ABSTRACT

Sharkovskii's Theorem Under Set-Valued Functions

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Sharkovskii's remarkable theorem from 1964 demonstrated significant results about periodic orbits of continuous functions on the real line. His work produced the Sharkovskii ordering, which is shown below. If $m \gg n$ in the Sharkovskii ordering and if f has a periodic orbit of period m, it must also have a periodic point of period n. While Sharkovskii worked with classical continuous functions, this paper expands Sharkovskii's theorem to a class of set-valued functions. In particular, we show that the ordering holds for upper semicontinuous set-valued functions with the strong intermediate value property.

The Sharkovskii ordering:

$$3 \gg 5 \gg 7 \gg ... \gg 3 \cdot 2^1 \gg 5 \cdot 2^1 \gg 7 \cdot 2^1 \gg ... \gg 3 \cdot 2^2 \gg 5 \cdot 2^2 \gg 7 \cdot 2^2 \gg ... \gg$$
$$... \gg 2^3 \gg 2^2 \gg 2 \gg 1$$

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SHARKOVSKII'S THEOREM UNDER SET-VALUED FUNCTIONS

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Contents

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INTRODUCTION

The Sharkovskii ordering:

$$3 \gg 5 \gg 7 \gg ... \gg 3 \cdot 2^1 \gg 5 \cdot 2^1 \gg 7 \cdot 2^1 \gg ... \gg 3 \cdot 2^2 \gg 5 \cdot 2^2 \gg 7 \cdot 2^2 \gg ... \gg$$

$$... \gg 2^3 \gg 2^2 \gg 2 \gg 1$$

This ordering of the positive integers is the result of work by Oleksandr Sharkovskii from 1964. In order to clarify the statement of the ordering, we define several terms. First, an *orbit of* f is a sequence of points $x_o, x_1, x_2, ...$ such that $x_{i+1} = f(x_i)$ for each i. It is a *periodic orbit with period* k if and only if $x_i = x_{i+k}$ for every i or, equivalently, $f^k(x_i) = x_i$. And, finally, k is a *prime* period if k is the least such integer. Each number in the ordering represents the period of an orbit on a classical continuous function $f : \mathbb{R} \to \mathbb{R}$. Then the Sharkovskii ordering means that if f has a periodic orbit of prime period m where $m \gg n$, then that same f also has a periodic orbit of prime period n.

This ordering is quite strong for two main reasons. First, the relation $\gg$ is transitive, so that if $m \gg n \gg p$, then $m \gg p$. For example, as 3 is the first ("greatest") number in the ordering, the existence of an orbit of period 3 implies the existence of orbits of all other prime periods. The other reason the ordering is strong is because it is sharp; there are maps that have orbits of prime period m but have no orbits of prime period n for $n \gg m$.

As mentioned, Sharkovskii wrote his paper original paper proving this result in 1964 [4]. Since Sharkovskii wrote in Russian, it was later kindly translated into English in 1995 [5]. However, this was not before Block, Guckenheimer, Misiurewicz,

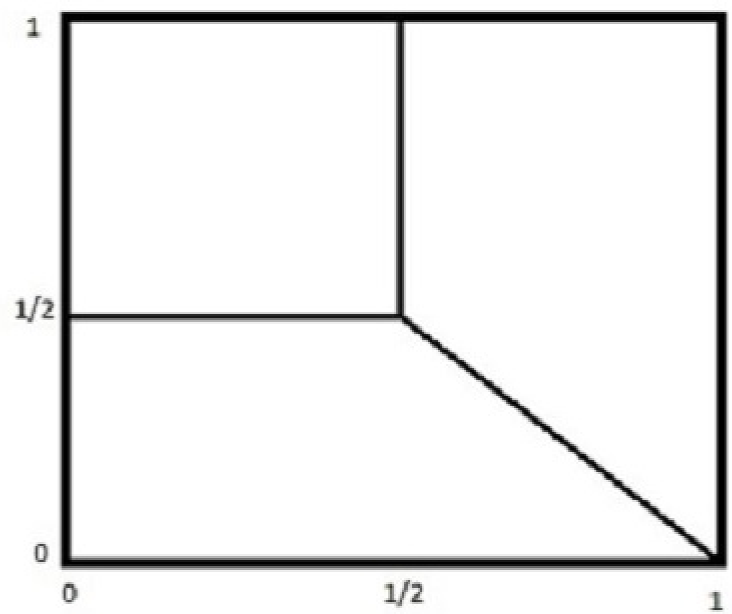

Figure 1: *A set-valued map with the standard intermediate value property and orbits with periods 1 and 3, but no orbit with period 2*

and Young gave a simpler and more conceptual proof in 1980 [2]. Later that same decade, Devaney included this proof in his book *An Introduction to Chaotic Dynamical Systems* [3]. The presentation of the proof in Devaney's book is the template for the extension of Sharkovskii's Theorem given in this thesis.

We extend Sharkovskii's theorem to a certain class of set-valued functions which we will specify below. A set-valued function $f : X \to Y$ differs from a classical function in principle so far as for $x \in X$, $f(x)$ may be any subset of points of Y rather than just a singular point. This expansion requires the modification of certain definitions pertaining to classical functions. For example, we no longer say $y = f(x)$ but rather $y \in f(x)$.

The set-valued map in Figure 1 has an orbit of prime period 3 (namely 0,1/2,1,0,...)

but no orbit of prime period 2. This is just one example that demonstrates that Sharkovskii's theorem does not hold for all set-valued functions. Accordingly, we specify a class of set-valued functions for which the theorem is true by applying two assumptions.

For a set-valued map f, we first assume that f is upper semicontinuous. Second, in order to resolve issues like the one presented in Figure 1, we require that f has the strong intermediate value property (strong IVP). Later in the paper, we show that these two properties imply two other important properties: connectivity and weak continuity.

In Chapter Two of this thesis we provide proof that, for a classical continuous function, the existence an orbit of prime period 3 implies the existence of orbits all other prime periods. In Chapter Three, we transition to set-valued functions. We detail several definitions, lemmas, and theorems pertaining to upper semicontinuous set-valued functions with the strong IVP. Finally, we show that the Sharkovskii ordering prevails on this class of set-valued functions. This is the main result of this thesis and is provided in Chapter Four.

CHAPTER TWO

Sharkovskii under Traditional Functions

This chapter provides the portion of the full proof given by Devaney [3] that shows that, on classical continuous functions, the existence of a prime period 3 orbit implies the existence of orbits of all other prime periods. Many of the definitions and ideas covered later in the set-valued setting have a simpler analog in the classical setting, so we introduce the ideas from the proof in the classical setting first. Accordingly, in this chapter we suppose f is a classical function (i.e. $f(x)$ is a single element of $\mathbb{R}$ for each x).

Definitions. Let $f : \mathbb{R} \to \mathbb{R}$. We define an *orbit of f* to be a sequence of points $x_o, x_1, x_2, ...$ such that $x_i \in \mathbb{R}$ and $x_{i+1} = f(x_i)$ for each i.

We say an orbit is a *periodic orbit with period k* if and only if $x_i = x_{i+k}$ for all $i \in \mathbb{N}$.

We say the period k of an orbit is *prime* if and only if k is the least such integer such that $x_i = x_{i+k}$ for each $i \in \mathbb{N}$.

We say that x_o is a *fixed point* if and only if under f, x_o is in an orbit with period 1, so that $f(x_o) = x_o$.

Notation. We denote the n-fold composition of f with itself by f^n. That is to say, $x_n = f^n(x_o)$ if and only if there are points $x_1, ..., x_{n-1}$ such that $x_i = f(x_{i-1})$ for all $i = 1, ..., n$.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ and let A and B be intervals. We say that A *f-covers* B if $B \subseteq f[A]$ and A f^n*-covers* B if $B \subseteq f^n[A]$.

Theorem 1. *Suppose f is a continuous map such that $f[a, b] \subseteq [a, b]$. Then f has a fixed point $x \in [a, b]$.*

Proof. Consider $g(x) = f(x) - x$. We have that $a = f(x_a)$ and $b = f(x_b)$ for some $x_a, x_b \in [a, b]$. Then $g(x_a) = a - x_a \leq 0$ and, likewise, $g(x_b) = b - x_b \geq 0$. By the intermediate value property, there exists an x_c between x_a and x_b where $0 = g(x_c)$. Thus $0 = f(x_c) - x_c$ and $x_c = f(x_c)$, so x_c is a fixed point of f. ■

Remark. It is a well-known fact that a map that is a composition of continuous functions is also continuous. It follows that the above theorem holds for compositions of continuous functions as well.

