we make use of both the nondecreasing properties and bounds on g_k that follow from the definition.

Lemma 2.9. *Suppose* (H0) *and* (H2) *hold and fix* $\rho^* > 0$. *Then given* $\lambda > 0$, *there is a* $\rho_2 \in (0,\rho^*)$ *and a* $\delta > 0$ *such that, for every* $(a_0,\ldots,a_{n-1}) \in [0,\infty)^n$ *with* $\sum_{i=1}^{n-1} a_i < \delta$,

$$\|T(u_0,\ldots,u_{n-1})\| \leq \|(u_0,\ldots,u_{n-1})\|,$$

for $(u_0,\ldots,u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Given $\lambda > 0$, pick and $\epsilon > 0$ so that $\lambda\epsilon < 4$. Then, by (H2), we can find a $\rho_2 \in (0,\rho^*)$ such that for $\sum_{i=0}^{n-1} u_i = \rho_2$ and $\sum_{i=1}^{n-1} a_i \leq \rho_2$, we have that

$$f(t, u_0 + a_0, \ldots, u_{n-1} + a_{n-1}) < \epsilon \sum_{i=0}^{n-1} (u_i + a_i)$$

for $t \in [0,1]$. Take $(u_0,\ldots,u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$ and suppose $\sum_{i=0}^{n-1} a_i \leq \rho_2$. Then, by the non-decreasing properties of f,

$$\begin{aligned}
A_{n-1}(u_0,\ldots,u_{n-1}) &= \lambda \int_0^1 G(t,s) f(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds \\
&\leq \lambda \int_0^1 G(t,s) f(s, \|u_0\|_\infty + a_0, \ldots, \|u_{n-1}\|_\infty + a_{n-1}) ds \\
&< \lambda\epsilon \left[\|(u_0, u_1, \ldots, u_{n-1})\| + \sum_{i=0}^{n-1} a_i \right] \int_0^1 G(t,s) ds \\
&\leq 2\lambda\epsilon \|(u_0,\ldots,u_{n-1})\| \int_0^1 G(t,s) ds \\
&\leq \frac{\lambda\epsilon}{4} \|(u_0,\ldots,u_{n-1})\|
\end{aligned}$$

for $t \in [0,1]$. Thus,

$$\|A_{n-1}(u_0,\ldots,u_{n-1})\|_\infty < \frac{1}{4}\lambda\epsilon \|(u_0,\ldots,u_{n-1})\|. \tag{2.34}$$

Next we examine A_k for each $k = 0, \dots, n-2$. To this end note that because of their nature as projections, each g_k, where $k = 0, \dots, n-2$, is nondecreasing in the last n variables. Moreover, there are $\delta_0, \dots, \delta_{n-2} > 0$, with

$$0 < \sum_{k=0}^{n-2} \delta_k < 8,$$

and a $q > 0$ such that for $(u_0, \dots, u_{n-1}) \in [0, \infty)^n$ with $\sum_{i=0}^{n-1} u_i < q$, we have

$$g_k(t, u_0, \dots, u_{n-1}) \le \delta_k \left(\sum_{i=0}^{n-1} u_i \right),$$

for each $k = 0, \dots, n-2$ and $t \in [0, 1]$. Pick ρ_2 small enough so that $\rho_2 < \frac{1}{2}q$. Since $\sum_{i=0}^{n-1}(u_i + a_i) < 2\rho_2 < q$, the above tells us that

$$g_k(t, u_0 + a_0, \dots, u_{n-1} + a_{n-1}) \le \delta_k \sum_{i=0}^{n-1}(u_i + a_i)$$

for $k = 1, \dots, n-2$. Let $\delta' < 1$ and set $\delta = \delta'\rho_2$. Then taking $\sum_{i=0}^{n-1} a_i < \delta$ and $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$, we see that

$$\begin{aligned} A_k(u_0, \dots, u_n)(t) &= \int_0^1 G(t, s) g_k(s, u_0(s) + sa_0, \dots, u_{n-1}(s) + sa_{n-1}) ds \\ &\le \delta_k [\|(u_0, \dots, u_{n-1})\| + \sum_{i=0}^{n-1} a_i] \int_0^1 G(t, s) ds \\ &\le \delta_k (1 + \delta') \|(u_0, \dots, u_{n-1})\| \int_0^1 G(t, s) ds \\ &\le \frac{\delta_k (1 + \delta')}{8} \|(u_0, \dots, u_{n-1})\|, \end{aligned}$$

for $k = 0, \dots, n-2$ and $t \in [0, 1]$. Thus, for each $k = 0, \dots, n-2$, we have the inequality

$$\|A_k(u_0, \dots, u_{n-1})\|_\infty \le \frac{\delta_k(1 + \delta')}{8} \|(u_0, \dots, u_{n-1})\|. \tag{2.35}$$

Combining inequalities (2.34) and (2.35), we have for $\sum_{i=0}^{n-1} a_i < \delta$ and $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$, that

$$\|T(u_0, \dots, u_{n-1})\| \le \left[\frac{(1 + \delta')}{8} \left(\sum_{k=0}^{n-2} \delta_k \right) + \frac{\epsilon\lambda}{4} \right] \|(u_0, \dots, u_{n-1})\|.$$

Taking ϵ and δ' small enough, gives $\|T(u_0, \dots, u_{n-1})\| \le \|(u_0, \dots, u_{n-1})\|$. $\square$

Lemma 2.10. *Suppose* (H0) *and* (H3) *hold and let* $(a_0, \dots, a_{n-1}) \in [0, \infty)^n$ *such that* $0 < \sum_{i=0}^{n-1} a_i < \delta$. *Then, for every* $\lambda > 0$, *there is a* $\rho_3 = \rho_3(\delta, \lambda)$ *such that, for every* $\rho \geq \rho_3$,

$$\|T(u_0, \dots, u_{n-1})\| \leq \|(u_0, \dots, u_{n-1})\|,$$

where $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_\rho$.

Proof. Each g_k, $k = 0, \dots n-2$, is nondecreasing in the last n variables as a by-product of their projective nature. Furthermore, there are $\eta_0, \dots, \eta_{n-2} > 0$, with

$$0 < \eta < \frac{8}{n-1},$$

where $\eta = \max\{\eta_0, \dots, \eta_{n-2}\}$, and a $p > 0$ such that for $(u_0, \dots, u_{n-1}) \in [0, \infty)^n$ with $\displaystyle\sum_{i=0}^{n-1} u_i > p$, we have

$$g_k(t, u_0, \dots, u_{n-1}) \leq \eta_k \left(\sum_{i=0}^{n-1} u_i \right),$$

for each $k = 0, \dots, n-2$, and $t \in [0, 1]$. Thus, given any $q_1 \geq p$, where p is defined above, we have

$$g_k(t, u_0 + ta_0, \dots, u_{n-1} + ta_{n-1}) \leq \eta_k \sum_{i=0}^{n-1} (u_i + a_i), \ \ k = 0, \dots, n-2,$$

for $(u_0, \dots, u_{n-1}) \in [0, \infty)^n$ with $\sum_{i=0}^{n-1} u_i \leq q_1$. Let $\epsilon > 0$ and pick q_1 large enough so that $\epsilon > \frac{\eta\delta}{q_1}$. Then, for $t \in [0, 1]$,

$$g_k(t, u_0 + ta_0, \dots, u_{n-1} + ta_{n-1}) \leq \eta \sum_{i=0}^{n-1} u_i + \eta \sum_{i=0}^{n-1} a_i = (\eta + \epsilon) \sum_{i=0}^{n-1} u_i,$$

for each $k = 0, \dots, n-2$. So, for any $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_{q_1}$,

$$\begin{aligned} A_k(u_0, \dots, u_{n-1})(t) &= \int_0^1 G(t, s) g_k(s, u_0(s) + sa_0, \dots, u_{n-1}(s) + sa_{n-1}) ds \\ &\leq (\eta + \epsilon) \|(u_0, \dots, u_{n-1})\| \int_0^1 G(t, s) ds \\ &\leq \frac{\eta + \epsilon}{8} \|(u_0, \dots, u_{n-1})\|, \end{aligned}$$

on $[0,1]$, yielding

$$\|A_k(u_0,\ldots,u_{n-1})\|_\infty < \frac{\eta+\epsilon}{8}\|(u_0,\ldots,u_{n-1})\| \tag{2.36}$$

for $k = 0,\ldots,n-2$.

Next we consider the operator, A_{n-1}. Let $\delta' > 0$. From (H0) and (H3), we can find a $q_2 > 0$ such that for every $(u_0,\ldots,u_{n-1}) \in [0,\infty)^n$ with $\sum_{i=0}^{n-1} u_i \geq q_2$, we have

$$f(t, u_0 + ta_0, \ldots, u_{n-1} + ta_{n-1}) < \delta' \left[\sum_{i=0}^{n-1}(u_i + a_i)\right],$$

for $t \in [0,1]$. Take $q_3 = \max\{\delta, q_2\}$ and note that $\sum_{i=0}^{n-1} a_i < \delta$. Then for each $(u_0,\ldots u_{n-1}) \in [0,\infty)^n$ with $\sum_{i=0}^{n-1} u_i \geq q_3$,

$$f(t, u_0 + ta_0, \ldots, u_{n-1} + ta_{n-1}) < \delta' \sum_{i=0}^{n-1} u_i + \delta'\delta \leq 2\delta' \sum_{i=0}^{n-1} u_i.$$

Thus for $(u_0,\ldots,u_{n-1}) \in C \cap \partial\Omega_{q_3}$,

$$\begin{aligned} A_{n-1}(u_0,\ldots,u_{n-1})(t) &< 2\delta'\lambda(\|(u_0,\ldots,u_{n-1})\| \int_0^1 G(t,s)ds \\ &\leq \frac{\delta'\lambda}{4}\|(u_0,\ldots,u_{n-1})\|, \end{aligned} \tag{2.37}$$

giving $\|A_{n-1}(u_0,\ldots,u_{n-1})\|_\infty \leq \frac{\delta'\lambda}{4}\|(u_0,\ldots,u_{n-1})\|$. Set $\rho_3 = \max\{q_1, q_3\}$ and take $\rho \geq \rho_3$. Then, by (2.36) and (2.37), given any $(u_0,\ldots,u_{n-1}) \in C \cap \partial\Omega_\rho$, we see that

$$\|T(u_0,\ldots,u_{n-1})\| < \left[\frac{(\eta+\epsilon)(n+1)}{8} + \frac{\delta'\lambda}{4}\right]\|(u_0,\ldots,u_{n-1})\|.$$

Therefore, if we pick ϵ and δ' small enough so that $\epsilon(n+1) + \lambda\epsilon < 8 - \eta(n+1)$, we have our desired result. □

As with Lemmas 2.7 and 2.8, Lemmas 2.9 and 2.10 give different sets for which we have the same inequality between $\|T(u_0,\ldots,u_{n-1})\|$ and $\|(u_), \ldots, u_{n-1})\|$. Notice, Lemma 2.9 uses a combination of assumptions and properties that only work for a $(u_0,\ldots,u_{n-1})$ with a sufficiently small norm. This leads to $\rho_2 < \rho^*$, and hence

$\Omega_{\rho_2} \subset \Omega_{\rho^*}$. In contrast, the assumptions and properties used in Lemma 2.10 rely on the norm of $(u_0, \ldots, u_{n-1})$ being sufficiently large to find a ρ_3. In fact, the results for Lemma 2.10 hold for any $\rho \geq \rho_3$, allowing us to assume, without loss of generality, $\rho_3 > \rho^*$. Similarly, Lemma 2.8 holds for any $\rho \leq \rho_1$. Hence, putting the four lemmas together produces four nested open sets, $\Omega_{\rho_1} \subset \Omega_{\rho_2} \subset \Omega_{\rho^*} \subset \Omega_{\rho_3}$, an important factor in using the Guo-Krasnosel'skii Fixed Point Theorem.

2.3.3 An Existence Result

Now that we have shown that the work done in Section 2.3 can be extended to produce a set of generalized lemmas analogous to those used in the sixth order problem, the extension process is almost complete. Just as before, the majority of the work is done in the preceding lemmas. All that remains is to assemble the above pieces in the proper manner so that we obtain an existence result for problems of the form (2.31)-(2.33).

