ABSTRACT

The Sixth-Order Krall Differential Expression and Self-Adjoint Operators

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We first provide an overview of classical GNK Theory for symmetric, or symmatrizable, differential expressions in $L^2((a,b);w)$. Then we review how this theory is applied to find a self-adjoint operator in $L^2_\mu(-1,1)$ generated by the sixth-order Lagrangian symmetric Krall differential equation, as done by S. M. Loveland. We later construct the self-adjoint operator generated by the Krall differential equation in the extended Hilbert space $L^2(-1,1) \oplus \mathbb{C}^2$ which has the Krall polynomials as (orthogonal) eigenfunctions. The theory we use to create this self-adjoint operator was developed recently by L. L. Littlejohn and R. Wellman as an application of the general Glazman-Krein-Naimark (GKN-EM) Theorem discovered by W. N. Everitt and L. Markus using complex symplectic geometry. In order to explicitly construct this self-adjoint operator, we use properties of functions in the maximal domain in $L^2(-1,1)$ of the Krall expression. As we will see, continuity, as a boundary condition, is forced by our construction of this self-adjoint operator. We also construct six additional examples of self-adjoint operators in an extended Hilbert space, three with a one-dimensional extension space and three with a two-dimensional extension space.

The Sixth-Order Krall Differential Expression and Self-Adjoint Operators

by

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To my parents

CHAPTER ONE

Classical GKN Theory

For more on this topic, see [1], [4], [16], [26], [27], [29], [30], [31], and [33]. In this chapter, I review classical GKN theory.

1.1 Symmetric Differential Expressions

Let us consider the ordinary differential expression $\ell[\cdot]$ of order $2n$ given by

$$\ell[y](x) := \sum_{j=0}^{n}(-1)^j \left(a_j(x)y^{(j)}(x)\right)^{(j)}, \quad x \in I, \tag{1.1}$$

where I is the open interval $I = (a, b)$ with $-\infty \leq a < b \leq \infty$ and n is a positive integer. In this chapter, we will assume that $a_k \in C^k(I, \mathbb{R})$ for $k = 0, 1, \ldots, n$ and $a_n(x) \neq 0$. In this chapter, we will study linear operators in $L^2(a, b)$ generated by (1.1). The differential expression $\ell[\cdot]$, as defined in (1.1), is called a *formally symmetric differential expression.*

We will now discuss regular and singular differential expressions.

Definition 1.1. The differential expression $\ell[\cdot]$ given in (1.1) is a *regular differential expression* if the interval I has finite length and the coefficients $\frac{1}{a_n}$, a_{n-1}, ... , $a_0 \in L(I)$, where $L(I)$ is defined as the set of Lebesgue measurable functions $f : I \to \mathbb{C}$ that are Lebesgue integrable on the interval I. On the other hand, $\ell[\cdot]$ is called a *singular differential expression* if it is not regular.

We can also classify the endpoints of $I = (a, b)$ as regular or singular endpoints.

Definition 1.2. The endpoint b is a *regular point* of the expression $\ell[\cdot]$ if $b < \infty$ and if there exists an $\varepsilon > 0$ sufficiently small so that the coefficients $\frac{1}{a_n}$, a_{n-1}, ... , $a_0 \in L(b - \varepsilon, b)$. If b is not a regular point of $\ell[\cdot]$, then it is a *singular point* of $\ell[\cdot]$.
We can similarly define the left endpoint a as a regular point of $\ell[\cdot]$ if $-\infty < a$

and there exists a sufficiently small $\varepsilon > 0$ such that $\frac{1}{a_n}$, a_{n-1}, ... , $a_0 \in L(a, a+\varepsilon)$. Otherwise, a is a singular point of $\ell[\cdot]$.

1.2 The Maximal and Minimal Operators Generated by $\ell[\cdot]$

We now define and discuss the maximal and minimal operators in $L^2(I)$ generated by the differential expression $\ell[\cdot]$ given in (1.1).

Definition 1.3. The operator $T_1 : L^2(I) \to L^2(I)$ defined by

$$\begin{aligned} T_1[y] =& \ell[y] \\ \mathcal{D}(T_1) =& \big\{ y : I \to \mathbb{C} | y^{(k)} \in AC_{loc}(I),\ k = 0,\ 1,\ \ldots, 2n-1;\ y,\ \ell[y] \in L^2(I) \big\} \end{aligned}$$

is called the *maximal operator* generated by the expression $\ell[\cdot]$ in the space $L^2(I)$.

The domain of the maximal operator, $\mathcal{D}(T_1)$, is the largest subspace in which T_1 can be defined as an operator from $L^2(I)$ into $L^2(I)$, so the word "maximal" is indeed appropriate.

Before we can define the minimal operator, we first need to define Green's formula and the sesquilinear form.