Lemma 1. *Suppose f is a continuous function whose domain contains an interval A and suppose A f-covers A. Then, for each $n \in \mathbb{N}$, there is a nested sequence of subintervals $A_n \subset A_{n-1} \subset ... \subset A_1 \subset A$ such that $f[A_i] = A_{i-1}$ and that $f^i[A_i] = A$ for all i.*

Proof. This is a proof by induction. Suppose f is a function whose domain contains an interval A and suppose A f-covers A. Then there is a subinterval $A_1 \subseteq A$ such that $f[A_1] = A$. Then $A_1 \subseteq f[A_1]$. Now assume A_i f-covers A_i so that, by definition, $A_i \subseteq f[A_i]$. It follows that there is a subinterval $A_{i+1} \subseteq A_i$ such that $f[A_{i+1}] = A_i$. We note that $f^i[A_i] = A$. Thus, by induction, there are intervals $A_1, ..., A_n$ that satisfy the conclusion of the lemma. ■

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic orbit of prime period 3, then f has a periodic orbit of all prime periods.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ and let $a < b < c \in \mathbb{R}$ such that $f(a) = b$, $f(b) = c$, and $f(c) = a$. The other case, in which $f(a) = c, f(c) = b, f(b) = a$, is similar. We set $I = [a, b]$ and $J = [b, c]$. We note that I f-covers J and J f-covers $I \cup J$, which we are assured by the continuity of f. Since $J \subset f(J)$, f has a fixed point by Theorem 1. Further, since J f-covers I, then there is subinterval $J' \subseteq J$ such that $f[J'] = I$. There is also $I' \subset I$ such that $f[I'] = J'$. Hence $[J'] \subseteq f^2[J']$. By Theorem 1, there is $x \in J'$ such that $f^2(x) = x$. Then there is $y \in I'$ such that $y = f(x)$ and $x = f(y)$. Thus $x, y, x, y, ...$ is a periodic orbit of prime period 2. We know that 2 is the prime period (i.e. $x \neq y$) because the only point in $I \cap J$ is b, which has period 3. Thus f has periodic orbits of periods 1, 2, and 3.

Fix $n \in \mathbb{N}$ so that $n > 3$. Let $A_n = J$. Then let $A_{n-1} \subseteq J$ be such that $f[A_{n-1}] = A_n$. We know that such an interval exists as J f-covers J. Continuing similarly and applying Lemma 1, we know that we may find subintervals $A_{n-2}, ..., A_2$ such that $f[A_{i-1}] = A_i$ and $f^{n-i}[A_i] = A_n$ for each i.

Now, since I f-covers J, then there is a subinterval $A_1 \subset I$ such that $f[A_1] = A_2$ and $f^{n-1}[A_1] = A_n = J$. Finally, as J f-covers I, there is a subinterval $A_o \subset J$ such that $f[A_o] = A_1$ and $f^n[A_o] = A_n$. Then, applying Theorem 1 to $f^n|_{[A_o]}$ there is a point $x_o \in A_o$ such that $f^n(x_o) = x_o$. It follows that there are $x_1, ..., x_{n-1}$ such that $x_i = f(x_{i-1})$ for each $i = 1, ..., n-1$ and $x_o = f(x_{n-1})$. Thus $x_o, x_1, ..., x_{n-1}, x_o, ...$ is a periodic orbit with period n. To see that n is prime, notice that x_1 is in I and

$x_o, x_2, ..., x_{n-1}$ are in J. Suppose x_1 is also in J. Then $x_1 = b$ and all points of the orbit are in J. We would also have that $x_o, x_1, x_2, x_3, ...$ is simply $a, b, c, a, ...,$ but $a \notin J$. A contradiction. Thus x_1 is not in J and, since all of the other points are in J, it follows that the period n of the orbit is prime. Then f has a periodic orbit of all prime periods. ■

CHAPTER THREE

Remarks on Set-Valued Functions

In this chapter, all functions f are presumed to be set-valued functions. As mentioned in the introduction, we are primarily interested in results about functions that are upper semicontinuous and have the strong intermediate value property. The groundwork laid here will be instrumental in demonstrating the results in Chapter Four.

Notation. The collection of all non-empty closed subsets of $[a, b]$ is denoted by $2^{[a,b]}$.

Definition. Let $f : [a, b] \to 2^{[c,d]}$. We say that f is *surjective* if for each $y \in [c, d]$ there is an $x \in [a, b]$ such that $y \in f(x)$.

Definition. A function $f : [a, b] \to 2^{[c,d]}$ is said to have the *intermediate value property* (IVP) if and only if, given $y_1 \in f(x_1)$, $y_2 \in f(x_2)$, and y between y_1 and y_2, there is an $x \in [x_1, x_2]$ such that $y \in f(x)$.

Definition. A function $f : [a, b] \to 2^{[c,d]}$ is said to have the *strong intermediate value property* (strong IVP) if and only if, given distinct points $y_1 \in f(x_1)$ and $y_2 \in f(x_2)$ and given y strictly between y_1 and y_2, there is $x \in (x_1, x_2)$ such that $y \in f(x)$.

Definition. We say that a sequence of ordered pairs (x_n, y_n) *converges to* (x, y) provided that the sequence x_n converges to x and the sequence y_n converges to y.

Definition. The function $f : [a, b] \to 2^{[c,d]}$ is said to be *upper semicontinuous* if and only if for every sequence $\{(x_n, y_n)\}$ where 1) $y_n \in f(x_n)$ for all $n \in \mathbb{N}$ and 2)

$\{(x_n, y_n)\}$ converges to (x, y), then $y \in f(x)$.

Remark. For an upper semicontinuous function f, the graph of f is closed where the graph of f is defined to be $\{(x, y) : y \in f(x)\}$. It follows that $f(x)$ is closed for each x.

Definitions. The function $f : [a, b] \to 2^{[c,d]}$ is said to be *weakly continuous from the left at* x if and only if it is upper semicontinuous and, for each $y \in f(x)$, there is a sequence $\{(x_n, y_n)\}$ that converges to (x, y) such that $x_n < x$ and $y_n \in f(x_n)$ for each n.

The function $f : [a, b] \to 2^{[c,d]}$ is said to be *weakly continuous from the right at* x if and only if it is upper semicontinuous and, for each $y \in f(x)$, there is a sequence $\{(x_n, y_n)\}$ that converges to (x, y) such that $x_n > x$ and $y_n \in f(x_n)$ for each n.

The function $f : [a, b] \to 2^{[c,d]}$ is said to be *weakly continuous at* x if and only if it is weakly continuous from both the right and the left at x. And, lastly, the function f is said to be *weakly continuous* if and only if it is weakly continuous at x for each $x \in (a, b)$.

Definitions. Let $f : [a, b] \to 2^{[c,d]}$ be an upper semicontinuous map. Suppose $x_o \in [a, b]$. Then the *limit superior of* f *as* x *goes to* x_o *from the right*, denoted $\limsup_{x \to x_o^+} f(x)$, is given by $\limsup_{x \to x_o^+} f(x) = \big\{y \in [c, d] : \exists \{(x_n, y_n)\}$ such that $x_n > x\ \forall n, y_n \in f(x_n)\ \forall n$, and $\{(x_n, y_n)\}$ converges to $(x_o, y)\big\}$.

Then the *limit superior of* f *as* x *goes to* x_o *from the left*, denoted $\limsup_{x \to x_o^-} f(x)$, is given by $\limsup_{x \to x_o^-} f(x) = \big\{y \in [c, d] : \exists \{(x_n, y_n)\}$ such that $x_n < x\ \forall n, y_n \in f(x_n)\ \forall n$, and $\{(x_n, y_n)\}$ converges to $(x_o, y)\big\}$.

Then the *limit superior of f as x goes to x_o*, denoted $\limsup_{x\to x_o} f(x)$, is given by $\limsup_{x\to x_o} f(x) = \{y \in [c,d] : \exists\{(x_n, y_n)\}$ such that $y_n \in f(x_n)$ $\forall n$ and $\{(x_n, y_n)\}$ converges to $(x_o, y)\}$. Equivalently, $\limsup_{x\to x_o} f(x) = \limsup_{x\to x_o^+} f(x) \cup \limsup_{x\to x_o^-} f(x)$.

Definition. We say a subset $A \subseteq \mathbb{R}$ is *connected* provided that, given $x, y \in A$ where $x < y$, then $z \in (x, y)$ is also in the set A. Equivalently, A is connected if and only if it is an interval.

Notation. We use the notation $f(x) < y$ to mean that for all $w \in f(x)$, $w < y$.

Theorem 3. *Suppose $f : [a,b] \to 2^{[c,d]}$ is an upper semicontinuous map. Then f has the intermediate value property (IVP) if and only if $f(x)$ is connected for each $x \in [a,b]$.*

Proof. Let $f : [a,b] \to 2^{[c,d]}$ be an upper semicontinuous map.

We first prove that the IVP implies connectivity by the contrapositive. Accordingly, we assume that, for some $x \in [a,b]$, $f(x)$ is not connected. Then we have $y_1, y_2 \in f(x)$ where $y_1 \neq y_2$ such that $(y_1, y_2) \cap f(x) = \emptyset$. Then for $x \in [x,x]$, we have $(y_1 + y_2)/2 \notin f(x)$. Then f does not have the intermediate value property. Thus, if f has the IVP, then $f(x)$ is connected for all $x \in [a,b]$.