Theorem 2.2. *Let f satisfy* (H0)-(H3). *Then there exists a $\Lambda > 0$ such that, given any $\lambda \geq \Lambda$, there is a $\delta > 0$ such that for every $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$, satisfying $0 < \sum_{i=1}^{n-1} a_i < \delta$, the system* (2.31)-(2.33) *has at least three positive solutions.*

Proof. Suppose f satisfies hypotheses (H0)-(H3) and let $\rho^* > 0$ be fixed. By Lemma 2.7, there is a $\Lambda > 0$ such that for every $\lambda \geq \Lambda$ and $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$,

$$\|T(u_0, \ldots, u_{n-1})\| \geq \|(u_0, \ldots, u_{n-1})\|, \text{ for } C \cap \partial\Omega_{\rho^*}.$$

Now, fix $\lambda \geq \Lambda$. Lemmas 2.8-2.10 then give that there is a $\delta > 0$ and a $\rho_1, \rho_2, \rho_3 > 0$ satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$ such that, for $(a_0, a_1, \ldots, a_{n-1}) \in [0, \infty)^n$ with $0 < \sum_{i=0}^{n-1} a_i < \delta$, we have

$$\|T(u_0, \ldots, u_{n-1})\| \geq \|(u_0, \ldots, u_{n-1})\|, \text{ for } (u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_1},$$

$$\|T(u_0, \ldots, u_{n-1})\| \leq \|(u_0, \ldots, u_{n-1})\|, \text{ for } (u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2},$$

$$\|T(u_0, \ldots, u_{n-1})\| \leq \|(u_0, \ldots, u_{n-1})\|, \text{ for } (u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_3}.$$

Applying the Guo-Krasnoselskii Fixed Point Theorem, we have the existence of three positive solutions, $(u_0, \dots, u_{n-1}), (v_0, \dots, v_{n-1}), (w_0, \dots, w_{n-1}) \in C$, such that,

$$\rho_1 < \|(u_0, \dots, u_{n-1})\| < \rho_2 < \|(v_0, \dots, v_{n-1})\| < \rho^* < \|((w_0, \dots, w_{n-1})\| < \rho_3.$$

□

2.4 Example

We conclude the chapter by incorporating an example from [26] illustrating an application of Theorem 2.2 for the case where $n = 2$. Let $a(t)$, $b(t)$, $c(t)$, and $d(t)$ all be nonnegative continuous functions such that $b(t)$, $c(t)$, and $d(t)$ are nonzero and $\|a\|_\infty$ and $\|b\|_\infty$ are bounded above by eight. Furthermore, suppose that $h_1, h_2 : [0, \infty) \to [0, \infty)$ are nondecreasing continuous functions such that

$$\lim_{u \to 0^+} \frac{h_i(u)}{u} = \lim_{u \to \infty} \frac{h_i(u)}{u} = 0, \ \ i = 1, 2,$$

and $h_1, h_2 \neq 0$ on $(0, \infty)$. Consider the system of second order boundary value problems

$$-v''(t) = a(t)u(t) + b(t)v(t), \ t \in (0, 1), \tag{2.38}$$

$$-u''(t) = c(t)h_1(u) + d(t)h_2(v), \ t \in (0, 1), \tag{2.39}$$

$$u(0) = v(0) = 0, \tag{2.40}$$

$$u(1) = a, \ v(1) = b. \tag{2.41}$$

Notice that if we take $a(t) := 0$ and $b(t) := 1$, then (2.38)-(2.41) is of the same form as (2.27)-(2.30) where $n = 2$. Since assumptions (H0)-(H3) are clearly satisfied, applying Theorem 2.2 yields that (2.38)-(2.41) has at least three positive solutions.

CHAPTER THREE

Differential Equations with Focal Boundary Conditions

In this chapter, we focus on the multiplicity of solutions for even order boundary value problems of the form

$$u^{(2n)} = \lambda h\left(t, u, u'', \dots, u^{(2(n-1))}\right),\ t \in (0,1), \tag{3.1}$$

for $n \geq 2$, satisfying the boundary conditions

$$u^{(2k)}(0) = 0,\ \ k = 0, \dots, n-1, \tag{3.2}$$

$$u^{(2k+1)}(1) = (-1)^k a_k,\ \ k = 0, \dots, n-1, \tag{3.3}$$

where $\lambda, a_0, \dots, a_{n-1} \geq 0$, and $\sum_{k=0}^{n-1} a_k > 0$. Notice that the differential equation (3.1) is of even order and only includes even order derivatives, just like the problem in the previous chapter. The boundary conditions, however, have been altered to include odd order derivatives, making (3.1)-(3.3) a focal boundary value problem. Despite the fact that the differential equation has changed form, the methods used in Chapter 2 to guarantee solutions carry over to this problem; that is, with minor adjustments. Rather than diving directly into the $2n$th order problem, we first examine the case where $n = 2$. This lends to a better understanding of the generalized problem, and gives us an idea of what effects the new boundary conditions have on our technique. Once an existence result for the fourth order problem is in place, we extend the modified technique to show an analogous result for any even order boundary value problem of the form (3.1)-(3.3).

3.1 Fourth Order Problem

In this section, we consider fourth order differential equations of the form

$$u^{(4)}(t) = \lambda h(t, u, u''),\ \ t \in (0,1), \tag{3.4}$$

satisfying the boundary conditions,

$$u(0) = u''(0) = 0, \tag{3.5}$$

$$u'(1) = a, \; u'''(1) = -b, \tag{3.6}$$

where $\lambda, a, b \geq 0$, with $a + b > 0$, and $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$. We start by laying the groundwork for a sequence of lemmas which leads to various expansion and contraction estimates on a defined operator. The culmination of these lemmas then allows a triple application of the Guo-Krasnosel'skii fixed point theorem, yielding at least three positive solutions. This section, however, not only guarantees solutions to (3.4)-(3.6), it also serves as a stepping stone to formulating a main result for $2n$th order problems of a corresponding form. With this in mind, we pay close attention to how the methods of Chapter 2 interact with the fourth order problem, (3.4)-(3.6), as well as where alterations need to be made to accommodate any even order.

3.1.1 Preliminaries

To construct an existence result for the fourth order problem, we must first gather the proper tools. To do this, we turn to the techniques used in Chapter 2 which began by making a series of substitutions into the original problem. Hence, set $-u'' = v$, $v = g(t,u,v)$, and $h(t,u,-v) = f(t,u,v)$, and note that solutions of (3.4)-(3.6) are in one to one correspondence with solutions of the system of second order right focal boundary value problems,

$$-v'' = \lambda f(t,u,v), \; t \in (0,1), \tag{3.7}$$

$$-u'' = g(t,u,v), \; t \in (0,1), \tag{3.8}$$

$$u(0) = v(0) = 0, \tag{3.9}$$

$$u'(1) = a, \; v'(1) = b, \tag{3.10}$$

where $a, b, \lambda \geq 0$ and $a + b > 0$. In order to assure positive solutions of (3.4)-(3.6), certain constraints must be placed on f which lead to solutions of (3.7)-(3.10) being positive. To determine what these constraints should be, we start by assuming that u is a positive solution of (3.4)-(3.6) and work our way backwards. Now, the cone we eventually use to apply the Guo-Krasnosel'skii Fixed Point Theorem contains a concavity condition, allowing us to also assume that u is concave. Then $-v = u'' \leq 0$ on $[0, 1]$, giving that $v = g(t, u, v)$ is nonnegative. Moreover, since u'' is nonpositive, u' is decreasing on $[0, 1]$. Taking this along with the boundary conditions $u'(1) = a > 0$ and $u''(0) = 0$ into consideration, and then appealing to some basic calculus techniques, observe that u' is concave. This leads to u'' being convex on $[0, 1]$. Hence $u^{(4)} \geq 0$, and thus $f(t, u, v)$ is also a nonnegative function. In fact, this is why we confined h to be nonnegative in the original problem. Thus, by constraining f to be nonnegative and reversing the above process, we see that if solutions exist, they must be positive. With this in mind, we place the following assumptions on f:

(G0) $f : [0, 1] \times [0, \infty)^2 \to [0, \infty)$ is a continuous function which is nondecreasing in the last two variables.

(G1) There exist $\alpha_1, \beta_1 \in (0, 1)$, with $\alpha_1 < \beta_1$, such that, given $(u, v) \in [0, \infty)^2$ with $u + v > 0$, there is a $k > 0$ such that

$$f(t, u, v) > k$$

for $t \in [\alpha_1, \beta_1]$.

(G2) $\displaystyle\lim_{u+v \to 0^+} \frac{f(t, u, v)}{(u + v)} = 0$ uniformly for $t \in [0, 1]$.

(G3) $\displaystyle\lim_{u+v \to \infty} \frac{f(t, u, v)}{(u + v)} = 0$ uniformly for $t \in [0, 1]$.

Continuing in the spirit of Chapter 2, we transform the above system, (3.7)-(3.10), into the system of second order differential equations

$$-u'' = f(t, u + ta, v + tb),\ t \in (0, 1), \tag{3.11}$$

$$-v'' = \lambda f(t, u + ta, v + tb),\ t \in (0, 1), \tag{3.12}$$

satisfying the homogeneous right focal boundary conditions,

$$u(0) = u'(1) = 0, \tag{3.13}$$

$$v(0) = v'(1) = 0. \tag{3.14}$$

As before, the main result given at the end of this section is actually for this system of boundary value problems. However, this problem is essentially the same problem as (3.4)-(3.6) and as such, if we can find a solution to (3.11)-(3.14), we automatically have a solution to (3.4)-(3.6). Therefore, note that solutions of the system (3.11)-(3.14) are of the form

$$u(t) = \int_0^1 G(t, s) g(s, u(s) + sa, v(s) + sb) ds,$$
$$v(t) = \lambda \int_0^1 G(t, s) f(s, u(s) + sa, v(s) + sb) ds,$$

where $G(t, s)$ denotes the Green's function

$$G(t, s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1. \end{cases}$$

Although the technique used here is somewhat similar to the one used in Chapter 2, notice it leads to a different Green's function. This is due to the change in boundary conditions and the resulting system of second order differential equations. One of the advantages of this transformation process is that we will continue to use this particular Green's function throughout the remainder of this chapter. Now, $G(t, s)$ is clearly nonnegative, as are both f and g, reaffirming the positivity of solutions. Some other useful properties of $G(t,s)$ are that

$$\max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{2}, \tag{3.15}$$

and

$$\max_{t\in[0,1]} \int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| ds = 1.$$

Both of these equations play a vital role in establishing the complete continuity of an appropriate operator, which we define momentarily. Moreover, the interplay between equation (3.15) and and the bounds on both γ and η that appear in the proofs of the final two lemmas play a key factor in making sure the estimates provided by the next section are small enough. In fact, this is where we get the bounds, $\gamma < 2$ and $\eta < 2$. In addition, it is well known form the properties of G, if $x \in C[0,1]$ and $y(t) = \int_0^1 G(t,s)x(s)ds$, then $y \in C^{(2)}[0,1]$ and $y(0) = y'(0) = 0$.

All we need to do now is find a cone contained in a Banach space with a completely continuous operator so that we can eventually apply the Guo-Krasnosel'skii Fixed Point Theorem. The space used throughout the previous chapter, works just as efficiently for this problem. Thus, we again take $(X, \|\cdot\|)$ to be the Banach Space

$$X = C([0,1];\mathbb{R}) \times C([0,1];\mathbb{R})$$

endowed with the norm

$$\|(u,v)\| = \|u\|_\infty + \|v\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Of course this space is specific to the case where $n = 2$, however, it is easy to see how it can be extended for the general problem. The cone used in the previous chapter, however, was defined with the boundary conditions of that particular problem in mind. Fortunately, a slight alteration of that cone gives a new cone, specific to our new boundary value problem. That is let $C \subset X$ denote the set

$$C = \{(u,v) \in X : (u,v)(0) = (0,0) \text{ and } u, v, \text{ are nonnegative and concave}\}.$$

It is easy to see that C remains a cone under these modifications, by directly showing it satisfies the conditions of the definition.

Finally, define the operator $T : X \to X$ by

$$T(u,v) = (A_1(u,v), A_2(u,v)),$$

where

$$A_1(u,v)(t) = \int_0^1 G(t,s)g(s,u(s)+sa,v(s)+sb)ds$$
$$A_2(u,v)(t) = \lambda \int_0^1 G(t,s)f(s,u(s)+sa,v(s)+sb)ds$$

and note that solutions of (3.11)-(3.14) are merely fixed points of T. In the following lemma, we give two properties of T which play an integral part in allowing us to apply the designated fixed point theorem.

Lemma 3.1. *T is a completely continuous operator and $T : C \to C$.*

Showing that T preserves the cone, C, is straight forward. The proof of complete continuity involves a standard application of the Arzela-Ascoli Theorem completely analogous to the proof given in Lemma 3.1.

3.1.2 Lemmas

In Chapter 2, we saw that the formulation of each main result followed directly from piecing together a sequence of lemmas. This problem is no different. If we can produce four new lemmas of a corresponding nature to those of the previous chapter, the sought after existence result will effortlessly ensue. Fortunately, assumptions (G0) and (G3) were formulated with this in mind. Although they vary from the assumptions in Chapter 2, the requirements placed on f play similar roles.