Definition 1.4. For $f, g \in \mathcal{D}(T_1)$, the *sesquilinear form* $[f, g](\cdot)$ generated by $\ell[\cdot]$ is defined by

$$\begin{aligned} [f,g](x) = \sum_{j=1}^{n} \sum_{m=1}^{j} (-1)^{m+j} \Big\{ & \left(a_j(x)\overline{g}^{(j)}(x)\right)^{(j-m)} f^{(m-1)}(x) \\ & - \left(a_j(x) f^{(j)}(x)\right)^{(j-m)} \overline{g}^{(m-1)}(x) \Big\}. \qquad (1.2) \end{aligned}$$

Definition 1.5. For $[\alpha, \beta] \subset (a, b)$, the *Green's formula* for $\ell[\cdot]$ is given by

$$\int_{\alpha}^{\beta} \{\ell[f]\overline{g} - \ell\overline{g}]f\}\, dx = [f, g]\Big|_{\alpha}^{\beta} \quad (f, g \in \mathcal{D}(T_1)) \qquad (1.3)$$

Notice that for all $f, g \in \mathcal{D}(T_1)$ and $a < x < b$, we have

$$[f, g](x) = -\overline{[f, g]}(x).$$

We also note that, by definition of $\mathcal{D}(T_1)$ and Hölder's inequality, the limits $[f,g](b) := \lim_{x\to b^-}[f,g](x)$ and $[f,g](a) := \lim_{x\to a^+}[f,g](x)$ exist and are finite. It will be seen later that Green's formula is important for determining all the self-adjoint extensions of the minimal operator generated by $\ell[\cdot]$ in $L^2(I)$.

Since $\mathcal{D}(T_1)$ is dense in $L^2(I)$, we know that the adjoint of T_1, denoted by T_1^*, exists.

Definition 1.6. We call $T_0 := T_1^*$ the *minimal operator* generated by $\ell[\cdot]$ in $L^2(I)$.

Observe that if L is any densely defined linear operator in $L^2(I)$ such that $L \subset T_1$, then $T_1^* \subset L^*$, and hence it makes sense to call T_0 "minimal." Before we give a constructive description of $\mathcal{D}(T_0)$, we first need a few more definitions and results.

Let $\mathcal{D}_0'$ be the densely defined subspace of all functions $f \in \mathcal{D}(T_1)$ which have compact support in the interval I. Then the restriction of the maximal operator T_1 to the subspace $\mathcal{D}_0'$ is denoted by T_0'.

Theorem 1.7. *The operator T_0' defined above is symmetric in $L^2(I)$.*

See [26, p. 61].

Let H be a Hilbert space with inner product $(\cdot,\cdot)_H$. For the rest of the section, let $S : H \to H$ be a linear operator with domain $\mathcal{D}(S)$.

Definition 1.8. The linear operator S is called *symmetric* in H if $\mathcal{D}(S)$ is dense in the Hilbert space H and if

$$(Sx, y)_H = (x, Sy)_H$$

for all $x, y \in \mathcal{D}(S)$.

Note that an operator S is symmetric in H if and only if $S \subset S^*$. Now we define a self-adjoint operator:

Definition 1.9. A linear operator $S : H \to H$ is *self-adjoint* in the Hilbert space H if $\mathcal{D}(S)$ is dense in H and $S = S^*$.

Definition 1.10. Let $\{x_n\} \subset \mathcal{D}(S)$. If the conditions $x_n \to x$ and $Sx_n \to y$, for some $y \in H$, imply $x \in \mathcal{D}(S)$ and $Sx = y$, then we say that the operator S is *closed*.

If S is a densely defined operator, then its adjoint S^* is a closed operator. Since we defined the minimal operator T_0 as the adjoint of the operator T_1, we see that the minimal operator is indeed a closed operator.

Definition 1.11. The linear operator S is *closable* if there exists a closed, linear extension of S.

Definition 1.12. A closed linear extension S' of a closable linear operator S is said to be *minimal* if every closed linear extension of S is a closed linear extension of S'. This minimal extension S' of S, if it exists, is called the *closure* of S, and S is said to *admit a closure* if S' exists. See [21, p. 537].

The notation for the closure of an operator S is $\overline{S}$.

Theorem 1.13. *Let H be a Hilbert space. Then if $T : H \to H$ is a symmetric operator, it admits a closure. Furthermore, the closure $\overline{T}$ is symmetric in the Hilbert space H.*

The proof of this theorem can be found in [26, p. 13]. As a consequence of Theorem 1.13, we see that the operator T_0' has a symmetric closure $\overline{T_0'}$. Furthermore, we have the following theorem:

Theorem 1.14. $\overline{(T_0')}^* = T_1$.

Naimark proves this result in [26, p. 18]. Combining Theorem 1.14 with the well-known fact that a closed, densely defined operator A in the Hilbert space H has the property that the adjoint of the adjoint of A is A, i.e. $A^{**} = A$, we obtain:

Theorem 1.15. $T_0 = \overline{T_0'}$ *and* $T_0^* = T_1$. *In particular, the minimal operator* T_0 *and the maximal operator* T_1 *are closed operators, each being the adjoint of the other.*

An immediate consequence of this theorem is that if A is a symmetric extension of the minimal operator T_0 in $L^2(I)$, then $A \subset T_1$, where T_1 is the maximal operator, since

$$T_0 \subset A \subset A^* \subset T_0^* = T_1.$$

Indeed, A has the same form as the differential expression $\ell[\cdot]$ and is a restriction of T_1, the maximal operator. This means that, in this case, $A[y] = \ell[y]$ for all $y \in \mathcal{D}(T_0)$.