We now assume that $f(x)$ is connected for all $x \in [a,b]$, but that f does not have the intermediate value property. Then there exist distinct points $x_1 < x_2 \in [a,b]$ with $y_1 \in f(x_1)$, $y_2 \in f(x_2)$ and y between y_1 and y_2 such that $y \notin f(x)$ for every $x \in [x_1, x_2]$. We suppose $y_1 < y_2$ where the case $y_2 > y_1$ is similar. We note that, by the connectivity of $f(x)$, for all $x \in [x_1, x_2]$, either $f(x) > y$ or $f(x) < y$. Consider the following set: $B = \{x \in [x_1, x_2]\colon\ f(z) < y \text{ for all } z \in [x_1, x]\}$. Note that B is not empty as $x_1 \in B$. Consider $x_s = \sup B$. We will show that $x_s \in B$. Due to the upper semicontinuity of f, if $w \in f(x_s)$, then $w \le y$. Yet $y \notin f(x_s)$ and $f(x_s)$ is connected, so $f(x_s) < y$. Thus x_s is in B. Then for all $w \in f(x_s)$, $w < y$. Then $\limsup_{x \to x_s^-} f(x) \le y$. Since $x_s = \sup B$, then for each $n \in N$ there exists $x_n \in (x_s, x_s+1/n)$ such that $w > y$ for some $w \in f(x_n)$. It follows that $f(x_n) > y$ for each n. Then $\limsup_{x \to x_s^+} f(x) \ge y$. Therefore $y \in \limsup_{x \to x_s} f(x) \subset f(x_s)$. A contradiction. Therefore if $f(x)$ is connected for every $x \in [a, b]$, it follows that f has the intermediate value property.

■

Theorem 4. *Let $f : [a, b] \to 2^{[c,d]}$. Then f has the strong intermediate value property if and only if f is weakly continuous and $f(x)$ is connected for each $x \in [a, b]$.*

Proof. We first assume that f has the strong intermediate value property. Then by Theorem 3, we know that $f(x)$ is connected for each $x \in [a, b]$ and we need only show that f is weakly continuous. To see that f is weakly continuous from the right, let $x \in (a, b)$ and $y \in f(x)$. First, note that if there exists a sequence $\{x_n\}$ converging to x such that there is some N for which $n \ge N$ implies that $y \in f(x_n)$, then the sequence $\{(x_n, y)\}$ satisfies the criteria of the definition of weak continuity from the right. Now suppose no such sequence exists and fix $n \in \mathbb{N}$. It may be true that there exists $w \in f(x + 1/n)$ such that $w \in [y - 1/n, y + 1/n]$, in which case we let $x_n = x + 1/n$ and $y_n = w$. If not, then for all $u \in f(x + 1/n)$, $u \notin [y - 1/n, y + 1/n]$. Since $f(x + 1/n)$ is connected, either $f(x + 1/n) > y + 1/n$ or $f(x + 1/n) < y + 1/n$. The two cases are similar; we assume the former. By the strong IVP, there exists some $x_n \in (x, x+1/n)$ such that $(y+1/n) \in f(x_n)$. Whichever method is used to determine x_n for each $n \in N$, we have that $x_n \in (x, x+1/n]$ and $y_n \in [y-1/n, y+1/n]$ and $y_n \in f(x_n)$. Therefore, since for all n we have chosen $x_n > x$ and $y_n \in f(x_n)$ so that $\{(x_n, y_n)\}$ converges to (x, y), we can conclude that f is weakly continuous from the right. By a similar argument, f is weakly continuous from the left and thus f is weakly continuous at x.

We now assume $f(x)$ is connected for each $x \in [a, b]$ and that f is weakly continuous. We wish to show that f has the strong IVP. From Theorem 3, we know that f has the standard IVP. We let $x_1 < x_2$, $y_1 \in f(x_1)$, and $y_2 \in f(x_2)$ where $y_1 \neq y_2$. We assume $y_1 < y_2$ (the other case is similar) and that $y \in (y_1, y_2)$. Take $\epsilon > 0$ such that $\epsilon < min\big\{|y-y_1|, |y-y_2|, 1/2|x_1-x_2|\big\}$. By weak continuity, there exist $(\tilde{x_1}, \tilde{y_1})$ and $(\tilde{x_2}, \tilde{y_2})$ such that $\tilde{x_1} \in (x_1, x_1+\epsilon)$, $\tilde{x_2} \in (x_2-\epsilon, x_2)$ and $\tilde{y_1} \in (y_1-\epsilon, y_1+\epsilon)$, $\tilde{y_2} \in (y_2-\epsilon, y_2+\epsilon)$ and such that $\tilde{y_1} \in f(\tilde{x_1})$, $\tilde{y_2} \in f(\tilde{x_2})$. Then $\tilde{y_1} < y < \tilde{y_2}$ and $\tilde{x_1} < \tilde{x_2}$. By the IVP, there exists $x \in [\tilde{x_1}, \tilde{x_2}]$ such that $y \in f(x)$. Note that $x \in (x_1, x_2)$ and that therefore f has the strong IVP. ■

Definition. Suppose $f : [a, b] \to 2^{[c,d]}$ is an upper semicontinuous map with the intermediate value property, and suppose I and J are closed subintervals of $[a, b]$ and $[c, d]$ respectively. Then f is said to have an I, J *restriction* if and only if there is a surjective, upper semicontinuous map $f|_I^J : I \to 2^J$ with the intermediate value property such that $f|_I^J(x) = f(x) \cap J$ for each $x \in I$.

Theorem 5. *Suppose* $f : [a, b] \to 2^{[c,d]}$ *is a surjective, upper semicontinuous map with the intermediate value property, and suppose* I *and* J *are subintervals of* $[a, b]$ *and*

$[c,d]$ respectively such that $J \subset f[I]$. Then f has an I, J restriction if and only if $f(x) \cap J \neq \emptyset$ for each $x \in I$.

Proof. We first assume that f has a well-defined, non-empty I, J restriction $f|_I^J$. Then $f|_I^J(x) = f(x) \cap J$ where $f|_I^J$ is a surjective, upper semicontinuous map with the intermediate value property. Since $f|_I^J$ is defined on its whole domain I, it follows that $f|_I^J(x) \neq \emptyset$, and hence that $f(x) \cap J \neq \emptyset$ for each $x \in I$.

Now, we assume only that $f(x) \cap J \neq \emptyset$ for each $x \in I$. We set $f|_I^J(x) = f(x) \cap J$ for every $x \in I$. Since $J \subseteq f[I]$, then $f|_I^J$ is surjective. Note that $f|_I^J$ inherits the IVP from f and, since I and J are closed, upper semicontinuity as well. Then f has an I, J restriction $f|_I^J$. ■

Theorem 6. *Suppose $f : [a,b] \to 2^{[c,d]}$ is an upper semicontinuous map with the intermediate value property, and suppose I and J are subintervals of $[a,b]$ and $[c,d]$, respectively, such that $J \subset f[I]$. Then there is a subinterval $\tilde{I} \subseteq I$ such that f has an $\tilde{I}, J$ restriction.*

Proof. Suppose $f : [a,b] \to 2^{[c,d]}$ is an upper semicontinuous map with the intermediate value property, and suppose I and J are subintervals of $[a,b]$ and $[c,d]$ respectively such that $J \subset f[I]$. Here we denote the maximum and minimum of J as J_h and J_l, respectively. We note that if $J_h, J_l \in f(x)$ for any $x \in I$, then we set $\tilde{I} = \{x\}$ and we are done. Rather suppose for all $x \in I$, $f(x) \cap \{J_l, J_h\}$ does not contain both J_l and J_h. Put $\tilde{p} = \inf\{x \in [a,b] : f(x) \cap \{J_l, J_h\} \neq \emptyset\}$. Then either $J_l \in f(\tilde{p})$ or $J_h \in f(\tilde{p})$. We assume the former, while the case for the latter is similar. Put $p = \sup\{x \in [a,b] : f(z) < J_h \ \forall z \leq x, \ \ J_l \in f(x)\}$. Then put $q = \inf\{x \in [a,b] : J_h \in f(x)\}$. Note that, since f is upper semicontinuous, $J_l \in f(p)$ and $J_h \in f(q)$. Lastly, put $\tilde{I} = [p,q]$ and consider x in the interior of $\tilde{I}$. We see that $f(x) < J_h$ by our selection of q. To see that $f(x) > J_l$, we can suppose to contrary that $w \leq J_l$ for some $w \in f(x)$ and see that by the intermediate value property, there is some $c \in (x,q]$ with $J_l \in f(c)$. This is a contradiction to our choice of p. Likewise, if $J_l \in f(x)$, this contradicts our choice of p. Then $f(x) > J_l$. Consequently, $J_l < f(x) < J_h$. Recall that $J_l \in f(p)$ and $J_h \in f(q)$. Thus we are assured that $f|_{\tilde{I}}^{J} = f(x) \cap J \neq \emptyset$. Then f has an $\tilde{I}, J$ restriction by Theorem 4.