The first two lemmas each give a set having a lower bound on T in terms of $\|(u,v)\|$, using only assumptions (G0) and (G1). Since we are concerned with finding lower estimates, the proofs rely solely on properties pertaining to f, allowing us to momentarily disregard the function, g. Furthermore, due to the similarities between assumptions (G0) and (H0), and between (G1) and (H1), Lemmas 3.2 and 3.3 have statements and proofs that are very close to the the corresponding lemmas for the previous problem. One of the more obvious differences being that in these proofs, we evaluate the Green's function at $t = 1$, rather than $t = 2^{-1}$, to ensure a nonzero integral.

Lemma 3.2. *Suppose* (G0) *and* (G1) *hold and let* $\rho^* > 0$. *Then there is a* $\Lambda > 0$ *such that, for every* $\lambda \geq \Lambda$ *and* $(a, b) \in [0, \infty)^2$, *we have*

$$\|T(u, v)\| \geq \|(u, v)\|$$

for $(u, v) \in C \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and $(u, v) \in C \cap \partial\Omega_{\rho^*}$. Set $r = \alpha_1(1 - \beta_1)$ and take $\Lambda \geq \left[rM \int_{\alpha_1}^{\beta_1} G(1, s)ds\right]^{-1}$, where

$$M = \inf\left\{\frac{f(t, ra_1, ra_2)}{r(a_1 + a_2)} : t \in [\alpha_1, \beta_1],\ a_1, a_2 > 0,\ \text{and } (a_1 + a_2) = \rho^*\right\}.$$

The fact that M exists and is nonzero follows from (G1). Then, the concavity lemma, Lemma 1.1, combined with nondecreasing property on f from (G0) give that,

$$\begin{aligned}
\|T(u, v)\| &\geq \|A_2(u, v)\|_\infty \\
&\geq \lambda \int_0^1 G(1, s) f(s, u(s) + sa, v(s) + sb)ds \\
&\geq \lambda \int_{\alpha_1}^{\beta_1} G(1, s) f(s, r\|u\|_\infty, r\|v\|_\infty)ds \\
&\geq \lambda r\|(u, v)\| \int_{\alpha_1}^{\beta_1} G(1, s)\frac{f(s, r\|u\|_\infty, r\|v\|_\infty)}{r\|(u, v)\|}ds \\
&\geq \Lambda rM\|(u, v)\| \int_{\alpha_1}^{\beta_1} G(1, s)ds \\
&\geq \|(u, v)\|
\end{aligned}$$

for $\lambda \geq \Lambda$, completing the proof. □

Lemma 3.3. *Fix* $\Lambda > 0$ *and suppose* (G0) *and* (G1) *hold. Then, for every* $\lambda \geq \Lambda$ *and* $(a, b) \in [0, \infty)^2$ *with* $a + b > 0$*, there is a* $\rho_1 = \rho_1(\Lambda, a, b)$ *such that, for every* $\rho \leq \rho_1$*,*

$$\|T(u, v)\| \geq \|(u, v)\|,$$

for $(u, v) \in C \cap \partial\Omega_\rho$*.*

Proof. By (G1) and the nondecreasing properties of f, there is a $k > 0$ such that,

$$f(t, u(t) + ta, v(t) + tb) \geq f(t, \alpha_1 a, \alpha_1 b,) > k,$$

for $t \in [\alpha_1, \beta_1]$. Take $\rho_1 = \Lambda k \int_{\alpha_1}^{\beta_1} G(1, s)ds$. Then for all $(u, v) \in C \cap \Omega_\rho$, where $\rho \leq \rho_1$,

$$\begin{aligned} \|T(u, v)\| &\geq \lambda \int_{\alpha_1}^{\beta_1} G(1, s) f(s, \alpha_1 a, \alpha_1 b) ds \\ &\geq \lambda k \|(u, v)\| \int_{\alpha_1}^{\beta_1} \frac{G(1, s)}{\|(u, v)\|} ds \\ &\geq \|(u, v)\|. \end{aligned}$$

□

After noting how (G0) and (G1) are used in the above proofs, we see that they can be readily modified to accommodate the $2n$th order problem for any $n \geq 2$, by simply making f a function of $n + 1$ variables and adapting the assumptions accordingly. As a result, g automatically extends to begin a function of $n + 1$ variable.

In contrast to the above lemmas, the remaining two lemmas each give a distinct set for which we have upper bounds, rather than lower bounds, on T in terms of $\|(u, v)\|$. These estimates, however, are slightly more arduous to come by and as such, require the remaining assumptions, (G2) and (G3). Lemma 3.4 uses assumption (G2) to gain a sufficiently small bound on A_2, while the relationship between the bound on the Green's function, given in equation (3.15) and the constraint eventually placed on γ, lends to a sufficiently small upper bound on A_1. This works similarly for Lemma 3.5, by replacing (G2) with (G3), and γ with η.

Lemma 3.4. *Suppose* (G0) *and* (G2) *hold and fix* $\rho^* > 0$. *Then given* $\lambda > 0$, *there is a* $\rho_2 \in (0, \rho^*)$ *and a* $\delta > 0$, *such that, for every* $(a, b) \in [0, \infty)^2$ *with* $a + b < \delta$,

$$\|T(u, v)\| \le \|(u, v)\|,$$

for $(u, v) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Given $\lambda > 0$, pick an $\epsilon > 0$ so that $\lambda\epsilon < 1$. Then, by (G2), we can find a $\rho_2 \in (0, \rho^*)$ such that, for $u + v = \rho_2$ and $a + b \le \rho_2$, we have

$$f(t, u + a, v + b) < \epsilon[(u + a) + (v + b)],$$

for $t \in [0, 1]$. Take $(u, v) \in C \cap \partial\Omega_{\rho_2}$ and suppose $a + b \le \rho_2$. Then, for $t \in [0, 1]$,

$$\begin{aligned}
A_2(u, v)(t) &= \lambda \int_0^1 G(t, s) f(s, u(s) + sa, v(s) + sb) ds \\
&\le \lambda \int_0^1 G(t, s) f(s, \|u\|_\infty + a, \|v\|_\infty + b) ds \\
&\le \lambda\epsilon[\|(u, v)\| + (a + b)] \int_0^1 G(t, s) ds \\
&\le 2\lambda\epsilon\|(u, v)\| \int_0^1 G(t, s) ds \\
&\le \lambda\epsilon\|(u, v)\|,
\end{aligned}$$

by equation (3.15). Thus,

$$\|A_2(u, v, w)\|_\infty \le \lambda\epsilon\|(u, v)\|.$$

Next, we consider $A_1(u, v)$. To this end note that g is nondecreasing in the last 2 variables and there are both a $\gamma \in (0, 2)$ and a $q > 0$, such that for $(u, v) \in [0, \infty)^2$ with $u + v < q$,

$$g(t, u, v) \le \gamma(u + v),$$

for $t \in [0,1]$ as a result of g's projective nature. Pick ρ_2 so that $\rho_2 < 2^{-1}q$. Then $[(u+a)+(v+b)] < 2\rho_2 < q$, which, by the above, gives,

$$g(t, u+a, v+b) \le \gamma[(u+a)+(v+b)]$$

for $t \in [0,1]$. Let $\delta' < 1$ and set $\delta = \delta'\rho_2$. Then, by taking $a+b < \delta$ and $(u,v) \in C \cap \partial\Omega_{\rho_2}$, we have

$$\begin{aligned} A_1(u,v)(t) &= \int_0^1 G(t,s)g(s, u(s)+sa, v(s)+sb)ds \\ &\le \gamma[\|(u,v)\| + (a+b)] \int_0^1 G(t,s)ds \\ &\le \gamma(1+\delta')\|(u,v)\| \int_0^1 G(t,s)ds \\ &\le \frac{\gamma(1+\delta')}{2}\|(u,v)\| \end{aligned}$$

for $t \in [0,1]$, giving

$$\|A_1(u,v)\|_\infty \le \frac{\gamma(1+\delta')}{2}\|(u,v)\|.$$

Letting $a+b < \delta$ and $(u,v) \in C \cap \partial\Omega_{\rho_2}$ gives

$$\|T(u,v)\| = \|A_1(u,v)\|_\infty + \|A_2(u,v)\|_\infty \le \left[\frac{\gamma(1+\delta')}{2} + \epsilon\lambda\right]\|(u,v)\|.$$

Therefore, if we take both ϵ and δ' to be sufficiently small, we have the desired result. □

Lemma 3.5. *Suppose assumptions* (G0) *and* (G3) *hold. Let* $\delta > 0$ *be given and suppose* $0 < a+b < \delta$ *Then for every* $\lambda > 0$*, there is a* $\rho_3 = \rho_3(\delta, \lambda)$ *such that, for every* $\rho \ge \rho_3$*,*

$$\|T(u,v)\| \le \|(u,v)\|,$$

where $(u,v) \in C \cap \partial\Omega_\rho$*.*

Proof. By the nondecreasing property naturally associated with g and the fact that there are both a $\eta \in (0,2)$ and a $p > 0$, such that for $(u,v) \in [0,\infty)^2$ with $u+v > p$,

$$g(t,u,v) \le \eta(u+v),$$

for $t \in [0,1]$, given any $q_1 \geq p$, where p is as above, we have for $t \in [0,1]$

$$g(t, u+ta, v+tb) \leq \eta[(u+a)+(v+b)]$$

where $(u,v) \in [0,\infty)^2$ with $u+v \geq q_1$. Let $\epsilon > 0$ and pick q_1 large enough so that $\epsilon > \frac{\eta\delta}{q_1}$. Then,

$$\begin{aligned} g(t, u+ta, v+tb) &\leq \eta(u+v) + \eta(a+b) \\ &\leq \eta(u+v) + \epsilon(u+v) \\ &= (\eta+\epsilon)(u+v). \end{aligned}$$

So, for any $(u,v) \in C \cap \partial\Omega_{q_1}$ and $t \in [0,1]$, we have

$$\begin{aligned} A_1(u,v)(t) &= \int_0^1 G(t,s)g(s, u(s)+as, v(s)+bs)ds \\ &\leq (\eta+\epsilon)\|(u,v)\| \int_0^1 G(t,s)ds \\ &\leq \frac{\eta+\epsilon}{2}\|(u,v)\|, \end{aligned}$$

giving $\|A_1(u,v)\|_\infty < \frac{\eta+\epsilon}{2}\|(u,v)\|$.

Next, we examine $A_2(u,v)$. Let $\delta' > 0$. By (G0) and (G3), there is a $q_2 > 0$ such that, for every $(u,v) \in [0,\infty)^2$ with $u+v \geq q_2$, we have

$$f(t, u+ta, v+tb) \leq \delta'[(u+a)+(v+b)],$$

for $t \in [0,1]$. Let $q_3 = \max\{\delta, q_2\}$. By the above and the fact that $a+b < \delta$, if $(u,v) \in [0,\infty)^2$ with $u+v \geq q_3$,

$$\begin{aligned} f(t, u+ta, v+tb) &\leq \delta'(u+v) + \delta'\delta \\ &\leq 2\delta'(u+v). \end{aligned}$$

Thus, for $(u,v) \in C \cap \partial\Omega_{q_3}$ and $t \in [0,1]$,

$$\begin{aligned} A_3(u,v,w) &\leq 2\delta'\lambda\|(u,v,w)\| \int_0^1 G(t,s)ds \\ &\leq \delta'\lambda\|(u,v)\|, \end{aligned}$$

giving,

$$\|A_3(u,v,w)\|_\infty \leq \delta'\lambda\|(u,v)\|.$$

Take $\rho_3 = \max\{q_1, q_3\}$ and let $\rho \geq \rho_3$. Then, given $(u,v) \in C \cap \partial\Omega_\rho$, we have

$$\|T(u,v,)\| \leq \left(\frac{\epsilon + \eta + 2\lambda\delta'}{2}\right)\|(u,v)\|.$$

Since we can pick ϵ and δ' small enough so that $\epsilon - 2\delta'\lambda \leq 2 - \eta$, we have our desired result. □

When considering any jump in order, the number of functions in the form of g also increases by a number corresponding to the order of the problem. This, in turn, leads to additional operators of the form A_1 and thus, additional operators needing sufficiently small upper bounds. So, when we move to the generalized problem it is desirable to give the existence of a set of γ's and η's with bounds that satisfy similar inequalities pertaining to each new g.