The next theorem characterizes the minimal operator in terms of the sesquilinear form defined in (1.2). We have

Theorem 1.16. *The domain* $\mathcal{D}(T_0)$ *of the minimal operator* T_0 *in* $L^2(I)$ *consists of all* $f \in \mathcal{D}(T_1)$ *satisfying*

$$[f, g]\,(x)\Big|_a^b = 0,$$

for all $g \in \mathcal{D}(T_1)$.

Naimark gives a proof of this theorem in [26, p. 70]. The condition given in Theorem 1.16 can be modified to a simplified condition if either one or both of the endpoints a, b of I are regular. For example, if a, the left endpoint of the interval I, is regular, then $f \in \mathcal{D}(T_0)$ if

(i) $f^{(k)}(a) = 0$ for $k = 0, 1, 2, \ldots, 2n-1$ and

(ii) $[f, g](b) = 0$ for all $g \in \mathcal{D}(T_1)$.

A proof of this modified statement can be found in [26, p. 71].

Note that each of the theorems from this chapter so far can be directly extended to Lagrangian symmetrizable differential expressions of the form

$$\eta[y](x) = \frac{1}{w}\sum_{j=0}^{n}(-1)^j \left(a_j(x)y^{(j)}(x)\right)^{(j)}, \quad x \in I, \tag{1.4}$$

where $w(x) \in C^{2n}(I)$ and $w(x) > 0$ for all $x \in I$. The operator $w(x)\eta[y]$ is formally symmetric, and the function $w(x)$ is called a *symmetry factor* for the expression $\eta[\cdot]$. The Hilbert space used for self-adjoint extension theory in this case would be $L^2((a,b),w)$ and the maximal operator T_1 in $L^2((a,b),w)$, generated by $\eta[\cdot]$ is defined to be

$$\begin{aligned} T_1[y] &= \eta[y] \\ \mathcal{D}(T_1) &= \big\{y : (a,b) \to \mathbb{C} \mid y^{(i)} \in AC_{loc}(a,b),\ j = 0,1,\ldots,2n-1; \\ &\qquad\qquad\qquad\qquad\qquad y, \eta[y] \in L^2((a,b),w)\big\}. \end{aligned}$$

1.3 Examples of Lagrangian Symmetrizable Differential Expressions

Example 1.17. Consider the Jacobi expression defined by

$$\tau_J[y] := -(1-x^2)y'' + \Big(\alpha - \beta + (\alpha+\beta+2)x\Big)y' + ky,$$

where $x \in (-1,\, 1)$ and $\alpha, \beta > -1$. The n^{th} Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ satisfies the equation

$$\tau_J[P_n^{(\alpha,\beta)}] = \Big(k + n(n+\alpha+\beta+1)\Big)P_n^{(\alpha,\beta)}.$$

The Jacobi expression cannot be put in the symmetric form (1.1) directly. However, multiplying τ_J by the symmetry factor $w(x) = (1-x)^\alpha(1+x)^\beta$ yields the following:

$$\begin{aligned} \ell_J[y](x) :=& (1-x)^\alpha(1+x)^\beta\tau_J[y](x) \\ =& -\Big((1-x)^{\alpha+1}(1+x)^{\beta+1}y'(a)\Big)' + k(1-x)^\alpha(1+x)^\beta y(x). \end{aligned}$$

Notice that the correct setting for the Jacobi expression is not $L^2(-1,1)$ (unless $\alpha = \beta = 0$), but rather the weighted Lebesgue space $L^2((-1,1);(1-x)^\alpha(1+x)^\beta)$. In this space, the maximal and minimal operators associated with the Jacobi expression are generated by

$$\tau_J[\cdot] = (1-x)^{-\alpha}(1+x)^{-\beta}\ell_J[\cdot].$$

Example 1.18. Now consider the Laguerre expression defined by

$$\tau_L[y] := -xy'' + (x - 1 - \alpha)y' + ky,$$

where $x \in (0, \infty)$ and $\alpha > -1$. Then the n^{th} Laguerre polynomial $L_n^{(\alpha)}(x)$ satisfies

$$\tau_L[L_n^{(\alpha)}] = (n + k)L_n^{(\alpha)}.$$

As with the Jacobi expression, the Laguerre expression requires multiplication by a symmetry factor in order to be written as a symmetric expression. In this case, the symmetry factor is $w(x) = x^\alpha e^{-x}$, which yields

$$\begin{aligned} \ell_L[y](x) :=& x^\alpha e^{-x} \tau_L[y](x) \\ =& -\left(x^{\alpha+1} e^{-x} y'(x)\right)' + kx^\alpha e^{-x} y(x). \end{aligned}$$