■

We now bring several definitions and theorems from the last chapter to the set-valued setting.

Definitions. For set-valued map f, we now say that an *orbit of* f to be a sequence of points $x_o, x_1, x_2, ...$ such that $x_{i+1} \in f(x_i)$ for each i.

The definitions of a *periodic orbit with period* k and *prime period* k remain the same as before:

We say an orbit is a *periodic orbit with period* k if and only if $x_i = x_{i+k}$ for all $i \in \mathbb{N}$.

We say the period of an orbit k is *prime* if and only if k is the least such integer such that $x_i = x_{i+k}$ for all $i \in \mathbb{N}$.

Theorem 7. *Suppose* f *is an upper semicontinuous map with the intermediate value property and suppose* $f[a,b] \subseteq [a,b]$. *Then* $x \in f(x)$ *for some* $x \in [a,b]$.

Proof. Consider $g(x) = f(x) - x$. We have that $a \in f(x_a)$ and $b \in f(x_b)$ for some $x_a, x_b \in [a, b]$. Then $a - x_a \leq 0$ and $a - x_a \in g(x_a)$ and, likewise, $b - x_b \geq 0$ and $b - x_b \in g(x_b)$. By the intermediate value property, there exists x_c between x_a and x_b where $0 \in g(x_c)$. Thus $0 \in f(x_c) - x_c$ and $x_c \in f(x_c)$. ■

Definition. Let $f : [a, b] \to 2^{[c,d]}$ be a upper semicontinuous map and let A and B be intervals. We say that A *f-covers* B if $B \subseteq f[A]$ and A f^n*-covers* B if $B \subseteq f^n[A]$.

Lemma 2. *Suppose f is an upper semicontinuous map in which A_o is an interval that f-covers itself. Then, for each $n \in N$ there is a nested sequence of intervals $A_1, A_2, ..., A_n$ such that $A_n \subset A_{n-1} \subset ... \subset A_1 \subset A_o$ where f has an A_i,A_{i-1} restriction for each $i = 1, 2, ..., n$.*

Proof. Fix $n \in N$. By applying Theorem 6 to the fact that A_o f-covers itself, we know there is a subinterval $A_1 \subset A_o$ such that f has an A_1, A_o restriction. We see that, since A_1 also f-covers itself, we may find a subinterval $A_2 \subset A_1$ such that f has an A_2, A_1 restriction. Proceeding inductively we find the intervals $A_1, A_2, ..., A_n$ that satisfy the conclusion of the lemma. ■

Lemma 3. *Suppose f is an upper semicontinuous map in from $[a, b] \to 2^{[a,b]}$. Let $A_o, A_1, ..., A_n \subset [a, b]$ such that f has an A_i, A_{i-1} restriction for $i = 1, 2, ..., n$. Then $f|_{A_1}^{A_o} \circ f|_{A_2}^{A_1} \circ ... \circ f|_{A_n}^{A_{n-1}}(x)$ is a subset of $f^n|_{A_n}^{A_o}(x)$ for each $x \in A_n$.*

Proof. Suppose $x \in A_n$. We let $z \in f|_{A_1}^{A_o} \circ f|_{A_2}^{A_1} \circ ... \circ f|_{A_n}^{A_{n-1}}(x)$. Then there exist points $p_1, p_2, ..., p_n$ such that $p_n = x$, $p_i \in A_i$, $p_{i-1} \in f|_{A_i}^{A_{i-1}}(p_i)$ for all i, and $z \in f(p_1)$. Thus

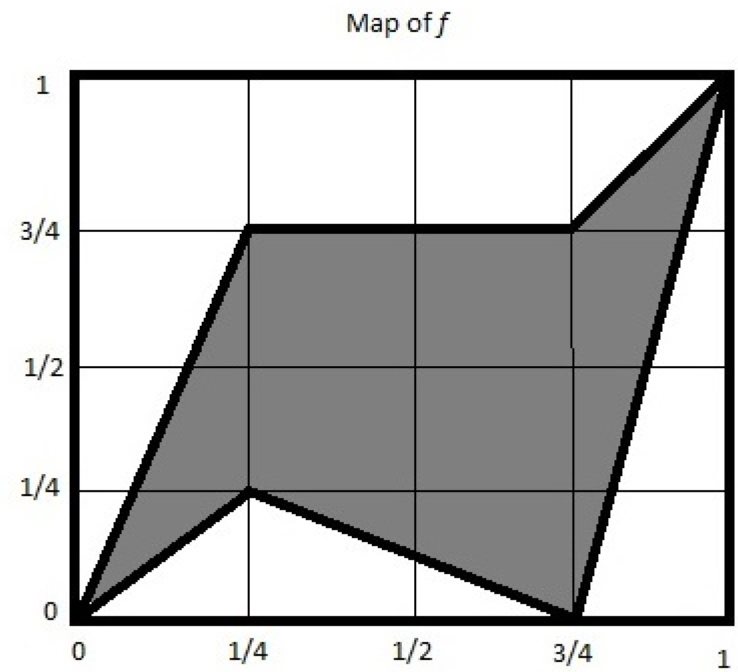

Figure 2: *A map where if* $A_o = A_1 = A_2 = [0, 1/4]$*, then* $f|_{A_1}^{A_o} \circ f|_{A_2}^{A_1}(1/4) = \{1/4\}$ *but where* $f^2|_{A_2}^{A_o}(1/4) = [0, 1/4]$

$p_{i-1} \in f(p_i)$ for all i. Consequently, $z \in f^n(x)$. And, finally, as $z \in A_o$ and $x \in A_n$, then $z \in f^n|_{A_n}^{A_o}(x)$. ■

We note that the converse is not necessarily always true, as shown by Figure 2.

CHAPTER FOUR

Sharkovskii under Set-Valued Functions

We now demonstrate Sharkovskii's ordering on upper semicontinuous set-valued functions with the strong IVP. Recall that the complete Sharkovskii ordering is as follows:

$$3 \gg 5 \gg 7 \gg ... \gg 3 \cdot 2^1 \gg 5 \cdot 2^1 \gg 7 \cdot 2^1 \gg ... \gg 3 \cdot 2^2 \gg 5 \cdot 2^2 \gg 7 \cdot 2^2 \gg ... \gg$$
$$... \gg 2^3 \gg 2^2 \gg 2 \gg 1$$

In order to prove that the existence of prime period m implies the existence of a prime period n for all $m \gg n$, we break the ordering into four lemmas, some of which are broken down further into multiple steps. Lemma 4 is the proof from Chapter Two, now under set-valued functions, in which $m = 3$. Lemma 5 is the case where m is odd and $m > 3$. Lemma 6 is the case where m is a power of 2. And Lemma 7 is the case where $m = x \cdot 2^a$ such that x is odd. Together, the four lemmas combined are the proof of the entire ordering, as stated in Theorem 8.

Notation. In the following proofs, we introduce the symbol $A \to B$ to mean that A f-covers B. And we use the symbol $A \mapsto B$ to mean that f maps A to B by an A, B restriction.

Theorem 8. *Let f be an upper semicontinuous function that has the strong intermediate value property. Let $m \in \mathbb{N}$ and assume f has a periodic orbit of prime period m. Then for n with $m \gg n$ in the Sharkovskii ordering, f also has a periodic orbit of period n.*

Proof. The orderings proven in the following Lemmas 4, 5, 6, and 7 may be combined as proof. The only omitted portion from some of the proofs is that the existence of a periodic orbit with any period m implies an orbit of period 1. This is an immediate consequence of the IVP and Theorem 7. ■

Lemma 4. *Let f be an upper semicontinuous function that has the strong intermediate value property. Suppose f has a periodic orbit of prime period 3. Then f has a periodic orbit of all other prime periods.*

Proof. Let a, b, c be elements of $\mathbb{R}$ where $a < b < c$ and $b \in f(a)$, $c \in f(b)$, and $a \in f(c)$. The other case, where $c \in f(a)$, $a \in f(b)$, and $b \in f(c)$, is similar. We set $I = [a, b]$ and $J = [b, c]$. We note that I f-covers J and J f-covers $I \cup J$. This is assured by the intermediate value property. We then have a fixed point in J, as J f-covers itself (see Theorem 7). Now, in pursuit of a periodic orbit with a prime period 2, we make note of the facts that $c \in f(b)$ and $a \in f(c)$ and apply the strong IVP to see that there exists a point $l \in (b, c)$ such that $b \in f(l)$. Set $L = [l, c]$. L f-covers I, hence by Theorem 6 there is a subinterval $\tilde{L} \subseteq L$ such that f has an $\tilde{L}, I$ restriction. Further, I f-covers $\tilde{L}$ since I f-covers J and $\tilde{L} \subset J$. It follows from Theorem 7 that there is an $x \in \tilde{L}$ such that $x \in f \circ f|_{\tilde{L}}^{I}(x)$. Then there is a $y \in I$ such that $y \in f(x)$ and $x \in f(y)$. Since I and $\tilde{L}$ are disjoint, $x \neq y$. So $x, y, x, y, ...$ is a periodic orbit with prime period 2. Thus f has points of prime periods 1, 2, and 3.