3.1.3 An Existence Result

Now that we have the desired lemmas, we are ready to give an existence result for the fourth order problem, (3.11)-(3.14). Although the boundary conditions for this problem differ from those in the previous chapter, the lemmas in Section 3.1.2 produce a similar set of estimates. Hence, by simply placing the four lemmas together in a manner analogous to that in the previous chapter and then applying the Guo-Krasnosel'skii Fixed Point Theorem, we have the existence of at least three positive solutions to the second order system of equations, (3.11)-(3.14).

Theorem 3.1. *Let f satisfy* (G0)-(G3). *Then there exists a $\Lambda > 0$ such that, for any $\lambda \geq \Lambda$, there is a $\delta > 0$ such that, for every $a, b \geq 0$ with $0 < a + b < \delta$, the system* (3.11)-(3.14) *has at least three positive solutions.*

Proof. Suppose f satisfies hypotheses (G0)-(G5) and fix $\rho^* > 0$. By Lemma 3.2, there is a $\Lambda > 0$ such that for every $\lambda \geq \Lambda$ and $a, b \geq 0$

$$\|T(u,v)\| \geq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho^*}.$$

Now, fix $\lambda \geq \Lambda$. Lemmas 3.3-3.5 then give a $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$, with $\rho_1 < \rho_2 < \rho^* < \rho_3$, such that for $(a,b) \in [0,\infty)^2$ satisfying $0 < a + b < \delta$, we have

$$\|T(u,v)\| \geq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_1},$$
$$\|T(u,v)\| \leq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_2},$$
$$\|T(u,v)\| \leq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_3}.$$

Therefore, by appealing to the Guo-Krasnosel'skii Fixed Point Theorem, there exist at least three positive soltuions $(u_1,v_1),(u_2,v_2),(u_3,v_3) \in C$ of (3.11)-(3.14) such that,

$$\rho_1 < \|(u_1,v_1)\| < \rho_2\|(u_2,v_2)\| < \rho^* < \|(u_3,v_3)\| < \rho_3.$$

□

3.2 Even Order Problem

Now that we have established an existence result for the case where $n = 2$, we are ready to consider the even order differential equation,

$$u^{(2n)} = \lambda h\left(t, u, u'', \dots, u^{(2(n-1))}\right), \ t \in (0,1), \tag{3.16}$$

for some $n \geq 2$, satisfying the boundary conditions,

$$u^{(2k)}(0) = 0, \ \ k = 0, \dots, n-1, \tag{3.17}$$
$$u^{(2k+1)}(1) = (-1)^k a_k, \ \ k = 0, \dots, n-1, \tag{3.18}$$

where $\lambda, a_0, \dots a_{n-1} \geq 0$, with $\sum_{i=0}^{n-1} a_i$, and $h : [0,1] \times \prod_{j=0}^{n-1} (-1)^j [0,\infty) \to (-1)^n [0,\infty)$. As we mentioned earlier, in the move to a more generalized even order problem, some adjustments to the steps taken in the preceding chapter are needed in order to achieve a similar existence result. We already alluded to what those modifications should be in terms of the basic assumptions. Now all we need to do is take these new assumptions, along with the insight provided by Section 3.1.2, and construct four additional lemmas having estimates similar to those of previous problems. Then, as our method has consistently shown us, a generalized existence result will follow.

3.2.1 Preliminaries

Following our pre-established technique, we start by gathering the preliminary information needed for the main result. This process begins by turning the $2n$th order problem, (3.16)-(3.18), into a system of second order ordinary differential equations under homogeneous right focal boundary conditions via a series of substitutions followed by a transformation. As this problem is of higher order, obtaining what substitutions are needed is more tedious than those for the fourth order problem. The method, however, is the same. For each $k = 0, \ldots, n-1$, set $u_k = (-1)^k u^{(2k)}$. Then, for $t \in [0, 1]$, if we let

$$f(t, u_0, -u_1, \ldots, (-1)^{n-1}u_{n-1}) = (-1)^n h(t, u_0, u_1, \ldots, u_{n-1}),$$

and

$$u_{k+1} = g_k(t, u_0, \ldots, u_{n-1}),\ k = 0, \ldots, n-1,$$

solutions to (3.16)-(3.18) are in one-to-one correspondence to solutions of the system of second order differential equations,

$$-u''_{n-1} = \lambda f(t, u_0, \ldots, u_{n-1}),\ \ t \in (0, 1), \tag{3.19}$$

$$-u''_k = g_k(t, u_0, \ldots, u_{n-1}),\ \ k = 0, \ldots, n-2,\ t \in (0, 1), \tag{3.20}$$

$$u_k(0) = 0,\ \ k = 0, \ldots, n-1, \tag{3.21}$$

$$u'_k(1) = a_k,\ \ k = 0, \ldots, n-1. \tag{3.22}$$

In our search for positive solutions, we need to make certain restrictions on the sign of the function f, which result in restrictions on the sign of each g_k. To figure out what these restrictions should be, we rely on the reverse reasoning used in previous sections. We start by assuming that u is a positive, concave solution of (3.16)-(3.18). A simple calculus argument, the same one used in Section 3.1.1, tells us that $u' > 0$ is a decreasing, concave function and that $u'' \leq 0$ is a decreasing convex function. Repeating this argument, we see that the sign of $u^{(2k)}$ has an oscillatory nature similar to the function in Section 2.3.1. That is, if k is even, $u^{(2k)}$ is a nonnegative concave function, and if k is odd, $u^{(2k)}$ is a nonpositive, convex function. But, $u_k = (-1)^k u^{(2k)}$ for each $k = 0, \ldots, n-1$, giving that each u_k is nonnegative. Now,

$$u_{k+1} = (-1)^{k+1} u^{(2(k+1))} = (-1)(-1)^k \left[u^{(2k)}\right]'' = -u_k'' = g_k(t, u_0, \ldots, u_{n-1})$$

for $k = 0, \ldots n-2$. Hence each g_k must be nonnegative. Furthermore, since

$$\lambda f(t, u_0, -u_1, \ldots, (-1)^{n-1} u_{n-1}) = (-1)^n \lambda h(t, u_0, u_1, \ldots, u_{n-1}) = u^{(2n)},$$

we have that f must also be nonnegative. In fact, this is what led to confining the sign of h in the original problem. Reversing this logic and constraining f to be nonnegative, results in giving solutions that are positive, that is, if they exist. With this in mind, the following assumptions are placed on f:

(G0) $f : [0,1] \times [0,\infty)^n \to [0,\infty)$ is a continuous function which is nondecreasing in the last n variables.

(G1) There exists $\alpha_1, \beta_1 \in (0,1)$, with $\alpha_1 < \beta_1$, such that given $(u_0, \ldots, u_{n-1}) \in [0,\infty)^n$ with $\sum_{k=0}^{n-1} u_k \geq 0$, there is a $c > 0$ such that

$$f(t, u_0, \ldots, u_{n-1}) > c$$

for $t \in [\alpha_1, \beta_1]$.

(G2) Let $z = \sum_{k=0}^{n-1} u_k$. Then $\lim_{z \to 0^+} \dfrac{f(t, u_0, \ldots, u_{n-1})}{u_0 + \ldots + u_{n-1}} = 0$ uniformly for $t \in [0,1]$.

(G3) Let $z = \sum_{k=0}^{n-1} u_k$. Then $\lim_{z\to\infty} \dfrac{f(t,u_0,\ldots,u_{n-1})}{u_0+\ldots+u_{n-1}} = 0$ uniformly for $t \in [0,1]$.

Continuing to adhere to the techniques previously established, note that problem (3.19)-(3.22) can be transformed into the system of second order ordinary differential equations,

$$-u''_{n-1} = \lambda f(t, u_0 + ta_0, \ldots, u_{n-1} + ta_{n-1}),\ t \in (0,1), \tag{3.23}$$

$$-u''_k = g_k(t, u_0 + ta_0, \ldots, u_{n-1} + ta_{n-1}),\ \ k = 0,\ldots,n-2,\ t \in (0,1), \tag{3.24}$$

satisfying the homogeneous right focal boundary conditions,

$$u_k(0) = u'_k(1) = 0,\ \ k = 0,\ldots,n-1. \tag{3.25}$$

We reiterate that this is the actual system we establish the existence of solutions for. However, since this system is simply a transformation of system (3.19)-(3.22), a system whose solutions are in one-to-one correspondence with the problem, (3.16)-(3.18), then we automatically have solutions to our original problem. Taking this into consideration, note that solutions of the system (3.23)-(3.25) are of the form

$$u_{n-1}(t) = \lambda \int_0^1 G(t,s) f(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds,$$
$$u_k(t) = \int_0^1 G(t,s) g_k(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds,\ k = 0,\ldots,n-2,$$

where $G(t,s)$ denotes the Green's function,

$$G(t,s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1. \end{cases}$$

Since this is the same Green's function as the one used for the fourth order problem, it carries the same advantageous properties, which we restate here for convenience. It is, of course, positive, which reinforces the positivity of solutions. We also have an upperbound on the integral given by,

$$\max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{2},$$

where the maximum occurs at $t = 1$. This bound plays a similar role in establishing the estimates for this problem as it did for the fourth order problem. It also has a working relationship with the two that appears as a bound in the final two proofs of the next section.

For our Banach space, we will once again rely on the space $(X, \|\cdot\|)$, where

$$X = \prod_{j=1}^{n} C([0,1]; \mathbb{R}),$$

endowed with the norm,

$$\|(u_0, \ldots, u_{n-1})\| = \|u_0\|_\infty + \ldots + \|u_{n-1}\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Notice, this is just the generalized version of the space used in the fourth order problem. In fact, it is the same space used throughout the previous chapter. In Section 3.1.2, we tweaked the cone from Chapter 2 so that it pertained to the boundary conditions of the fourth order right focal problem. By extending the set so that it deals with an arbitrary n, we get the set $C \in X$, defined by

$$C = \{(u_0, \ldots, u_{n-1}) \in X | (u_0, \ldots, u_{n-1})(0) = (0, \ldots, 0) \text{ and } u_0, \ldots, u_{n-1} \text{ are nonnegative and concave for}\}.$$

It is easy to see, by definition, that under this particular extension, C remains a cone.

We conclude with defining the operator $T : X \to X$ by

$$T(u_0, \ldots, u_{n-1}) = (A_0(u_0, \ldots, u_{n-1}), \ldots, A_{n-1}(u_0, ..., u_{n-1})),$$

where

$$A_{n-1}(u_0, \ldots, u_{n-1})(t) = \lambda \int_0^1 G(t,s) f(s, u_0(s) + a_0 s, \ldots, u_{n-1}(s) + s a_{n-1}) ds,$$
$$A_k(u-0, \ldots, u_{n-1})(t) = \int_0^1 G(t,s) g(s, u_0(s) + a_0 s, \ldots, u_{n-1}(s) + s u_{n-1}) ds,$$

for $k = 0, \dots, n-2$ and $t \in [0,1]$. Note, solutions of (3.23)-(3.25) are fixed points of T, generating our interest in the Guo-Krasnosel'skii Fixed Point Theorem. The following Lemma yields two properties of T which factor into the ability to apply this particular fixed point theorem to our problem.

Lemma 3.6. *T is a completely continuous operator and $T : C \to C$.*

The proof that T is completely continuous uses an Arzela Ascoli argument similar to the one used in Lemma 2.1. The fact that T preserves the cone C follows directly from the definition of cone.

3.2.2 Lemmas

Now that the preliminaries are in place, our next step is to construct a set of four lemmas which lead to an existence result. We were careful in the previous section to consider what effects the generalized n would have on our methods and thus, made sure to account for these affects when formulating the new hypotheses. In this section, we show how a new quartet of lemmas follows from these newly modified assumptions, giving results very close in nature to the lemmas of the previous sections.

We start, in the first two lemmas, by providing two distinct sets for which we have the relationship,

$$\|T(u_0, \dots, u_{n-1})\| \geq \|(u_0, \dots, u_{n-1})\|.$$

Since these lemmas only deal with lower estimates on T, it is sufficient to find sets which have sufficiently large bounds on A_{n-1}, hence, we only deal with the function, f. This idea is not unlike the first two lemmas of the previous sections, and as such, the proofs follow in a similar fashion.