The proper setting for the Laguerre expression is the space $L^2((0, \infty); x^\alpha e^{-x})$ and the maximal and minimal operators in this space associated with the Laguerre expression are generated by

$$\tau_L[\cdot] = x^{-\alpha} e^x \ell_L[\cdot].$$

Example 1.19. For our last example, consider the Hermite expression defined by

$$\tau_H[y] := -y'' + 2xy' + ky,$$

where $x \in (-\infty, \infty)$. Then the n^{th} Hermite polynomial $H_n(x)$ satisfies the equation

$$\tau_H[H_n] = (k + 2n)H_n.$$

Like the Jacobi and Laguerre expressions, the Hermite expression must be multiplied by a symmetry factor, $w(x) = e^{-x^2}$, to obtain

$$\begin{aligned} \ell_H[y](x) :=& e^{-x^2} \tau_H[y](x) \\ =& -\left(e^{-x^2} y'(x)\right)' + ke^{-x^2} y(x). \end{aligned}$$

The proper setting to use is the space $L^2((-\infty,\infty);e^{-x^2})$. The maximal and minimal operators in this space associated with the Hermite expression are generated by the expression

$$\tau_H[\cdot] = e^{x^2}\ell_H[\cdot].$$

1.4 Deficiency Spaces and von Neumman Formula

Let S be a symmetric operator in H, a Hilbert space. Define

$$\mathcal{D}_+ := \{f \in \mathcal{D}(S^*) \mid S^*f = if\}$$

and

$$\mathcal{D}_- := \{f \in \mathcal{D}(S^*) \mid S^*f = -if\},$$

where $i = \sqrt{-1}$.

Definition 1.20. The space $\mathcal{D}_+$ is called the *positive deficiency space* of S and $\mathcal{D}_-$ is called the *negative deficiency space* of S. The dimension of $\mathcal{D}_+$ is the *positive deficiency index* of S and the dimension of $\mathcal{D}_-$ is called the *negative deficiency index* of S.

We use the notation $n_\pm := \dim \mathcal{D}_\pm$. It is the case that any complex number λ with $Im(\lambda) > 0$ can be used in place of i in the above definitions.

If S is a symmetric operator in the Hilbert space H, then we define a new inner product on the space $\mathcal{D}(S^*)$ by

$$(x, y)^* := (x, y) + (S^*x, S^*y).$$

It can be shown that $\mathcal{D}(S^*)$ is a Hilbert space with this inner product ([4, p. 1225]). Now we are equipped to state von Neumann's formula.

Theorem 1.21. *Let A be a symmetric operator in a Hilbert space H. Then $\mathcal{D}(\overline{S})$, $\mathcal{D}_+$, and $\mathcal{D}_-$ are closed orthogonal subspaces in $(\mathcal{D}(S^*), (\cdot,\cdot)^*)$ and*

$$\mathcal{D}(S^*) = \mathcal{D}(\overline{S}) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-. \tag{1.5}$$

The proof of this result can be found in [4, p. 1227].

In the special case when S is the minimal operator T_0, then equation (1.5) becomes

$$\mathcal{D}(T_1) = \mathcal{D}(T_0) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-.$$

In the next result, which can be found in [4], we will see that the deficiency spaces play an important role in determining self-adjoint extensions of the minimal operator T_0.

Theorem 1.22. *Let S be a symmetric operator in a Hilbert space H. Let $\mathcal{D}' \subset \mathcal{D}_+ \oplus \mathcal{D}_-$ be a closed subspace and define $\mathcal{D}$ by $\mathcal{D} := \mathcal{D}\left(\overline{S}\right) \oplus \mathcal{D}'$. Then, the restriction of S^* to the subspace $\mathcal{D}$ is self-adjoint if and only if $\mathcal{D}'$ is the graph of an isometry that maps $\mathcal{D}_+$ onto $\mathcal{D}_-$.*

An important consequence of Theorem 1.22 is the following theorem:

Theorem 1.23. *Let S be a symmetric operator in a Hilbert space H. Then S has self-adjoint extensions in H if and only if its deficiency indices are equal, in other words, $n_+ = n_-$. Additionally, if $n_+ = n_- = 0$, then the only self-adjoint extension of S is $\overline{S} = S^*$.*

The proof of this result can be found in [4, p. 1230].

Now we move back to finding self-adjoint extensions of the minimal operator T_0 in $L^2(I)$. Note that since for any $\lambda \in \mathbb{C}$, the equation $\ell[y] = \lambda y$ has a basis of $2n$ solutions, the deficiency indices n_+ and n_- of T_0 are both finite. Since the coefficients a_k of $\ell[\cdot]$ as defined in (1.1) are real-valued, a function f is a solution of $\ell[y] = iy$ if and only if $\overline{f}$ is a solution of $\ell[y] = -iy$. Likewise, if $\{f_1, f_2, \ldots, f_m\}$ is a basis for $\mathcal{D}_+$, then $\left\{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m\right\}$ must in fact be a basis for $\mathcal{D}_-$. Therefore, the positive and negative deficiency indices are equal. However, in general, if the coefficients of the expression $\ell[\cdot]$ are complex-valued, then the deficiency indices are not necessarily equal. Now we state a result relating the properties of the endpoints a and b of the interval I to the deficiency indices n_+ and n_-.