Now suppose n is a positive integer greater than 3. We construct a sequence of intervals $A_o, A_1, ..., A_n$ as follows: First, let $A_o = I$. By the strong IVP, we may find $l \in (b, c)$ such that $b \in f(l)$. Note that $[l, c]$ f-covers I. By Theorem 6, we

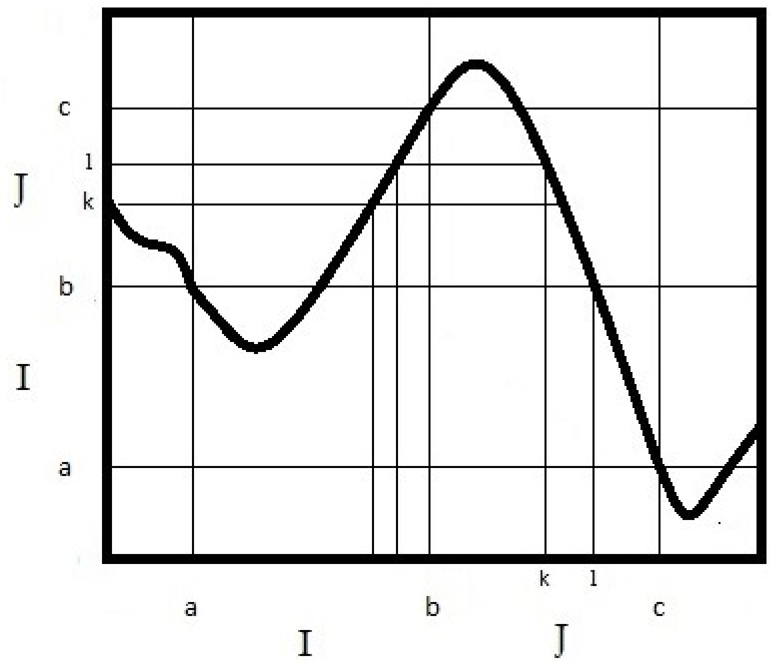

Figure 3: *Finding k and l on a map of period 3*

can set $A_1 \subseteq [l, c]$ such that f has an A_1, A_o restriction. Since $[b, l]$ f-covers J and $A_1 \subset J$, then $[b, l]$ f-covers A_1. Then we may find $A_2 \subseteq [b, l]$ such that f has an A_2, A_1 restriction. Note that $A_2 \subset J$ and $c \notin A_2$. Now as $b \in f(l)$ and $c \in f(b)$ with $l \in (b, c)$, then by the strong IVP, there is a $k \in (b, l)$ such that $l \in f(k)$. See Figure 3. Then $[k, l]$ f-covers $[b, l]$ and consequently also f-covers A_2. Set $A_3 \subseteq [k, l]$ such that f has an A_3, A_2 restriction. Here, note that $A_3 \subset J$ and $c, b \notin A_3$. Proceeding inductively by a similar argument, there exists subintervals $A_4, ..., A_{n-1} \subset J$ where $b, c \notin A_i$ and f has an A_i, A_{i-1} for $3 < i \leq n-1$. Now, as $A_{n-1} \subset J$ with $b, c \notin A_{n-1}$, then A_{n-1} is in the interior of J with some endpoints $h < k$. By using the strong IVP and the fact that I f-covers J, we can find points in the interior of I and a subinterval A_n therein such that f has an A_n, A_{n-1} restriction. Then consider now the composition of restrictions $g(x) = f|_{A_1}^{A_o} \circ f|_{A_2}^{A_1} \circ ... \circ f|_{A_n}^{A_{n-1}}(x)$. Since each restriction is a surjective, upper semicontinuous map with the IVP, then the composition of restrictions also has these properties. Therefore, as $A_n \subset A_o$, $g(x)$ meets all of the requirements of Theorem 7 and there is a fixed point $x \in A_n$ such that $x \in g(x)$. Thus, there are $x_o, x_1, ..., x_n$ such that $x_i \in A_i$ and $x_{i-1} \in f|_{A_i}^{A_{i-1}}(x_i)$ for $i = 1, 2, ..., n$ and $x = x_o = x_n$. Hence, $x_o, x_1, ..., x_n, ...$ is a periodic orbit of period n. We can be assured that n is the prime period as every n^{th} element is in the interior of I and all other elements are in J, which is disjoint from the interior of I.

Then f has a periodic orbit of all prime periods $n \in \mathbb{N}$.

■

Lemma 5. *Let f be an upper semicontinuous function that has the strong intermediate value property. Let $m \in \mathbb{N}$ and assume f has a periodic orbit of prime period m where $m > 3$ is odd. Then for $n \in \mathbb{N}$ with $m \gg n$ in the Sharkovskii ordering, f also has a periodic orbit of period n.*

Proof. Step 1: We demonstrate several crucial facts about minimum periods of periodic orbits. This results in the construction the critical figures Figure 5 and Figure 6.

Take $x_1 < x_2 < ... < x_m$ to be an orbit of distinct points of f where m is odd and assume f has no periodic orbits of odd periods smaller than m. We clarify our naming of the elements of the orbit by saying that it is not necessarily true that $x_i \in f(x_{i-1})$. In fact, this cannot always be the case, since for the x_j such that $x_j \in f(x_m)$, $x_j < x_m$. Let $\bar{f}$ be the subfunction of f that generates the orbit, and whose domain and range are exactly the orbit: such that $\bar{f}(x_g) = x_h$ for x_g, x_h in our orbit.

We now turn to consider the largest element x_i of the orbit such that $\bar{f}(x_i) > x_i$, and let $\theta_1 = A_1 = [x_i, x_{i+1}]$. By this choice of x_i, we are assured that $\bar{f}(x_{i+1}) < x_{i+1}$. Also note that as $m > 2$, $\bar{f} \circ \bar{f}(x_i) \neq x_i$. Consequently, from the IVP, we now have that A_1 f-covers itself and some other interval of the form $[x_t, x_{t+1}]$. Let θ_2 be the union all intervals of the form $[x_t, x_{t+1}]$ f-covered by A_1. Since A_1 f-covers itself, $\theta_1 \subset \theta_2$. Now let θ_3 be the union of all intervals which are f-covered by any interval $[x_t, x_{t+1}] \subset \theta_2$. Proceeding inductively, we may construct intervals $\theta_2, \theta_3, ...$ with the property that for all j, θ_{j+1} is the union of the intervals of the form $[x_t, x_{t+1}]$ that are f-covered by a subinterval of θ_j of the form $[x_t, x_{t+1}]$. Note here that if $A_{j+1} \subset \theta_{j+1}$ is of the form $[x_t, x_t + 1]$, then there are subintervals $A_2, ..., A_j$ of $\theta_2, ..., \theta_j$ of the form $[x_t, x_{t+1}]$ such that $A_1 \to A_2 \to ... \to A_j \to A_{j+1}$.

We claim also that $\theta_{j+1} \supseteq \theta_j$ for all j, which can be shown by induction. We have already shown for the initial case that $\theta_2 \supset \theta_1$. Since θ_3 contains all intervals of the form $[x_t, x_{t+1}]$ that are f-covered by θ_2, which in turn contains all intervals of the form $[x_t, x_{t+1}]$ that are f-covered by θ_1, then $\theta_3 \supset \theta_2$. Proceeding inductively, $\theta_{j+1} \supset \theta_j$ for each j. Now, since we have finitely many intervals of the form $[x_t, x_{t+1}]$, it follows that there exists some l such that $\theta_{l+1} = \theta_l$. This θ_l must contain our entire orbit and all intervals $[x_t, x_{t+1}]$ on the interior of our orbit.