Lemma 3.7. *Suppose* (G0) *and* (G1) *hold and let $\rho^* > 0$. Then there exists a $\Lambda > 0$*

such that, for every $\lambda \geq \Lambda$ *and* $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$,

$$\|T(u_0, \ldots, u_{n-1})\| \geq \|(u_0, \ldots, u_{n-1})\|,$$

where $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and take $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho^*}$. Set $r = \alpha_1(1 - \beta_1)$ and

$$\Lambda = \left[rM \int_{\alpha_1}^{\beta_1} G(1, s)\, ds \right]^{-1},$$

where

$$M = \inf \left\{ \frac{f(t, rz_0, \ldots, rz_{n-1})}{r(z_0 + \cdots + z_{n-1})} : t \in [\alpha_1, \beta_1], (z_0, \ldots, z_{n-1}) \in (0, \infty)^n \text{ and } \sum_{i=0}^{n-1} z_i = \rho^* \right\}.$$

The existence of a positive M follows from assumption (G1). The combination of the concavity lemma, Lemma 1.1, with the nondecreasing property placed of f, tells us that

$$\begin{aligned}
\|T(u_0, \ldots, u_{n-1})\| &\geq \|A_{n-1}(u_0, \ldots, u_{n-1})\|_\infty \\
&\geq \lambda \int_0^1 G(1, s) f(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds \\
&\geq \lambda \int_{\alpha_1}^{\beta_1} G(1, s) f(s, r\|u_0\|_\infty, \ldots, r\|u_{n-1}\|_\infty) ds \\
&\geq \lambda r \|(u_0, \ldots, u_{n-1})\| \int_{\alpha_1}^{\beta_1} G(1, s) \frac{f(s, r\|u_0\|_\infty, \ldots, r\|u_{n-1}\|_\infty)}{r\|(u_0, \ldots, u_{n-1})\|} ds \\
&\geq \Lambda r M \|(u_0, \ldots, u_{n-1})\| \int_{\alpha_1}^{\beta_1} G(1, s) ds \\
&\geq \|(u_0, \ldots, u_{n-1})\|,
\end{aligned}$$

for $\lambda \geq \Lambda$. □

Lemma 3.8. *Fix* $\Lambda > 0$ *and suppose* (G0) *and* (G1) *hold. Then, for every* $\lambda \geq \Lambda$ *and* $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$ *with* $\sum_{i=0}^{n-1} a_i < 0$, *there is a* $\rho_1 = \rho_1(\Lambda, a_0, \ldots, a_{n-1})$ *such that, for every* $\rho \leq \rho_1$, *we have*

$$\|T(u_0, \ldots, u_{n-1})\| \geq \|(u_0, \ldots, u_{n-1})\|,$$

for $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$.

Proof. By (G1) and the nondecreasing property on f, there is a $c > 0$, such that,

$$f(t, u_0(t) + ta_0, \ldots, u_{n-1}(t) + ta_{n-1}) \geq f(t, \alpha_1 a_0, \ldots, \alpha_1 a_{n-1}) > c,$$

for $t \in [\alpha_1, \beta_1]$. Take $\rho_1 = \Lambda c \int_{\alpha_1}^{\beta_1} G(1, s)ds$. Then for all $(u_0, \ldots, u_{n-1}) \in C \cap \Omega_\rho$, where $\rho \leq \rho_1$,

$$\begin{aligned}
\|T(u_0, \ldots, u_{n-1})\| &\geq \lambda \int_{\alpha_1}^{\beta_1} G(1, s) f(s, \alpha_1 a_0, \ldots, \alpha_1 a_{n-1}) ds \\
&\geq \Lambda c \|(u_0, \ldots, u_{n-1})\| \int_{\alpha_1}^{\beta_1} \frac{G(1, s)}{\|(u_0, \ldots, u_{n-1})\|} ds \\
&\geq \|(u_0, \ldots, u_{n-1})\|.
\end{aligned}$$

□

The final two lemmas result in two additional sets that give the reverse inequality between $\|T(u_0, \ldots, u_{n-1})\|$ and $\|(u_0, \ldots, u_{n-1})\|$. Because each of these deals with an upper bound on T, we must find sets having upper bounds on each of the A_k, where $k = 0, \ldots, n-1$. This requires the use of the remaining hypotheses. Fortunately, when the bounds on each A_k are summed for $k = 0, \ldots n-2$, we can keep the sum sufficiently small by taking advantage of how each g_k, $k = 0, \ldots, n-2$ is defined. Assumptions (G2) and (G3) play a similar role on the bound of A_{n-1}, in their respective lemma. This allows us two new lemmas yielding similar conclusions to the corresponding lemmas in previous problems.

Lemma 3.9. *Suppose* (G0) *and* (G2) *hold and fix* $\rho^* > 0$. *Then, given* $\lambda > 0$, *there is a* $\rho_2 \in (0, \rho^*)$ *and a* $\delta > 0$, *such that for every* $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$, *with* $0 < \sum_{i=1}^{n-1} a_i < \delta$,

$$\|T(u_0, \ldots, u_{n-1})\| \leq \|(u_0, \ldots, u_{n-1})\|,$$

for $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Given $\lambda > 0$, pick $\epsilon > 0$ so that $\lambda\epsilon < 1$. Then, by (G2), there is a $\rho_2 \in (0, \rho^*)$ such that, for $\sum_{i=0}^{n-1} u_i = \rho_2$ and $0 < \sum_{i=1}^{n-1} a_i \leq \rho_2$, we have that,

$$f(t, u_0 + a_0, \ldots, u_{n-1} + a_{n-1}) < \epsilon \sum_{i=0}^{n-1}(u_i + a_i)$$

for $t \in [0, 1]$. Take $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$ and suppose $\sum_{i=0}^{n-1} a_i \leq \rho_2$. Then, by (G0),

$$\begin{aligned} A_{n-1}(u_0, \ldots, u_{n-1})(t) &= \lambda \int_0^1 G(t, s) f(s, u_0(s) + sa_0, \ldots, u_{n-1}(s) + sa_{n-1}) ds \\ &\leq \lambda \int_0^1 G(t, s) f(s, \|u_0\|_\infty + a_0, \ldots, \|u_{n-1}\|_\infty + a_{n-1}) ds \\ &< \lambda\epsilon \left[\|(u_0, \ldots, u_{n-1})\| + \sum_{i=0}^{n-1} a_i \right] \int_0^1 G(t, s) ds \\ &\leq 2\lambda\epsilon \|(u_0, \ldots, u_{n-1})\| \int_0^1 G(t, s) ds \\ &\leq \lambda\epsilon \|(u_0, \ldots, u_{n-1})\|, \end{aligned}$$

for $t \in [0, 1]$. Thus,

$$\|A_{n-1}(u_0, \ldots, u_{n-1})\|_\infty \leq \lambda\epsilon \|(u_0, \ldots, u_{n-1})\|.$$

Now we turn our attention to the remaining A_k's. First recall that each g_k, $k = 0, \ldots, n-2$ is nondecreasing in the last n variables as a result of their nature as projections. Furthermore, there are $\gamma_0, \ldots, \gamma_{n-2}$, with

$$0 < \sum_{k=0}^{n-2} \gamma_k < 2,$$

and a $q > 0$ such that, for $(u_0, \ldots, u_{n-1}) \in [0, \infty)^n$ with $\sum_{i=0}^{n-1} u_i < q$,

$$g_k(t, u_0, \ldots, u_{n-1}) \leq \gamma_k \left(\sum_{i=0}^{n-1} u_i \right),$$

for each $k = 0, \ldots, n-2$, and $t \in [0, 1]$. Pick ρ_2 so that $\rho_2 < 2^{-1}q$. Then

$$\sum_{i=0}^{n-1}(u_i + a_i) < 2\rho_2 < q.$$

Hence, by the above,

$$g_k(t, u_0 + a_0, \ldots, u_{n-1} + a_{n-1}) \leq \gamma_k \sum_{i=0}^{n-1} (u_i + a_i),$$

for $k = 1, \ldots, n-2$, and $t \in [0, 2]$. Let $\gamma' < 1$ and set $\delta = \gamma' \rho_2$. Then for $\sum_{i=0}^{n-1} a_i < \delta$ and $(u_0, ..., u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$,

$$\begin{aligned}
A_k(u_0, \ldots, u_n)(t) &= \int_0^1 G(t, s) g_k(s, u_0(s) + s a_0, \ldots, u_{n-1}(s) + s a_{n-1}) ds \\
&\leq \gamma_k \left[\|(u_0, \ldots, u_{n-1})\| + \sum_{i=0}^{n-1} a_i \right] \int_0^1 G(t, s) ds \\
&\leq \gamma_k (1 + \delta') \|(u_0, \ldots, u_{n-1})\| \int_0^1 G(t, s) ds \\
&\leq \frac{\gamma_k (1 + \delta')}{2} \|(u_0, \ldots, u_{n-1})\|,
\end{aligned}$$

for $k = 0, \ldots, n-2$, and $t \in [0, 1]$. Thus, for each $k = 0, \ldots, n-2$, we have the inequality,

$$\|A_k(u_0, \ldots, u_{n-1})\|_\infty \leq \frac{\gamma_k (1 + \delta')}{2} \|(u_0, \ldots, u_{n-1})\|.$$

So, if we take $\sum_{i=0}^{n-1} a_i < \delta$ and $(u_0, \ldots, u_{n-1}) \in C \cap \Omega_{\rho_2}$, then

$$\|T(u_0, \ldots, u_{n-1})\| \leq \left[\left(\frac{1 + \gamma'}{2} \right) \sum_{k=0}^{n-2} \gamma_k + \epsilon \lambda \right] \|(u_0, \ldots, u_{n-1})\|.$$

Therefore, if we choose ϵ and γ' sufficiently small, the desired result follows. □

Lemma 3.10. *Suppose* (G0) *and* (G3) *hold and let* $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$ *satisfying,* $0 < \sum_{i=0}^{n-1} a_k < \delta$, *where* $\delta > 0$ *is given. Then, for every* $\lambda > 0$, *there is a* $\rho_3 = \rho_3(\delta, \lambda)$ *such that for every* $\rho \geq \rho_3$,

$$\|T(u_0, \ldots, u_{n-1})\| \leq \|(u_0, \ldots, u_{n-1})\|,$$

where $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$.

Proof. Due to their inherent nature, each g_k for $k = 0, \ldots, n-2$ is nondecreasing in the last n variables and there are $\eta_0, \ldots, \eta_{n-2}$, satisfying

$$0 < \eta < \frac{2}{n-1},$$

where $\eta = \max\{\eta_0, \dots \eta_{n-2}\}$, and a $p > 0$ such that, for $(u_0, \dots, u_{n-1}) \in [0, \infty)^n$ with $\sum_{i=0}^{n-1} u_i > p$, we have

$$g_k(t, u_0, \dots, u_{n-1}) \leq \eta_k \left(\sum_{i=0}^{n-1} u_i \right),$$

for each $k = 0, \dots, n-2$, and $t \in [0, 1]$. Thus, given any $q_1 \geq p$, where p is defined above, we have for $(u_0, \dots, u_{n-1}) \in [0, \infty)^n$, with $\sum_{i=0}^{n-1} u_i \leq q_1$,

$$g_k(t, u_0 + ta_0, \dots, u_{n-1} + ta_{n-1}) \leq \eta_k \sum_{i=0}^{n-1} (u_i + a_i),$$

for each $k = 0, \dots, n-2$, and $t \in [0, 1]$. Let $\epsilon > 0$ and pick q_1 large enough so that $\epsilon > \frac{\eta\delta}{q_1}$, then

$$g_k(t, u_0 + ta_0, \dots, u_{n-1} + ta_{n-1}) \leq \eta \sum_{i=0}^{n-1} u_i + \eta \sum_{i=0}^{n-1} a_i = (\eta + \epsilon) \sum_{i=0}^{n-1} u_i$$

for each $k = 0, \dots, n-2$ and $t \in [0, 1]$. So for any $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_{q_1}$,

$$\begin{aligned} A_k(u_0, \dots, u_{n-1})(t) &= \int_0^1 G(t, s) g_k(s, u_0(s) + sa_0, \dots, u_{n-1}(s) + sa_{n-1}) ds \\ &\leq (\eta + \epsilon) \|(u_0, \dots, u_{n-1})\| \int_0^1 G(t, s) ds \\ &\leq \frac{\eta + \epsilon}{2} \|(u_0, \dots, u_{n-1})\|, \end{aligned}$$

for $k = 0, \dots, n-2$, and $t \in [0, 1]$. Thus, for each $k = 0, \dots, n-2$, we have the inequality,

$$\|A_k(u_0, \dots, u_{n-1})\|_\infty < \frac{\eta + \epsilon}{2} \|(u_0, \dots, u_{n-1})\|.$$

Now we examine A_{n-1}. Let $\delta' > 0$. By (G0) and (G3), we can find a $q_2 > 0$ such that for every $(u_0, \dots, u_{n-1}) \in [0, \infty)^n$, with $\sum_{i=0}^{n-1} u_i \geq q_2$, we have

$$f(t, u_0 + ta_0, \dots, u_{n-1} + ta_{n-1}) < \delta' \left[\sum_{i=0}^{n-1} (u_i + a_i) \right].$$