Theorem 1.24. *Let T_0 be the minimal operator in $L^2(I)$ generated by the expression $\ell[\cdot]$, where $I = (a, b)$.*

(i) If both a and b are regular endpoints, then $n_\pm = 2n$.

(ii) If one of the endpoints is singular, then $0 \leq n_+ = n_- \leq 2n$. Indeed, for any integer m such that $0 \leq m \leq 2n$, $\ell[\cdot]$ can be constructed so that $n_\pm = m$. If exactly one of the endpoints a or b is singular, then $n \leq n_+ = n_- \leq 2n$.

The proof of part (i) can be found in [26, p. 66] and the proof of part (ii) can be found in [26, p. 69 and 71].

Now let $c \in I$. Then c is a regular point of $\ell[\cdot]$. Let T_0^- be the minimal operator generated by $\ell[\cdot]$ on the interval (a, c) and let T_0^+ be the minimal operator generated by $\ell[\cdot]$ on the interval (c, b). Also let (m_-, m_-) denote the deficiency indices of T_0^- in $L^2(a, c)$ and let (m_+, m_+) denote the deficiency indices for T_0^+ in $L^2(c, b)$.

Theorem 1.25. *The deficiency index of the minimal operator T_0 in $L^2(a, b)$ is (m, m) where*

$$m = m_+ + m_- - 2n$$

and $2n$ is the order of the expression $\ell[\cdot]$. In addition, m is independent of the choice of $c \in I$.

The proof of this theorem can be found in [14, p. 353]. Since the point $c \in I$ is a regular point of $\ell[\cdot]$, all solutions of the equation $\ell[y] = \pm iy$ must belong to $L^2(c - \varepsilon, c]$ for all $0 < \varepsilon < c - a$. Thus, m_- is equal to the number of solutions of $\ell[y] = \pm iy$ that are in $L^2(a, a + \delta]$ for sufficiently small δ. Likewise, m_+ is the same as the number of solutions of $\ell[y] = \pm iy$ that are in $L^2[b - \delta, b)$ for sufficiently small $\delta > 0$. This provides motivation for the next few definitions.

Definition 1.26. The expression $\ell[\cdot]$ is said to be in the *limit-p case at* $x = a$ in $L^2(I)$ if there are exactly p solutions of $\ell[y] = \pm iy$ that are in $L^2(a, a+\varepsilon)$ for some sufficiently small $\varepsilon > 0$. On the other hand, $\ell[\cdot]$ is said to be in the *limit-q case at* $x = b$ in $L^2(I)$ if there are exactly q solutions of $\ell[y] = \pm iy$ that are in $L^2(b-\varepsilon, b)$ for some sufficiently small $\varepsilon > 0$.

If the order of the expression $\ell[\cdot]$ is two, then the limit-2 case is often called the *limit circle case* and the limit-1 case is often called the *limit point case*. This notation comes from Weyl's paper [32]. The limit circle and limit point terminology come from the geometry Weyl used in his analysis of the number of L^2 solutions of the second-order Sturm-Liouville equation. In this second-order case, he showed that if $\ell[y] = \lambda_0 y$ is limit-point at the endpoint a or b for a certain $\lambda_0 \in \mathbb{C}$, then $\ell[y] = \lambda y$ is limit-point at a or b for all $\lambda \in \mathbb{C}$. Weyl also showed an analogous result for the limit-circle case when $\ell[\cdot]$ is order 2.

Combining the definitions of limit-p case and limit-q case with Theorem 1.25, we can now find the deficiency index of the minimal operator T_0 in $L^2(I)$ once we determine the limit case for both endpoints a and b of the interval I. The Method of Frobenius may be used to determine the number of Lebesgue square-integrable solutions near a regular singular endpoint, which will be defined below. For more on the Method of Frobenius, see [17, p. 396-404].

Consider the differential equation

$$L[y](x) = \sum_{j=0}^{n} b_j(x) y^{(j)}(x) = 0, \tag{1.6}$$

for $x \in J$ where $J \subset \mathbb{R}$ is an open interval, $b_j : J \to \mathbb{R}$ for $j = 0, 1 \ldots, n$, and $b_j(x) \neq 0$ for all $x \in J$. Suppose $a, b \in J$ with $a < b$.

Definition 1.27. If $x = a > -\infty$ and if

$$\frac{(x-a)^n L[y](x)}{b_n(x)} = \sum_{j=0}^{n} (x-a)^j c_j(x) y^{(j)}(x), \tag{1.7}$$

where $c_n(x) \equiv 1$ and each $c_j(x)$ is analytic in some neighborhood of $x = a$ for $j = 0, 1, \ldots, n-1$, then $x = a$ is called a *regular singular point* of $L[\cdot]$. There is an analogous definition for $x = b < \infty$ as a regular singular point.