Now, since m is odd, there must be more points of $x_1, x_2, ..., x_m$ on one side of the interior of A_1, A_1^o, than the other. Consequently, for whichever side of A_1^o that has more points of $x_1, x_2, ..., x_m$, at least one point on that side is mapped by $\bar{f}$ to a point on that same side. And since x_1, under finitely many iterations of $\bar{f}$, maps to x_m, it follows that at least one point of $x_1, ..., x_i$ that is on the left of A_1^o maps to a

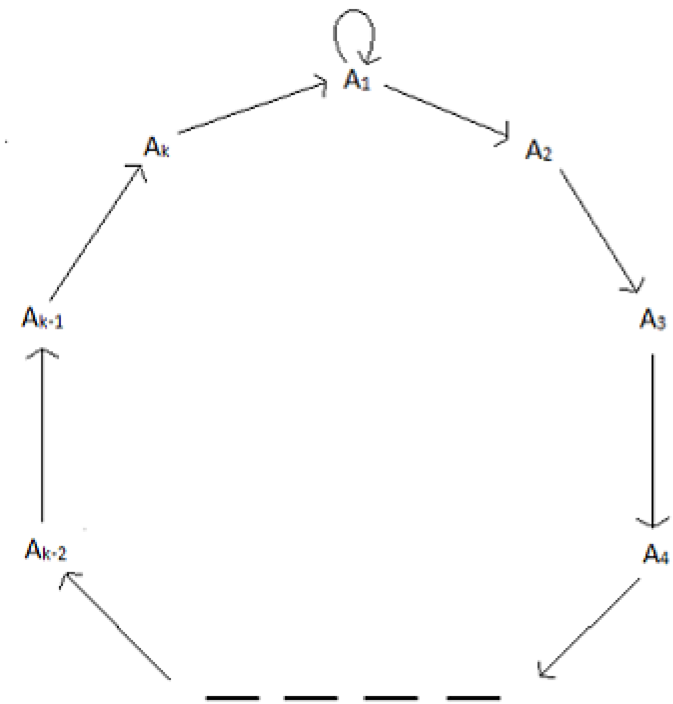

Figure 4: *The sequence of intervals* A_1 *to* A_k *(or* A_{m-1}*)*

point on the right of A_1^o. Similarly, at least one point $x_{i+1}, ..., x_m$ on the right of A_1^o maps to a point on the left of A_1^o. It follows that, besides A_1 itself, there is another interval of the form $[x_t, x_{t+1}]$ that will f-cover A_1.

Hence, as depicted in Figure 4, a sequence of intervals $A_1 \to A_2 \to ... \to A_k \to A_1$ exists in which each A_j is of the form $[x_t, x_{t+1}]$ and $A_1 \neq A_2$. Note that since there are only $m-1$ intervals of the form $[x_t, x_{t+1}]$, it cannot be that $k > m-1$. We will show by contradiction that $k = m-1$.

Suppose $k < m-1$. Then either $A_1 \to A_2 \to ... \to A_k \to A_1$ or $A_1 \to A_2 \to ... \to A_k \to A_1 \to A_1$ will yield a periodic orbit of odd period $q < m$. Yet we chose m to be the smallest odd periodic orbit of f. This contradiction stands even if q is not the prime period of the orbit. Thus $k = m - 1$.

Since k is the smallest integer for which there is a sequence $A_1 \to A_2 \to ... \to A_k \to A_1$, we can make several strong claims about the limited range of A_j for $j = 1, 2, ..., k-1$. First, we claim that it is not possible that A_j f-covers any of $A_{j+2}, A_{j+3}, ..., A_k, A_1$, as this would shortcut the orbit found in Figure 4 and would contradict the minimality of k. Second, by similar reasoning, the only interval that may f-cover A_1 (besides A_1 itself) is A_k. Now, as A_1 f-covers A_1 and A_2 and not A_j for $j = 3, ..., k$, then A_1 and A_2 must be adjacent. Recall that $A_1 = [x_i, x_{i+1}]$ and $\bar{f}(x_i) > (x_i)$, then either $\bar{f}(x_i) = x_{i+1}$ or $\bar{f}(x_i) = x_{i+2}$. We will see that the cases are similar, but for now we take the former to be true. Then we have $\bar{f}(x_{i+1}) = x_{i-1}$ and $A_2 \leq A_1$. We know that A_2 may only f-cover A_2 and A_3, and that one end point of A_2, x_i, has $\bar{f}(x_i) = x_{i+1}$, so it follows that x_{i-1} has either $\bar{f}(x_{i-1}) = x_{i-2}$ or $\bar{f}(x_{i-1}) = x_{i+2}$. If it is the former, then by the strong IVP, A_2 f-covers A_1, so it must be the latter. Then A_2 f-covers only A_3. Proceeding inductively we get the Figure 5. In the other case where $\bar{f}(x_i) = x_{i+2}$, the mappings of the orbit under $\bar{f}$ would be in the opposite direction.

We can now use Figure 5 to expand Figure 4 to Figure 6. We will make heavy use of Figure 5 and Figure 6 in the next steps.

Step 2: We prove that f has a periodic orbit of prime period $n \in \mathbb{N}$ where $n > m$.

Suppose $n > m$. Notice that since $A_1 \to A_2 \to ... \to A_{m-1} \to A_1$ and $A_1 \to A_1 \to ... \to A_1$, we may construct a sequence of the form $A_1^{(1)} \to A_2^{(2)} \to ... \to A_{m-1}^{(m-1)} \to A_1^{(m)} \to A_1^{(m+1)} \to ... \to A_1^{(n)}$ with exactly n mappings. Since $A_1^{(n-1)} \to A_1^{(n)}$, by