Take $q_3 = \max\{\delta, q_2\}$. Noting that $\sum_{i=0}^{n-1} a_i < \delta$, for each $(u_0, \dots, u_{n-1}) \in [0, \infty)^n$ satisfying $\sum_{i=0}^{n-1} u_i \geq q_3$, we have

$$f(t, u_0 + ta_0, \dots, u_{n-1} + ta_{n-1}) < \delta' \sum_{i=0}^{n-1} u_i + \delta'\delta \leq 2\delta' \sum_{i=0}^{n-1} u_i.$$

Thus, for $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{q_3}$,

$$A_{n-1}(u_0, \ldots, u_{n-1}) < 2\delta'\lambda(\|(u_0, \ldots, u_{n-1})\| \int_0^1 G(t,s)ds$$
$$\leq \delta'\lambda\|(u_0, \ldots, u_{n-1})\|,$$

giving that

$$\|A_{n-1}(u_0, \ldots, u_{n-1})\|_\infty \leq \delta'\lambda\|(u_0, \ldots, u_{n-1})\|.$$

Set $\rho_3 = \max\{q_1, q_3\}$ and take $\rho \geq \rho_3$. Then given $(u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$,

$$\|T(u_0, \ldots, u_{n-1})\| < \left[\frac{(\eta+\epsilon)(n+1)}{2} + \delta'\lambda\right] \|(u_0, \ldots, u_{n-1})\|.$$

Therefore if we pick ϵ and δ' small enough so that $\epsilon(n-1) < 2 - \eta(n-1)$, we have our desired result. □

3.2.3 An Existence Result

We conclude this chapter be giving an existence theorem for the system of second order ordinary differential equation (3.23)-(3.24) satisfying the right focal boundary conditions (3.25). The bulk of the changes needing to be made to acquire such a result were taken care of by the previous sections. After restating the theorem to account for any $n \geq 2$, we combine the four lemmas to allow a multiple application of the Guo-Krasnosel'skii Fixed Point Theorem. It then follows that the system (3.23)-(3.25), and hence the $2n$th order boundary value problem (3.16)-(3.18), has at least three positive solutions.

Theorem 3.2. *Let f satisfy* (G0)-(G3). *Then, there exists a $\Lambda > 0$ such that, given any $\lambda \geq \Lambda$, there is a $\delta > 0$ such that, for every $(a_0, \ldots a_{n-1}) \in [0, \infty)^n$, with $0 < \sum_{i=1}^{n-1} a_i < \delta$, the system* (3.23)-(3.25) *has at least three positive solutions.*

Proof. Suppose f satisfies hypotheses (G0)-(G3), and fix $\rho^* > 0$. By Lemma 3.7, there is a $\Lambda > 0$ such that, for every $\lambda \geq \Lambda$ and $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$,

$$\|T(u_0, \ldots, u_{n-1})\| \geq \|(u_0, \ldots, u_{n-1})\|, \text{ for } (u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega^*_\rho$$

for $\lambda \geq \Lambda$ and $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$. Now, fix $\lambda \geq \Lambda$. Then, by Lemmas 3.8-3.10, there is a $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$, with $\rho_1 < \rho_2 < \rho^* < \rho_3$, such that for $(a_0, \ldots, a_{n-1}) \in [0, \infty)^n$ satisfying $0 < \sum_{i=0}^{n-1} a_i < \delta$, we have

$$\|T(u_0, \ldots, u_{n-1})\| \geq \|(u_0, \ldots, u_{n-1})\|, \text{ for } (u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_1},$$

$$\|T(u_0, \ldots, u_{n-1})\| \leq \|(u_0, \ldots, u_{n-1})\|, \text{ for } (u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2},$$

$$\|T(u_0, \ldots, u_{n-1})\| \leq \|(u_0, \ldots, u_{n-1})\|, \text{ for } (u_0, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_3}.$$

Applying the Guo-Krasnoselskii Fixed Point Theorem, we have the existence of three positive solutions $(u_0, \ldots, u_{n-1}), (v_0, \ldots, v_{n-1}), (w_0, \ldots, w_{n-1}) \in C$, satisfying

$$\rho_1 < \|(u_0, \ldots, u_{n-1})\| < \rho_2 < \|(v_0, \ldots, v_{n-1})\| < \rho^* < \|(w_0, \ldots, w_{n-1})\| < \rho_3.$$

□

CHAPTER FOUR

Difference Equations

In this chapter, we focus on modifying the transformation technique used on the even order differential equations that appeared in previous chapters to hold for difference equations, by establishing an existence result for the fourth order discrete boundary value problem,

$$\Delta^4 u(t-2) = \lambda h(t, u(t), \Delta^2 u(t-1)),\ t \in (0, N+2)_{\mathbb{Z}}, \tag{4.1}$$

$$u(0) = 0,\ \Delta^2 u(-1) = 0, \tag{4.2}$$

$$u(N+2) = a,\ \Delta^2 u(N+1) = -b, \tag{4.3}$$

where $a, b, \lambda \geq 0$, $a+b > 0$, and $h : [0, N+2]_{\mathbb{Z}} \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$. We start by making a series of substitutions that ultimately transform the above difference equation into a system of second order difference equations satisfying homogeneous conjugate boundary conditions. We then show that the new system has multiple solutions by first constructing a sequence of four lemmas reminiscent of those in the previous chapters, and then appealing to the Guo-Krasnosel'skii Fixed Point Theorem.

4.1 Preliminaries

In order to construct an existence result for the fourth order difference equation (4.1)-(4.3), we start by setting up some useful preliminary work. First note that given any set $S \subset \mathbb{R}$, $S_{\mathbb{Z}}$ denotes the intersection of the set S with the integers; that is

$$S_{\mathbb{Z}} = S \cap \mathbb{Z}.$$

Since we are working with functions having a discrete domain, we use the forward difference operator, defined by

$$\Delta y(t) = y(t+1) - y(t).$$

Recall that in order to take the second derivative of a function, we just take the derivative of the function's derivative. The case is similar for the second difference of a function, denoted Δ^2. We simply take the forward difference of the forward difference, giving

$$\Delta^2 y(t) = \Delta(\Delta y(t)) = y(t+2) - 2y(t+1) + y(t).$$

Continuing with this pattern, we take the nth difference of a function by applying the forward difference n times. A simple inductive proof shows that

$$\Delta^n y(t) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} y(t+i).$$

Now that we have a basic understanding of the notation used in (4.1)-(4.3), we are ready to adapt the transformation method that has thus far proven quite efficient in allowing existence results for various differential equations to this particular difference equation. As before, we make substitutions in the original problem resulting in a system of second order boundary value problems. So, for $t \in [1, N+1]_{\mathbb{Z}}$, let $v(t) = -\Delta^2 u(t-1)$, $g(t, u(t), v(t)) = v(t)$, and $f(t, u, -v) = h(t, u, v)$. Then $-\Delta^2 u(t-1) = g(t, u(t), v(t))$, and

$$\begin{aligned} -\Delta^2 v(t-1) &= \Delta^4 u(t-2) \\ &= \lambda h(t, u(t), \Delta^2 u(t-1)) \\ &= \lambda f(t, u(t), -\Delta^2 u(t-1)) \\ &= \lambda f(t, u(t), v(t)). \end{aligned}$$

Applying these substitutions to the boundary conditions gives that $v(0) = -\Delta^2 u(-1)$ $= 0 = u(0)$ and $v(N+2) = -\Delta^2 u(N+1) = b$. Thus, solutions of the fourth order difference equation (4.1)-(4.3) are in one-to-one correspondence with solutions of the system of second order difference equations,

$$-\Delta^2 u(t-1) = g(t, u(t), v(t)),\ t \in (0, N+2)_{\mathbb{Z}}, \tag{4.4}$$

$$-\Delta^2 v(t-1) = \lambda f(t, u(t), v(t)),\ t \in (0, N+2)_{\mathbb{Z}}, \tag{4.5}$$

satisfying the conjugate boundary conditions,

$$u(0) = v(0) = 0, \tag{4.6}$$

$$u(N+2) = a,\ v(N+2) = b. \tag{4.7}$$

As we eventually wish to formulate a theorem that gives not just the existence of solutions, but existence of positive solutions, our next step is to consider what sign constraints should be placed on f to ensure positivity. To do this, we turn to our previously established reverse reasoning and assume that $u(t)$ is a positive concave solution on $[0, N+2]_{\mathbb{Z}}$. The ability to assume that the solution is concave follows from the selection of the cone we use to apply the Guo-Krasnosel'skii Fixed Point Theorem. Thus, we see that $v(t) = -\Delta^2 u(t) \geq 0$, leading us to confine g to having only nonnegative values on $[0, N+2]_{\mathbb{Z}}$. Using basic difference calculus, it is easy to see that $\Delta^2 u(t)$ is not only nonpositive on $[0, N+2]_{\mathbb{Z}}$, but convex as well. So $\Delta^4 u(t) \geq 0$, and hence, f must also be nonnegative. Notice that these signs fall in line with the boundary conditions of the original problem as well as the initial sign constraint made on h. With this in mind, we place the following requirements on the functions f:

(K0) $f : [0, N+2]_{\mathbb{Z}} \times [0, \infty)^2 \to [0, \infty)$ is a continuous function which is nondecreasing in the last two variables.

(K1) Suppose there is an $\alpha_1, \beta_1 \in (0, N+2)_{\mathbb{Z}}$, where $\alpha_1 < \beta_1$, such that given $(u, v) \in [0, \infty)^2$, there is a $k > 0$ such that

$$f(t, u, v) > k$$

for $t \in [\alpha_1, \beta_1]_{\mathbb{Z}}$.

(K2) $\displaystyle\lim_{u+v \to 0^+} \frac{f(t, u, v)}{u+v} = 0$ uniformly for $t \in [0, N+2]_{\mathbb{Z}}$.

(K3) $\displaystyle\lim_{u+v \to \infty} \frac{f(t, u, v)}{u+v} = 0$ uniformly for $t \in [0, N+2]_{\mathbb{Z}}$.

In [17], Kelly and Peterson showed that second order difference equations of the form

$$\Delta^2 w(t-1) = k(t), \tag{4.8}$$

$$w(0) = 0 = w(N+2), \tag{4.9}$$

have solutions of the form

$$w(t) = \sum_{s=1}^{N+1} G(t,s)k(s),$$

where $G(t,s)$ is the corresponding Green's function, defined momentarily, having properties that will prove useful to the establishment of our main result. Knowing this, we transform (4.3)-(4.7) to a system of second order difference equations having a similar form. That is, (4.3)-(4.7) becomes the system

$$-\Delta^2 u(t-1) = g\left(t, u(t) + \left(\frac{a}{N+2}\right)t, v(t) + \left(\frac{b}{N+2}\right)t\right),\ t \in (0, N+2)_{\mathbb{Z}}, \tag{4.10}$$

$$-\Delta^2 v(t-1) = \lambda f\left(t, u(t) + \left(\frac{a}{N+2}\right)t, v(t) + \left(\frac{b}{N+2}\right)t\right),\ t \in (0, N+2)_{\mathbb{Z}}, \tag{4.11}$$

satisfying the homogeneous conjugate boundary conditions,

$$u(0) = v(0) = 0, \tag{4.12}$$

$$u(N+2) = v(N+2) = 0. \tag{4.13}$$

Since (4.10)-(4.13) is merely a transformation of (4.3)-(4.7), solutions satisfying one system, automatically satisfy the other. Thus, it suffices to show that (4.10)-(4.13) has positive solutions. For simplicity set

$$A := \frac{a}{N+2} \text{ and } B := \frac{b}{N+2}$$

and notice that we transformed (4.3)-(4.7) to look like (4.8)-(4.9). As a result, we know that solutions are are of the form

$$u(t) = \sum_{s=1}^{N+1} G(t,s) g(s, u(s) + As, v(s) + Bs),$$
$$v(t) = \lambda \sum_{s=1}^{N+1} G(t,s) f(s, u(s) + As, v(s) + Bs),$$

where $G(t,s)$ denotes the Green's function,

$$G(t,s) = \frac{1}{N+2} \begin{cases} t(N+2-s), & 0 \leq t \leq s \leq N+1, \\ s(N+2-t), & 1 \leq s \leq t \leq N+2. \end{cases}$$

Now, $G(t,s)$ is nonnegative by inspection and since both f and g are non-negative by assumption, we see that u and v must also be positive. Another useful property of $G(t,s)$ is that

$$\max_{t \in [0,N+2]_{\mathbb{Z}}} \sum_{s=1}^{N+1} G(t,s) \leq \frac{(N+2)^2}{8}. \tag{4.14}$$

This bound plays a vital role in acquiring the estimates of the next section and is an underlying factor in the bound that appears on both γ and η in the proofs of Lemmas 4.4 and 4.5, respectively.