In the case where $a = -\infty$ (or where $b = \infty$), we use the transformation $x = \frac{1}{t}$ to express $L[\cdot]$ in the form

$$\sum_{j=0}^{n} t^j c_j(t) y^{(j)}(t),$$

where, as before, $c_n(t) \equiv 1$ and $c_j(t)$ is analytic in some neighborhood of $t = 0$ for each j. In this case, we say that $x = \infty$ is a regular singular point of $L[\cdot]$.

Definition 1.28. If an endpoint is not a regular singular endpoint of $L[\cdot]$, then we say it is an *irregular singular point.*

Frobenius created a useful tool for determining a basis of n solutions of the homogeneous equation (1.6), which expands each solution around a regular singular point. An important part of the Method of Frobenius is the indicial equation.

Definition 1.29. The *indicial equation* at $x = a$ associated with equation (1.7) is:

$$\sum_{j=0}^{n} P(r, j) c_j = 0, \tag{1.8}$$

where $c_j = c_j(a)$ and $P(r, j) = \dfrac{r!}{(r-j)!}$, for $j = 0, 1, \ldots, n$.

In this method, each of the n roots of (1.8) determines a solution of the differential equation given in (1.6), even when roots have a multiplicity greater than one.

1.4.1 Examples

Example 1.30. Legendre Example

Consider the Legendre differential equation given by

$$M_k^{(1)}[y](x) := \Big((1-x)^2 y'(x)\Big)' + ky \tag{1.9}$$

defined in the interval $(-1,1)$. Both endpoints $x = \pm 1$ are regular singular points of $M_k^{(1)}[\cdot]$, as will be shown. Near $x = -1$, we have

$$\begin{aligned}\frac{(x+1)^2 M_k^{(1)}[y](x)}{-(1-x^2)} =&(x+1)^2 y''(x) + (x+1)\left(\frac{-2x}{1-x}\right) y'(x) \\ &+ \left(\frac{-k(x+1)}{1-x}\right) y(x).\end{aligned}$$

So,

$$\begin{aligned}c_0(x) =&\frac{-k(x+1)}{1-x}, \\ c_1(x) =&\frac{-2x}{1-x}, \\ c_2(x) =&1.\end{aligned}$$

Near the endpoint $x = 1$, we have

$$\begin{aligned}\frac{(x-1)^2 M_k^{(1)}[y](x)}{-(1-x)^2} =&(x-1)^2 y''(x) + (x+1)\left(\frac{2x}{1+x}\right) y'(x) \\ &+ \left(\frac{k(x-1)}{x+1}\right) y(x),\end{aligned}$$

so we have

$$\begin{aligned}c_0(x) =&\frac{k(x-1)}{x+1}, \\ c_1(x) =&\frac{2x}{1+x}, \\ c_2(x) =&1.\end{aligned}$$

All points $x \in (-1,1)$ are regular points. Now we will solve the indicial equations at $x = -1$ and $x = 1$.

The indicial equation at $x = -1$ is

$$\begin{aligned}0 =&\sum_{j=0}^{2} P(r,j) c_j \\ =&\frac{r!}{(r-0)!}\left(\frac{-k(-1+1)}{1-(-1)}\right) + \frac{r!}{(r-1)!}\left(\frac{-2(-1)}{1-(-1)}\right) + \frac{r!}{(r-2)!} \\ =&r + r(r-1) = r^2.\end{aligned}$$

Thus, the indicial equation at $x=-1$ has a root of multiplicity 2 at $r=0$. Therefore, from the method of Frobenius, a basis of solutions of the differential equation $M_k^{(1)}[y] =$ around $x=-1$ is $\{y_1, y_2\}$ where

$$y_1(x) = \sum_{j=0}^{\infty} a_j (x+1)^j, \quad a_0 \neq 0,$$
$$y_2(x) = \ln|x+1| \sum_{j=0}^{\infty} a_j (x+1) + \sum_{j=0}^{\infty} b_j (x+1)^j, \quad b_0 \neq 0.$$

Both of these series converge for $|x+1| < 2$.

The indicial equation at $x=+1$ is

$$\begin{aligned} 0 =& \sum_{j=0}^{2} P(r,j) c_j \\ =& \frac{r!}{(r-0)!}\left(\frac{k(1-1)}{1+1)}\right) + \frac{r!}{(r-1)!}\left(\frac{2(1)}{1+1}\right) + \frac{r!}{(r-2)!} \\ =& r + r(r-1) \\ =& r^2. \end{aligned}$$

Hence, just as with $x=-1$, the indicial equation at $x=+1$ has a root of multiplicity 2 at $r=0$. Thus, a basis of solutions of $M_k^{(1)}[y]=0$ around $x=+1$ is $\{y_1, y_2\}$ where y_1 and y_2 are given by

$$y_1(x) = \sum_{j=0}^{\infty} a_j (x-1)^j, \quad a_0 \neq 0,$$
$$y_2(x) = \ln|x-1| \sum_{j=0}^{\infty} a_j (x-1)^j + \sum_{j=0}^{\infty} b_j (x-1)^j, \quad b_0 \neq 0.$$

Both of these series converge for $|x-1| < 2$.