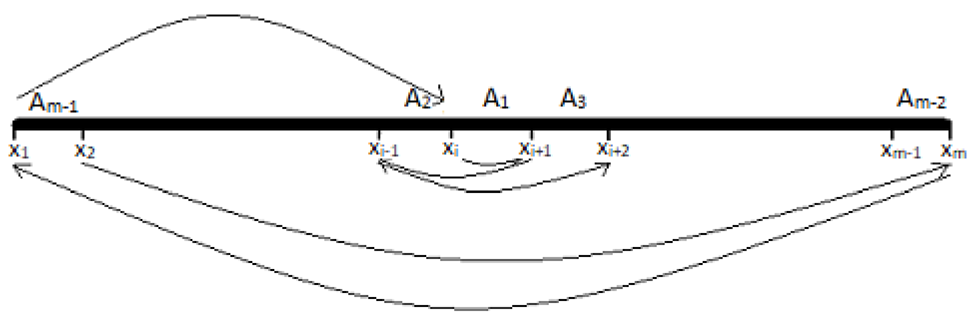

Figure 5: *Here the arrows indicate the orbital mapping of $\bar{f}$*

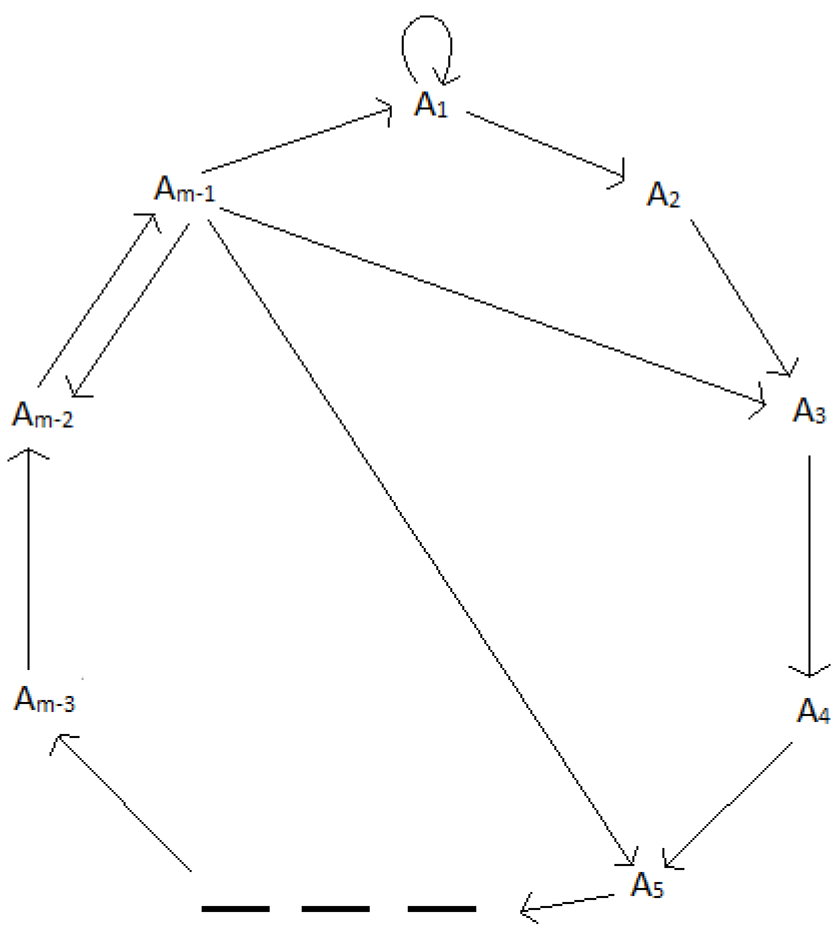

Figure 6: *Figure 4 expanded*

Theorem 6 we can find $\tilde{A}_1^{(n-1)} \subseteq A_1^{(n)}$ such that f has an $\tilde{A}_1^{(n-1)}, A_1^{(n)}$ restriction. We may proceed by similar method to find other subintervals $\tilde{A}_1^{(n-2)}, ..., \tilde{A}_1^{(m+1)}, \tilde{A}_1^{(m)}$ with similar restrictions. We now find $\tilde{A}_{m-1}^{(m-1)}$. Since $\bar{f}(x_1) = x_i$ and $\bar{f}(x_2) = x_m$, then by the strong IVP there exists a point $y \in (x_1, x_2)$ such that $x_{i+1} \in f(y)$. Then $[x_1, y]$ f-covers A_1 (and consequently also $\tilde{A}_1^{(m)}$) so we may choose $\tilde{A}_{m-1}^{(m-1)} \subseteq [x_1, y]$ such that f has an $\tilde{A}_{m-1}^{(m-1)}, \tilde{A}_1^{(m)}$ restriction. Proceeding, we can find $\tilde{A}_{m-2}^{(m-2)}, \tilde{A}_{m-3}^{(m-3)}, ..., \tilde{A}_1^{(1)}$ with corresponding restrictions. The complete sequence of restrictions is therefore: $\tilde{A}_1^{(1)} \mapsto \tilde{A}_2^{(2)} ... \mapsto \tilde{A}_{m-1}^{(m-1)} \mapsto \tilde{A}_1^{(m)} \mapsto \tilde{A}_1^{(m+1)} \mapsto ... \mapsto A_1^{(n)}$. Then by Theorem 7 we have a periodic orbit $p_1, p_2, ..., p_n, ...$ such that $p_{k+n} = p_k$ for each $k \in \mathbb{N}$. Now since $\tilde{A}_{m-1}^{(m-1)} \subseteq [x_1, y] \subset [x_1, x_2)$, then $\tilde{A}_{m-1}^{(m-1)}$ does not intersect any other subinterval $\tilde{A}_1^{(1)}, ..., \tilde{A}_{m-2}^{(m-2)}, \tilde{A}_1^{(m)}, ..., A_1^{(n)}$ in our sequence. It follows that $p_{m-1} \in \tilde{A}_{m-1}^{(m-1)}$ in our orbit is distinct from each of $p_1, ..., p_{m-2}, p_m, ..., p_n$ and the orbit $p_1, p_2, ..., p_n, ...$ has prime period n.

Step 3: We prove that f has a periodic of prime period $n \in \mathbb{N}$ where $n < m$ and n is even.

First note from Figure 6 that A_{m-1} f-covers all intervals A_i with i odd. So since A_{m-1} f-covers A_{m-2} and vice versa, we may find $\tilde{A}_{m-1} \subset A_{m-1}$ with $\tilde{A}_{m-1} \mapsto A_{m-2}$ and $\tilde{A}_{m-2} \subset A_{m-2}$ with $\tilde{A}_{m-2} \mapsto A_{m-1}$. Thus we have a periodic orbit of period 2, where 2 is the prime period since the intervals A_{m-1} and A_{m-2} are disjoint.

For $n = 4, ..., m-1$, we make use of the fact that $A_{m-1} \to A_{m-n} \to A_{m-n+1} \to ... \to A_{m-1}$. Therefore we may begin to find the corresponding subsets and restrictions, beginning with $\tilde{A}_{m-2} \subseteq A_{m-2}$ where $\tilde{A}_{m-2} \mapsto A_{m-1}$. We continue until we have $\tilde{A}_{m-n} \subseteq A_{m-n}$ where $\tilde{A}_{m-n} \mapsto \tilde{A}_{m-n+1}$. Now, we find 2 possible cases for the endpoints x_g, x_{g+1} of A_{m-n}. We have either $x_i \leq x_g < x_{g+1} < ... < x_m$ or $x_i < ... < x_g < x_{g+1} = x_m$. The second case, where $x_{g+1} = x_m$, is in fact the case $n = 2$ which is resolved above. In the first case, by the strong IVP, there are $u, v \in [x_1, x_2)$ where $x_g \in f(u), x_{g+1} \in f(v)$. It follows that $[u, v] \subset [x_1, x_2)$ f-covers A_{m-n}, and we can find a subset $\tilde{A}_{m-1} \subset [u, v]$ in the interior of A_{m-1} where $\tilde{A}_{m-1} \mapsto \tilde{A}_{m-n}$. It follows that by using the sequence of restrictions $\tilde{A}_{m-1} \mapsto \tilde{A}_{m-n} \mapsto ... \mapsto A_{m-1}$ we get a periodic orbit of period n. We are assured that n is prime as every n^{th} point is in the interior of A_{m-1}, and consequently not in any of the other intervals $A_{m-n}, ..., A_{m-2}$. We have now shown that if m is odd, then f has orbits of prime periods n where either $n > m$ or $n < m$ and n is even. ■

Lemma 6. *Let f be an upper semicontinuous function that has the strong intermediate value property. Let $m \in \mathbb{N}$ and assume f has a periodic orbit of prime period $m = 2^a$. Then f also has a periodic orbit of prime period $n = 2^b$ for $b < a$.*

Proof. Step 1: We first show that a map with any even period m also has a periodic orbit of prime period 2.

Assume now that f is a map with an orbit $x_1 < x_2 < ... < x_m$ of even period $m > 2$. We select x_i and A_1 as in Lemma 5. Consider the mappings under $\bar{f}$ of the points on either side of (x_i, x_{i+1}). We recall that by our choice of x_i, we have $\bar{f}(x_i) > x_i$ and $\bar{f}(x_{i+1}) < x_i$. Thus at least one point on both sides of (x_i, x_{i+1}) changes sides under $\bar{f}$. It could be that all points change sides under $\bar{f}$. We would then see that $[x_1, x_{i-1}]$ f-covers $[x_{i+1}, x_n]$ and vice versa. This gives us a period 2 orbit. It seems at first that it could also be the case that some points map to the same side of (x_i, x_{i+1}) under $\bar{f}$. If this is true, then at least one interval of the form $[x_t, x_{t+1}]$ f-covers A_1. Take k to be the minimum number of intervals to get a sequence $A_1 \to A_2 \to ... \to A_k \to A_1$. Since we cannot shortcut this sequence, by the same reasoning as above we could make a similar figure to Figure 5. If A_k is the leftmost interval, then this is the case we have already demonstrated. If A_k is the rightmost interval, then $\bar{f}(x_{g+1}) = x_i$ and $\bar{f}(x_g) = x_1$. But this would mean that all points switch sides under $\bar{f}$, a contradiction. Then a map with any even period m also has an orbit of prime period 2.

Step 2: We show the case where the existence of a periodic orbit of prime period $m = 2^a$ implies the existence of an orbit of prime period $n = 2^b$ for $b < a$.

Suppose $p_o, p_1, ..., p_{m-1}, p_o, ...$ is an orbit in f of distinct points of period $m = 2^a$. Then we will let $n = 2^b$ for some $b < a$ and show that f has a periodic orbit of period n as well. First note that if α is any factor of m and $\alpha \cdot \beta = m$, then f^α has a periodic orbit of period β, namely $x_o, x_\alpha, x_{2\alpha}, ..., x_{\beta\alpha}, ...$. Consider the function $g = f^{n/2} = f^{2^{b-1}}$. Note that g must have a periodic orbit of period 2^{a-b+1}. Since this orbit is even, then from Step 1 we are sure that g also has a prime period 2 orbit of distinct points q_1, q_2 where $q_1 \in g(q_2)$ and $q_2 \in g(q_1)$. It follows that there is a periodic orbit $r_o, r_1, r_2, ...$ of f such that $r_o = q_1$, $r_{2^{b-1}} = q_2$, and $r_{2^b} = q_1$. This orbit has a period of 2^b, which we want to be sure is prime. To that effect, we assume presently that it is not prime. Then the true prime period must be a factor of 2^b, which means it is period 2^c for some $c < b$. Then $r_o = r_{2^c} = r_{2 \cdot 2^c} = ... = r_{k \cdot 2^c}$ for all $k \in \mathbb{N}$. Yet for $k = 2^{(b-1)-c}$, then $r_o = r_{2^{(b-1)-c}.2^c} = r_{2^{b-1}}$. But we have that $r_o = q_1$ and $r_{2^{b-1}} = q_2$ where $q_1 \neq q_2$. Thus, from this contradiction we have that $n = 2^b$ is a prime period for the periodic orbit and we are done. ■

Lemma 7. *Let f be an upper semicontinuous function that has the strong intermediate value property. Assume f has a periodic orbit of prime period $m = x \cdot 2^a$ with x odd. Then for $n \in \mathbb{N}$ with $m \gg n$ in the Sharkovskii ordering, f also has a periodic orbit of period n.*

Proof. Based on the ordering, we need to demonstrate three possibilities for n. Each possibility is given as one of three steps: 1) $n = y \cdot 2^a$ where $y > x$ is odd, 2) $n = z \cdot 2^b$ for $b > a$ and any $z \in \mathbb{N}$, and 3) $n = 2^b$ for $b < a$.

Step 1: A prime period $m = x \cdot 2^a$ with odd $x > 1$ implies a prime period n where $n = y \cdot 2^a$ where $y > x$ is odd.

Assume f has a periodic orbit $p_o, p_1, p_2, ...$ of prime period $m = x \cdot 2^a$ where x is odd and $x > 1$. Fix $y \in \mathbb{N}$ such that $y > x$ and y is odd.