Since our main result hinges on the use of the Guo-Krasnosel'skii Fixed Point Theorem, we need a Banach space containing a cone as well as a completely continuous operator. In previous sections, our Banach space was the product space of a set of continuous functions. However, here we are working on a discrete set and hence our functions are automatically continuous. It suffices to let

$$Y = \{u(t) |\, u : [0, N+2]_{\mathbb{Z}} \to \mathbb{R}\}$$

and let $(X, \|\cdot\|)$ denote the Banach space $X = Y \times Y$, endowed with the norm

$$\|(u,v)\| = \|u\|_\infty + \|v\|_\infty,$$

where $\|u\|_\infty = \max_{t\in[0,N+2]_{\mathbb{Z}}} |u(t)|$. Define $C \subset X$ to be the cone

$$C = \{(u,v) \in X | (u,v)(0) = (u,v)(N+2) = (0,0) \text{ and } u,v \text{ are concave}\}.$$

A simple exercise shows that C is, in fact, a cone. Finally, define the operator $T : X \to X$ by $T(u,v) = (A_1(u,v), A_2(u,v))$, where

$$A_1 = \sum_{s=1}^{N+1} G(t,s)g(s, u(s) + As, v(s) + Bs),$$
$$A_2 = \lambda \sum_{s=1}^{N+1} G(t,s)f(s, u(s) + As, v(s) + Bs).$$

The following lemma reveals a couple of properties of T pertinent to allowing the application of the Guo-Krasnosel'skii Fixed Point Theorem.

Lemma 4.1. *T is a completely continuous cone preserving operator.*

4.2 Lemmas

Although this chapter centers on the establishment of an existence result for (4.1)-(4.3), that is not its sole purpose. We are also concerned with if and how our transformation technique can be molded to include another class of equations. This gives a better insight into the power of the technique. It also opens the door to a more vast collection of applications of this technique, which we will discuss in greater detail in the final chapter. With that said and our preliminaries in place, we move to constructing a sequence of four lemmas generating a set of estimates that work in unison to allow a triple application of the Guo-Krasnosel'skii Fixed Point Theorem. Notice, assumptions (K0)-(K3) are similar in nature to assumptions (H0)-(H3), used in Chapter 2, and (G0)-(G3), used in Chapter 3. As a result, the lemmas given in this section correspond with those in the previous chapters. The proofs, however, have some natural variation.

In the first two lemmas, involving differential equations, we find lower estimates on T in terms of $\|(u,v)\|$. As with the previous problems, these results require only the use of assumptions (K0) and (K1). In fact, the function g is irrelevant to these two lemmas, but will be useful in acquiring the upper bounds on T given in the final two lemmas. One of the more notable differences between this difference equation and the differential equations dealt with in Chapters 2 and 3 is that the solutions for this problem are in terms of summations, not integrals. However, integration and summation carry similar fundamental properties resulting in a sequence of inequalities reminiscent of those in the corresponding lemmas. We also use the greatest integer function to find a point where $G(t,s)$ is not zero. This has to do with the location of the maximum of $\sum G(t,s)$ as a function of t.

Lemma 4.2. *Suppose* (K0) *and* (K1) *hold and let* $\rho^* > 0$. *Then there is a* $\Lambda > 0$ *such that, for every* $\lambda \geq \Lambda$ *and* $(a,b) \in [0,\infty)^2$,

$$\|T(u,v)\| \geq \|(u,v)\|,$$

for $(u,v) \in C \cap \partial\Omega_{\rho*}$.

Proof. Let $\rho^* > 0$ and $(u,v) \in C \cap \partial\Omega_{\rho^*}$. Pick r so that both $u(t) \geq r\|u\|_\infty$ and $v(t) \geq r\|u\|_\infty$ for $t \in [\alpha_1, \beta_1]$ and set

$$\Lambda = \left[rM \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2} \right]\!\right], s \right) \right]^{-1},$$

where $[\![\cdot]\!]$ denotes the greatest integer function and

$$M = \inf\left\{ \frac{f(t, ra_1, ra_2)}{r(a_1+a_2)} : t \in [\alpha_1, \beta_1]_{\mathbb{Z}},\ a_1, a_2 > 0,\ \text{and } (a_1+a_2) = \rho^* \right\}.$$

Thus, appealing to the nondecreasing property placed on f, we see that

$$\begin{aligned}
\|T(u,v)\| &\geq \|A_2(u,v)\|_\infty \\
&\geq \lambda \sum_{s=1}^{N+1} G\left(\left[\!\left[\frac{N+2}{2} \right]\!\right], s \right) f(s, u(s)+As, v(s)+Bs) \\
&\geq \lambda \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2} \right]\!\right], s \right) f(s, r\|u\|_\infty, r\|v\|_\infty)
\end{aligned}$$

$$= \lambda r\|(u,v)\| \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2}\right]\!\right], s\right) \frac{f(s, r\|u\|_\infty, r\|v\|_\infty)}{r\|(u,v)\|}$$
$$\geq \lambda r M\|(u,v)\| \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2}\right]\!\right], s\right)$$
$$\geq \|(u,v)\|$$

for each $\lambda \geq \Lambda$, yielding the desired result. □

Lemma 4.3. *Let* (K0) *and* (K1) *hold and fix* $\Lambda > 0$. *Then, for every* $\lambda \geq \Lambda$ *and* $(a,b) \in [0,\infty)^2$, *with* $a+b > 0$, *there is a* $\rho_1 = \rho_1(\Lambda, a, b)$ *such that for every* $\rho \leq \rho_1$, *we have*

$$\|T(u,v)\| \geq \|(u,v)\|,$$

for $(u,v) \in C \cap \partial\Omega_\rho$.

Proof. By (K1) and the nondecreasing property on f, there is a $k > 0$ such that

$$f(t, u(t) + At, v(t) + Bt) \geq f(t, \alpha_1 A, \alpha_1 B) > k,$$

for $t \in [\alpha_1, \beta_1]_{\mathbb{Z}}$. Take $\rho_1 = \Lambda k \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2}\right]\!\right], s\right)$. Then, for $(u,v) \in C \cap \partial\Omega_\rho$, where $\rho \leq \rho_1$,

$$\|T(u,v)\| \geq \lambda \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2}\right]\!\right], s\right) f(s, \alpha_1 A, \alpha_1 B)$$
$$\geq \lambda k\|(u,v)\| \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2}\right]\!\right], s\right) (\|(u,v)\|)^{-1}$$
$$\geq \|(u,v)\|,$$

□

Now that we have sets that give lower estimates on T, we reorient our focus to finding sets giving the appropriate upper estimates. Although these are slightly more daunting to come by, they follow a pattern similar to the corresponding lemmas in previous chapters. First, we make note of the fact that these estimates require not only the additional assumptions made on f in (K2) and (K3), but also the ability to find appropriate bounds on the function, g.

In the earlier problems involving differential equations, the transformation process gave us variables of the form $u(t) + ta$ within the functions f and g. Now, for those problems, $t \in [0, 1]$, gave the inequality $u(t) + a \leq u(t) + ta$. This, in conjunction with the nondecreasing assumption, played a key role in both of the final proofs. The transformation for this problem, however, gives slightly more complex variables of the form $u(t) + \frac{a}{N+2}t$. Since t is no longer restricted to being between 0 and 1, we cannot drop it as we did previously, to make our inequality work. Fortunately, however, $t \in [0, N+2]_{\mathbb{Z}}$ and hence, $0 \leq \frac{t}{N+2} \leq 1$. So $u(t)+At \leq u(t)+a$, giving us a series of inequalities close to those in previous problems.

Lemma 4.4. *Suppose* (K0) *and* (K2) *hold and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$, *there is a* $\rho_2 \in (0, \rho^*)$ *and a* $\delta > 0$ *such that for every* $(a, b) \in [0, \infty)^2$, *with* $0 < a + b < \delta$, *we have*

$$\|T(u, v)\| \leq \|(u, v)\|,$$

for $(u, v) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Given $\lambda > 0$, pick $\epsilon > 0$ so that $\lambda\epsilon(N + 2)^2 < 4$. Then, by (K2), there is a $\rho_2 \in (0, \rho^*)$ such that for $u + v = \rho_2$ and $a + b \leq \rho_2$, we have

$$f(t, u + a, v + b) \leq \epsilon[(u + a) + (v + b)], \;\; t \in [0, N + 2]_{\mathbb{Z}}.$$

Take $(u, v) \in C \cap \partial\Omega_{\rho_2}$ and suppose $a + b < \rho_2$. Then,

$$\begin{aligned} A_2(u, v)(t) &= \lambda \sum_{s=1}^{N+1} G(t, s) f(s, u(s) + As, v(s) + Bs) \\ &\leq \lambda \sum_{s=1}^{N+1} G(t, s) f(s, u(s) + a, v(s) + b) \\ &\leq \lambda \sum_{s=1}^{N+1} G(t, s) f(s, \|u\|_\infty + a, \|v\|_\infty + b) \end{aligned}$$

$$\leq \lambda\epsilon[\|(u,v)\| + (a+b)] \sum_{s=1}^{N+1} G(t,s)$$

$$\leq 2\lambda\epsilon\|(u,v)\| \sum_{s=1}^{N+1} G(t,s)$$

$$\leq \frac{\lambda\epsilon(N+2)^2}{4}\|(u,v)\|$$

for $t \in [1, N+1]_{\mathbb{Z}}$, giving that

$$\|A_2(u,v)\|_\infty \leq \frac{\lambda\epsilon(N+2)^2}{4}\|(u,v)\|.$$

Next we consider A_1. First note that g is nondecreasing in the final two variables as a result of its projective nature. Moreover, there are both a

$$0 < \gamma < \frac{8}{(N+2)^2}$$

and a $q > 0$ such that for every $(u,v) \in [0,\infty)^2$, with $u + v < q$, we have

$$g(t,u,v) \leq \gamma(u+v),$$

for $t \in [0, N+2]_{\mathbb{Z}}$. Pick $\rho_2 < \frac{1}{2}q$. Then $[(u+a)+(v+b)] < 2\rho_2 < q$. Thus, the above gives that

$$g(t, u+a, v+b) \leq \gamma[(u+a)+(v+b)],$$

for $t \in [0, N+2]_{\mathbb{Z}}$. Let $\delta' < 1$ and set $\delta = \delta'\rho_2$. Take $a+b < \delta$ and $(u,v) \in C \cap \partial\Omega_{\rho_2}$. Then, for $t \in [1, N+1]_{\mathbb{Z}}$, we have

$$A_1(u,v)(t) = \sum_{s=1}^{N+1} G(t,s)g(s, u(s) + As, v(s) + Bs)$$

$$\leq \sum_{s=1}^{N+1} G(t,s)g(s, \|u\|_\infty + a, \|v\|_\infty + b)$$

$$\leq \gamma[\|(u,v)\| + (a+b)] \sum_{s=1}^{N+1} G(t,s)$$

$$\leq \gamma(1+\delta')\|(u,v)\| \sum_{s=1}^{N+1} G(t,s)$$

$$\leq \frac{\gamma(1+\delta')(N+2)^2}{8}\|(u,v)\|.$$

Therefore,

$$\|A_1(u,v)\|_\infty \leq \frac{\gamma(1+\delta')(N+2)^2}{8}\|(u,v)\|.$$

Letting $a+b<\delta$ and $(u,v)\in C\cap\partial\Omega_{\rho_2}$ yields

$$\|T(u,v)\| \leq (N+2)^2\left(\frac{\gamma(1+\delta')+2\lambda\epsilon}{8}\right)\|(u,v)\|.$$

Thus, if we select δ' and ϵ small enough so that $\gamma(1+\delta')+2\lambda\epsilon \leq 8(N+2)^{-2}$, we have our desired result. □

Lemma 4.5. *Let $\delta > 0$ be given and let $0 < a+b < \delta$. In addition, suppose assumptions* (K0) *and* (K3) *hold. Then, for every $\lambda > 0$, there is a $\rho_3 = \rho_3(\delta,\lambda)$ such that for $\rho \geq \rho_3$,*

$$\|T(u,v)\| \leq \|(u,v)\|,$$

where $(u,v)\in C\cap\partial\Omega_\rho$.