With these bases for solutions at $x=\pm 1$, it is clear that all solutions of $M_k^{(1)}[y]=0$ are Lebesgue square-integrable near the endpoints $x=\pm 1$. Therefore, $M_k^{(1)}[\cdot]$ is in the limit-circle case at both endpoints, and thus, by applying Theorem 1.25, we see that the deficiency index of the minimal operator generated by $M_k^{(1)}[\cdot]$ is $(2,2)$.

Example 1.31. Krall-Legendre

Consider the fourth-order Krall-Legendre differential expression $M_k^{(2)}[\cdot]$ defined by

$$M_k^{(2)}[y](x) = \big((1-x^2)^2y''(x)\big)'' - \big((4A(1-x^2)+8)y'(x)\big)' + ky(x)$$

on the interval $(-1,1)$. It will be shown that both $x=+1$ and $x=-1$ are regular singular endpoints, and all $x \in (-1,1)$ are regular points.

Since, near $x=1$, we can write

$$\begin{aligned}\frac{(x+1)^4 M_k^{(2)}[y](x)}{(1-x^2)^2} &= \left(\frac{k(x+1)^2}{(1-x)^2}\right) y(x) + \left(\frac{8Ax(x+1)}{(1-x)^2}\right)(x+1)y'(x) \\ &+ \left(\frac{4Ax^2+12x^2+4-4A}{(1-x)^2}\right)(x+1)^2y''(x) \\ &+ \left(\frac{-8x}{1-x}\right)(x+1)^3y'''(x) + (x+1)^4y^{(4)}(x),\end{aligned}$$

and we see that $x=-1$ is a regular singular endpoint of $M_k^{(2)}[\cdot]$ with

$$\begin{aligned}c_0(x) =& \frac{k(x+1)^2}{(1-x)^2},\\ c_1(x) =& \frac{8Ax(x+1)}{(1-x)^2},\\ c_2(x) =& \frac{4Ax^2+12x^2+4-4A}{(1-x)^2},\\ c_3(x) =& \frac{-8x}{1-x},\\ c_4(x) =& 1.\end{aligned}$$

Therefore, the indicial equation at $x=-1$ is

$$\begin{aligned}0 =& \sum_{j=0}^{4} P(r,j)c_j \\ =& \frac{r!}{(r-0)!}\left(\frac{k(-1+1)^2}{(1-(-1))^2}\right) + \frac{r!}{(r-1)!}\left(\frac{8A(-1)(-1+1)}{(1-(-1))^2}\right) \\ &+ \frac{r!}{(r-2)!}\left(\frac{4(A(-1)^2+12(-1)^2+4-4A}{(1-(-1))^2}\right) \\ &+ \frac{r!}{(r-3)!}\left(\frac{-8(-1)}{(1-(-1))}\right) + \frac{r!}{(r-4)!} \\ =& r(r-1)(4) + r(r-1)(r-2)(4) + r(r-1)(r-2)(r-3)\end{aligned}$$

$$=r(r-1)[4+(r-2)(4)+(r-2)(r-3)]$$

$$=r(r-1)(r^2-r-2)$$

$$=r(r-1)(r-2)(r+1).$$

So, a basis of solutions to $M_k^{(2)}[y](x)=0$ around $x=-1$ is given by $\{y_1,\, y_2,\, y_3,\, y_4\}$ where

$$y_1(x)=\sum_{j=0}^{\infty} a_j(x+1)^{j+2},$$

$$y_2(x)=\sum_{j=0}^{\infty} b_j(x+1)^{j+1}+\ln(x-1)\sum_{j=0}^{\infty} b'_j(x+1)^{j+1},$$

$$y_3(x)=\sum_{j=0}^{\infty} c_j(x+1)^{j}+2\ln(x+1)\sum_{j=0}^{\infty} c'_j(x+1)^{j+2}$$

$$y_4(x)=\sum_{j=0}^{\infty} d_j(x+1)^{j-1}+3\ln(x+1)\sum_{j=0}^{\infty} d'_j(x+1)^{j},$$

and each of these series converges for $|x+1|<2$. Note that the solutions y_1, y_2, and y_3 are Lebesgue-square integrable near the endpoint $x=-1$. However, the solution y_4 is not Lebesgue-square integrable near $x=-1$, we see that fourth-order Krall-Legendre differential expression $M_k^{(2)}[\cdot]$ is in the limit-3 case at the endpoint $x=-1$.