This is a proof by induction to show that the statement of Step (1) is true for all a. The base case is statement when $a = 0$. In this case, $m = x$ is odd and is a consequence of Lemma 5. Then we make the inductive hypothesis that the statement holds for $a = 0, 1, ..., t$. Suppose that f has a periodic orbit $x_o, x_1, x_2, ...$ of prime period $m = x \cdot 2^{t+1}$. Then f^2 has a periodic orbit, namely $x_o, x_2, x_4, ...$, of prime period $x \cdot 2^t$. It follows from the inductive hypothesis that f^2 also has a periodic orbit $y_o, y_2, y_4, ...$ of prime period $y \cdot 2^t$. We note here that since the period is prime, then there is at least one y_s in the orbit that is unique from the other points of the orbit so that $y_s \neq y_{s+k}$ for $k < n$. Now, there are points $y_1, y_3, y_5, ...$ such that $y_o, y_1, y_2, ...$ is periodic orbit of period $y \cdot 2^{t+1}$. To see that $y \cdot 2^{t+1}$ is a prime period, suppose to the contrary some $\tilde{n}$ is the true prime period. It must be that $\tilde{n}$ is a factor of $y \cdot 2^{t+1}$. If $\tilde{n}$ is odd, this is the case in Lemma 5 and we are done. If $\tilde{n}$ is even, then under $f^{\tilde{n}}$, $y_s, y_{s+\tilde{n}}, y_{s+2\cdot\tilde{n}}, ...$ is a periodic orbit of period 1 and $y_s = y_{s+\tilde{n}}$. This is a contradiction. Thus $y \cdot 2^{t+1}$ is the true prime period.

We have shown, by induction, that the statement of Step (1) is true for all a. Then f has a periodic orbit of prime period $n = y \cdot 2^a$ and we have completed Step (1).

Step 2: A prime period $m = x \cdot 2^a$ with $x > 1$ odd implies a prime period n where $n = z \cdot 2^b$ for $b > a$ and any odd $z \geq 3$.

Assume f has a periodic orbit $p_o, p_1, p_2, ...$ of prime period $m = x \cdot 2^a$ with x odd. Let $z \geq 3$ and let $b > a$. Consider f^{2^a}. This map has a prime period x orbit, namely $p_o, p_{2^a}, ...$ Since x is odd, by Lemma 5, f^{2^a} also has a periodic orbit $q_o, q_1, ..., q_5, q_o, ...$ of prime period $3 \cdot 2^1$. It follows that f has a periodic orbit $r_o, r_1, r_2, ...$ of period $3 \cdot 2^{a+1}$ such that $r_{k\cdot 2^a} = q_k$ for $k \in \mathbb{N}$. Suppose $3 \cdot 2^{a+1}$ is not the prime period of the orbit but rather some $\tilde{n} < 3 \cdot 2^{a+1}$. We note that $\tilde{n}$ must be a factor of $3 \cdot 2^{a+1}$ and thus there are four possible cases, all of which lead to contradictions: 1) $\tilde{n} = 3$, 2) $\tilde{n} = 3 \cdot 2^c$ for $c \leq a$, 3) $\tilde{n} = 2^c$ for $c \leq a$, and 4) $\tilde{n} = 2^{a+1}$. Case (1) is proven in Lemma 4. The other 3 cases are similar to one another.

Case (2): $\tilde{n} = 3 \cdot 2^c$ where $c \leq a$. Then $r_o, r_{2^c}, r_{2\cdot 2^c}, r_o, ...$ is a period 3 orbit of f^{2^c}. It follows that $r_o, r_{2^a}, r_{2\cdot 2^a}, ...$ is a period 3 orbit of f^{2^a}. Equivalently, $q_o, q_1, ...$ is a period 3 orbit of f^{2^a}. But $q_o, q_1, ..., q_5$ are distinct points. A contradiction.

Case (3): $\tilde{n} = 2^c$ for $c \leq a$. In f^{2^c}, we have that $r_o, r_{2^c}, r_{2\cdot 2^c}, r_o, ...$ is a period 1 orbit. Thus in f^{2^a}, it must be that $r_o, r_{2^a}, r_{2\cdot 2^a}, ...$ is a period 1 orbit. Again, as

$r_{k\cdot 2^a} = q_k$ but $q_o \neq q_1$, we have another contradiction.

Case (4): $\tilde{n} = 2^{a+1}$. In this case we find that $r_o, r_{2^a}, r_{2\cdot 2^a}, ...$ is a period 2 orbit in f^{2^a}. Yet if $r_o = r_{2\cdot 2^a}$, then $q_o = q_2$, a similar contradiction.

Then we may conclude that $3 \cdot 2^{a+1}$ is the prime period of the orbit. We may use similar reasoning to find a periodic orbit of prime period $3 \cdot 2^{a+2}$ and then $3 \cdot 2^{a+3}$ and so on until we have found an orbit of prime period $3 \cdot 2^b$. By Step (1) of this Lemma, f has a periodic orbit of all prime periods $y \cdot 2^b$ where $y > 3$ is odd. Thus f has prime period $z \cdot 2^b$.

Step 3: A prime period $m = x \cdot 2^a$ with $x > 1$ odd implies a prime period n where $n = 2^b$ for any $b \in \mathbb{N}$.

Fix $b \in \mathbb{N}$. From Step (2) of this lemma, we have that f also as a periodic orbit $p_o, p_1, ...$ of prime period $3 \cdot 2^c$ for some $c > b$. It follows that f^{2^c} has period 3 orbit, namely $p_o, p_{2^c}, p_{2\cdot 2^c}, p_o, ...$ Then f^{2^c} has a prime period 2 orbit $q_o, q_1, q_o, ...$ whereby $q_o \neq q_1$. Then f has a periodic orbit $q_o, q_a, ..., q_1, q_\alpha, ..., q_o$ of prime period 2^{c+1}. We are assured this period is prime by a similar argument to Step (2) of Lemma 6. Then by the statement of Lemma 6, we have a periodic orbit of prime period 2^b.

■

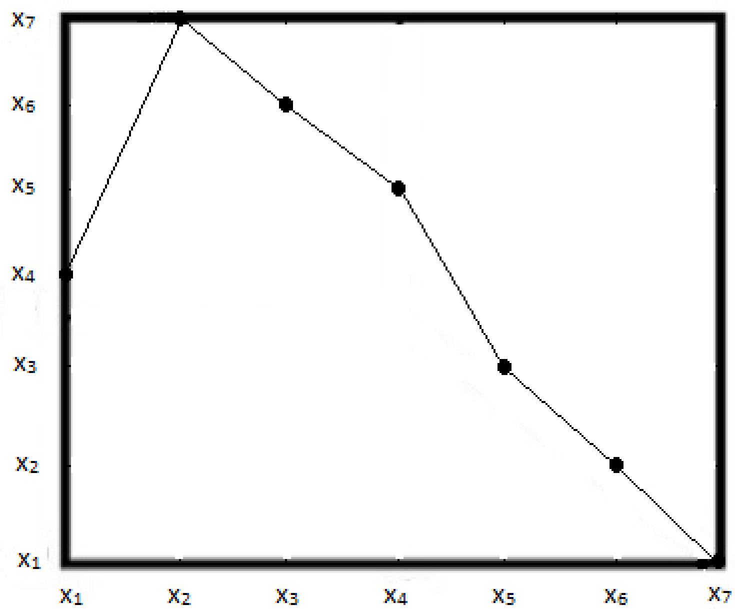

Figure 7: *A map with a period 7 orbit but no orbits with period 3 or 5*

CONCLUSION

Closing Remarks

We have proven that the Sharkovskii ordering prevails on a set-valued function provided that the function is upper semicontinuous and has the strong IVP.

We note that the same examples that show that Sharkovskii's ordering is sharp in the classical setting also work in the set-valued setting. This is true since classical continuous functions, when regarded as set-valued functions, are upper semicontinuous and have the strong IVP. One such example is in Figure 7.

References

[1] Lluís Alsedà, Jaume Llibre, and MichałMisiurewicz, *Combinatorial dynamics and entropy in dimension one*, Second ed., World Scientific (2000).

[2] L. Block, J. Guckenheimer, M. Misiurewicz, and L.-S. Young, *Periodic points and topological entropy of one-dimensional maps*, Global Theory of Dynamical Systems, Z. Nitecki and C. Robinson, eds., Lecture Notes in Mathematics, Vol. 819, Springer Verlag, Berlin, 1980, 18–34.

[3] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley Publishing Company, A.M. Wylde, Second ed., 1989, 60-68.

[4] A.N. Sharkovskii, *Coexistence of cycles of a continuous map of the line into itself*, Ukrain. Mat. Z. 16 (1964), pp. 61 - 71.

[5] A.N. Sharkovskii, *Coexistence of cycles of a contiuous map of the line into itself*, International Journal of Bifurcation and Chaos, Vol. 5 No. 5 (1995) pp. 1263 - 1273.