Proof. It follows from the construction of g that it is both nondecreasing in the last two variables, and there are both an

$$0<\eta<\frac{8}{(N+2)^2}$$

and a $p>0$ such that for $(u,v)\in[0,\infty)^2$, with $u+v>p$, we have

$$g(t,u,v)\leq \eta(u+v),$$

for $t\in[0,N+2]_{\mathbb{Z}}$. Thus, given any $q_1 \geq p$, where p is defined above, we have

$$g(t,u+a,v+b)\leq \eta[(u+a)+(v+b)]$$

where $u+v\geq q_1$. Let $\epsilon>0$ and pick q_1 large enough so that $\epsilon > \frac{\eta\delta}{q_1}$. Then

$$\begin{aligned} g(t,u+a,v+b) &\leq \eta(u+v)+\eta(a+b)\\ &\leq \eta(u+v)+\epsilon(u+v)\\ &= (\eta+\epsilon)(u+v), \end{aligned}$$

for $t \in [0, N+2]_{\mathbb{Z}}$. Therefore, for any $(u,v) \in C \cap \partial\Omega_{q_1}$, we have

$$\begin{aligned}
A_1(u,v)(t) &= \sum_{s=1}^{N+1} G(t,s)g(s, u(s) + As, v(s) + Bs) \\
&\le \sum_{s=1}^{N+1} G(t,s)g(s, u(s) + a, v(s) + b) \\
&\le \sum_{s=1}^{N+1} G(t,s)g(s, \|u\|_\infty + a, \|v\|_\infty + b) \\
&\le (\eta + \epsilon)\|(u,v)\| \sum_{s=1}^{N+1} G(t,s) \\
&\le \frac{(\eta+\epsilon)(N+2)^2}{8}\|(u,v)\|,
\end{aligned}$$

for $t \in [1, N+1]_{\mathbb{Z}}$, yielding

$$\|A_1(u,v)\|_\infty \le \frac{(\eta+\epsilon)(N+2)^2}{8}\|(u,v)\|.$$

Next we consider $A_2(u,v)$. Let $\delta' > 0$. By (K0) and (K3), there is a $q_2 > 0$ such that for every $(u,v) \in [0,\infty)^2$ with $u + v \ge q_2$,

$$f(t, u + At, v + Bt) \le f(t, u + a, v + b) \le \delta'[(u + a) + (v + b)].$$

Let $q_3 = \max\{\delta, q_2\}$ and recall $a + b < \delta$. Then for every $(u,v) \in [0,\infty)^2$, with $u + v \ge q_3$, we have

$$f(t, u + ta, v + tb) \le \delta'(u + v) + \delta'\delta \le 2\delta'(u + v),$$

for $t \in [0, N+2]_{\mathbb{Z}}$. It then follows that

$$\|A_2(u,v)\|_\infty \le \frac{\delta'\lambda(N+2)^2}{4}\|(u,v)\|$$

for $(u,v) \in C \cap \partial\Omega_{q_3}$. Therefore,

$$\|T(u,v)\| \le (N+2)^2 \left(\frac{\eta + \epsilon + 2\delta'\lambda}{8}\right)\|(u,v)\|.$$

Picking ϵ and δ' so that $\epsilon + 2\delta'\lambda \le 8(N+2)^{-2} - \eta$, gives the desired result.

$\square$

4.3 An Existence Result

Now that we have seen that the transformation process we used on various differential equations can be modified to produce a series of lemmas which give significant contraction and expansion estimates on T, we are ready to present an existence result for the second order system (4.10)-(4.13). Fortunately, since the Lemmas in Section 4.2 have conclusions analogous to the problems of the previous chapters, the statement and proof of the main result is virtually identical. As such, all that remains to be shown is how the four lemmas work in unison to allow the triple application of the Guo-Krasnosel'skii Fixed Point Theorem.

Theorem 4.1. *Let f satisfy* (K0)-(K3)*. Then there exists a $\Lambda > 0$ such that, for any $\lambda \geq \Lambda$, there is a $\delta > 0$ such that, for every $a, b \geq 0$ with $0 < a + b < \delta$, the system* (4.10)-(4.13) *has at least three positive solutions.*

Proof. Suppose f satisfies hypotheses (K0)-(K3) and fix $\rho^* > 0$. By Lemma 4.2, there is a $\Lambda > 0$ such that for every $\lambda \geq \Lambda$ and $a, b \geq 0$,

$$\|T(u,v)\| \geq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho^*}.$$

Now, fix $\lambda \geq \Lambda$. Lemmas 4.3-4.5 give that there is a $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$, with $\rho_1 < \rho_2 < \rho^* < \rho_3$, such that for $(a, b) \in [0, \infty)^2$, satisfying $0 < a + b < \delta$, we have

$$\|T(u,v)\| \geq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_1},$$

$$\|T(u,v)\| \leq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_2},$$

$$\|T(u,v)\| \leq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_3}.$$

Therefore, by appealing to the Guo-Krasnosel'skii Fixed Point Theorem, there exist three positive solutions, $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in C$ of (4.10)-(4.13) such that,

$$\rho_1 < \|(u_1, v_1)\| < \rho_2 < \|(u_2, v_2)\| < \rho^* < \|(u_3, v_3)\| < \rho_3.$$

□

CHAPTER FIVE

Future Work

The most elegant portion of this work does not lie in the existence results themselves, but in the ease of how the transformation method can by modified and adapted to work for such a variety of problems. This leaves one to wonder how far this technique can actually go. Some of the more obvious questions include extending the problem visited in Chapter 4 to the general case. Since we have seen that we can mold the method for analogous difference and differential equations satisfying conjugate boundary conditions, another natural question would be to see how the technique works on an arbitrary time scale.

Notice that the differential equation,

$$u^{(2n)} = \lambda h\left(t, u, u'' \ldots, u^{(2(n-1))}\right),\ t \in (0,1),\ n \geq 2, \tag{5.1}$$

visited in Chapters 2 and 3 only contains even derivatives, but must this necessarily be the case? If you take a further look at (5.1) under right focal boundary conditions, it seems that the addition of odd order derivatives is highly plausible. There would be a modification in the first hypothesis to include a nonincreasing property and both the transformation process and constructing the proper estimates would be more involved.

Throughout this work, we were developing the transformation technique and thus, we worked with less complicated boundary conditions to gain better insight into the actual method. Now that we have a good feel for how the transformation process works, another natural step would be to investigate more complicated boundary conditions. Furthermore, there is nothing restricting us to using the sup norm to find our estimates. Another interesting notion to consider is how the transformation process works with the construction of estimates on our operator under different norms. These are a few examples of the future direction of this work.

BIBLIOGRAPHY

[1] R. Agarwal, On fourth order boundary value problems arising in beam analysis, *Differ. Integral Equ.* **2** (1989), 91-110.

[2] R. Agarwal and D. O'Regan, Triple solutions to boundary value problems on time scales, *Appl. Math. Lett.* **13** (2005), 7-11.

[3] R. Avery, J. Henderson and D. O'Regan, Six Functionals Fixed Point Theorem, *Comm. Appl. Anal.* **12** (2008) 69-82.

[4] F. Bernis, On some nonlinear singular boundary value problems of higher order, *Nonlin. Anal.* **26** (1996), 1061-1078.

[5] E. Cetin and S. G. Topal, Higher order boundary value problems on time scales, *J. Math. Anal. App.* **334** (2007), 876-888.

[6] C. J. Chyan, J. Henderson and H. C. Lo, Existence of triple solutions for a class of boundary value problems, *Panamer. Math. J.* **11** 53-60, 2001.

[7] C. J. Chyan and J. Henderson, Existence of triple solutions of higher order boundary value problems, *Comm. Appl. Nonlin. Anal.* **6** (1997), 37-47.

[8] C. J. Chyan and P. J. Y. Wong. Multiple positive solutions of conjugate boundary value problems on time scales, *Taiwanese J. Math.* **11** (2007), 421-445.

[9] D. R. Dunninger and H. Wang, Existence and multiplicity of positive solutions for elliptic systems, *Nonlin. Anal.* **29** (1997), 1051-1060.

[10] D. R. Dunninger, Multiplicity of positive solutions for a nonlinear fourth order equation, *Ann. Polon. Math.* **77** (2001), 161-168.

[11] Z. Du, C. Xue, and W. Ge, Triple solutions for a higher-order difference equation, *JIPAM. J. Inequal. Pure Appl. Math.* **6** (2005), 11.

[12] J. Ehme and A. Lanz, Uniqueness and existence for $2n$th order Lidstone boundary value problems, *Panamer. Math. J.* **17** (2007), 75-81.

[13] D. G. Figueiredo, Positive solutions of semilinear elliptic problems. Differential equations (S ao Paulo, 1981), pp. 34-87, Lecture Notes in Math., 957, Springer, Berlin-New York, 1982.

[14] D. D. Hai, Positive solutions for semilinear elliptic equations in annular domains, *Nonlin. Anal.* **37** (1999), 1051-1058.

[15] J. Henderson and H. Wang, Positive solutions for nonlinear eigenvalue problems, *J. Math. Anal. Appl.* **208** (1997),252-259.

[16] L. Hu and L. Wang, Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations, *J. Math. Anal. Appl.* **335** 2007, 1052-1060.

[17] W. Kelley and A. Peterson, *Difference Equations: an Introduction with Applications,* Academic Press, San Diego, 2001.

[18] C. B. Kim and Y. Lee, Existence and mulitiplicity results for nonlinear boundary value problems, *Comp. Math. Appl.* **55** (2008), 2870-2886.

[19] M. G. Lee and S. S. Lin, On the positive solutions for semilinear elliptic equations on annular domain with nonhomogeneous Dirichlet boundary condition, *J. Math. Anal. Appl.* **181** (1994), 348-361.

[20] C. Li, L. Liu and Y. Wu, Multiple positive solutions for nonlinear second order m-point boundary value problems, *J. Nonlin. Funct. Anal. Differ. Equ.* **1** (2007), 121-135.

[21] X. Liu, W. Jiang and Y. Guo, Multi-point boundary value problems for higher order differential equations, *Appl. Math. E-Notes*, **4** (2004), 106-113.

[22] Y Liu, Multiple positive solutions of nonlinear singular boundary value problem for fourth-order equations, *Appl. Math. Lett.* **17** (2004), 747-757.

[23] R. Ma, Existence of positive solutions of a fourth-order boundary value problem, *Appl. Math. Comput.* **168** (2005), 1219-1231.

[24] Y. Ma, Existence of positive solutions of Lidstone boundary value problems, *J. Math. Anal. Appl.* **314** (2006), 97-108.

[25] J. Marcos, S. Lorca and P. Ubilla, Local superlinearity for elliptic systems involving parameters, *J. Differential Equations* **211** (2005), 1-19.

[26] J. Marcos, S. Lorca and P. Ubilla, Multiplicity of solutions for a class of nonhomogeneous fourth-order boundary value problems, *Appl. Math. Lett.* **21** (2008), 279-186.

[27] C. Song and P. Weng, Multiple positive solutions for p-Laplacian functional dynamic equations on time scales, *Nonlin. Anal.* **68** (2008), 208-215.

[28] B. Sun, W. Ge and D. Zhao, Existence of multiple positive solutions for a class of p-Laplacian boundary value problem with dependence on the first order derivative, *Far East J. Appl. Math.* **29** (2007), 273-282.

[29] H. R. Sun and W. T. Li, Multiple positive solutions for even order Sturm-Liouville boundary value problems, *Acta Math. Sci. Ser. A Chin. Ed.* **26** (2006), 700-706.

[30] Y. Wang and W. Ge, Existence of triple positive solutions for multi-point boundary value problems with a one dimensional p-Laplacian, *Comput. Math. Appl.* **54** (2007), 793-807.

[31] Y. Wang and W. Ge, Multiple positive solutions for a nonlinear even-order m-point boundary value problem. II, *Tamsui Oxf. J. Math. Sci.* **22** (2006), 143-158.

[32] Y. Wang, On $2n$th-order Lidstone boundary value problems, *J. Math. Anal. Appl.* **312** (2005), 383-400.

[33] J. R.L. Webb, Fixed point index and its application to positive solutions of nonlocal boundary value problems, *Seminar of Mathematical Analysis*, Univ. Sevilla Secr. Publ., Seville, (2006), 181-205.

[34] Y. Xu, L. Li and L. Debnath. A necessary and sufficient condition for the existence of positive solutions of singular boundary value problems, *Appl. Math. Lett.* **18** (2005), 881-889.

[35] Y. R. Yang and S. S. Cheng, Positive solutions of a Lidstone boundary value problem with variable coefficient function, *J. Appl. Math. Comput.* **27** (2008), 411-419.

[36] M. Y. Zhang, Multiple positive solutions of boundary value problems for a class of functional differential equations, *Gongcheng Shuxue Xuebao* **24** (2007), 735-749.

[37] D. Zhao, H. Wang and W. Ge. Existence of triple positive solutions to a class of p-Laplacian boundary value problems, *J. Math. Anal. Appl.* **328** (2007)972-983.