There is a similar analysis at the endpoint $x=+1$. In fact, $M_k^{(2)}[\cdot]$ is in the limit-3 case at $x=+1$ as well. Hence, since $M_k^{(2)}[\cdot]$ is in the limit-3 case at both endpoints, we see from Theorem 1.25 that the deficiency index of the minimal operator generated by the differential expression $M_k^{(2)}[\cdot]$ in the Hilbert space $L^2(-1,1)$ is $(2,2)$.

1.5 The Classical Glazman-Krein-Naimark Theorem

Before we state the important Glazman-Krein-Naimark theorem, we first need to state one more definition. Let X be a vector space and $M_1\subset M_2$ be subspaces of X.

Definition 1.32. The set $\{x_1, x_2, \dots, x_n\} \subset M_2$ is said to be *linearly independent modulo M_1* if

$$\sum_{j=1}^{n} \alpha_j x_j \in M_1$$

implies that $\alpha_j = 0$, for $j = 1, 2, \dots, n$.

Then if $A \subset M_2$ is a maximal linearly independent set modulo M_1 and $\beta = card(A)$, then $\dim M_2 = \beta \mod M_1$.

It can easily be seen that if the set $\{x_1, x_2, \dots, x_n\} \subset M_2$ is a linearly independent set, then it is a maximal linearly independent set modulo M_1 if and only if

$$M_2 = M_1 \dotplus \operatorname{span}\{x_1, x_2, \dots, x_n\}.$$

If $\{x_1, x_2, \dots, x_n\} \subset M_2$ is linearly independent modulo M_1, then it is also a linearly independent set in the vector space X; however, the converse is not necessarily true. We now state the important Glazman-Krien-Naimark theorem, the proof of which can be found in [26].

Theorem 1.33 (GKN Theorem). *Suppose the deficiency index of the minimal operator T_0 in $L^2(a, b)$ generated by the expression $\ell[\cdot]$ is (m, m).*

(i) Let S be a self-adjoint extension of T_0 in $L^2(a, b)$. Then there exists a set $\{w_1, w_2, \dots, w_m\} \subset \mathcal{D}(S)$ that is linearly independent modulo $\mathcal{D}(T_0)$ such that

$$\begin{aligned} S[y] &= \ell[y] \\ \mathcal{D}(S) &= \left\{ y \in \mathcal{D}(T_1) \mid [w_j, y]\Big|_a^b = 0,\ j = 1, 2, \dots, m \right\}. \end{aligned} \tag{1.10}$$

Here, $[\cdot, \cdot]$ is the sesquilinear form defined in (1.2).

(ii) Suppose $\{w_1, w_2, \dots, w_m\} \subset \mathcal{D}(T)$ is linearly independent modulo $\mathcal{D}(T_0)$ with

$$[w_j, w_k]\big|_a^b = 0, \quad j = 1, 2, \dots, m.$$

Define an operator S in $L^2(a,b)$ by

$$\begin{aligned} S[y] =& \ell[y] \\ \mathcal{D}(S) =& \left\{ y \in \mathcal{D}(T_1) \mid [w_j, y]\Big|_a^b = 0,\ j = 1,\, 2,\, \ldots,\, m \right\}. \end{aligned}$$

Then S is a self-adjoint extension of T_0.

This theorem provides a recipe for constructing self-adjoint extensions of the minimal operator.

Definition 1.34. The conditions

$$[w_j,\, w_k]|_a^b = 0, \quad j = 1,\, 2,\, \ldots,\, m$$

given in (1.10) are known as Glazman-Naimark *boundary conditions* and the functional

$$[w_j, \cdot\,]\Big|_a^b : \mathcal{D}(T) \to \mathbb{C} \tag{1.11}$$

is called a *boundary value* for T_0. It is called a *separated boundary condition* such that $[w_j, y]\Big|_a^b = 0$ is independent of a or b for all $y \in \mathcal{D}(S)$. If (1.11) is not a separated boundary condition, it is then called a *mixed boundary condition.*

The GKN theorem can be generalized for arbitrary symmetric operators, as in [4].

CHAPTER TWO

The Everitt-Littlejohn-Loveland Approach to the Spectral Study of the Krall Sixth-Order Differential Expression

For more on this topic, see [25] and [8].

2.1 Nonclassical Differential Equations

In 1938, H. L. Krall [19], the father of Allan Krall and the supervisor of Littlejohn at Pennsylvania State University, found three fourth-order nonclassical differential equations of the form

$$\sum_{i=1}^{4}\left(\sum_{j=0}^{i}\ell_{ij}x^j\right)y^{(i)}(x)=\lambda_n y(x)$$

having orthogonal polynomial solutions $\{p_n\}$ corresponding to $\{\lambda_n\}$, where, necessarily,

$$\lambda_n=\sum_{j=0}^{4}P(n,j)\ell_{ij},$$

and

$$P(n,j)=\frac{n!}{(n-j)!}.$$

The three differential equations were classified as follows:

Table 2.1. Three fourth-order nonclassical differential equations

Weight	Interval	Name
$e^{-x}+\frac{1}{A}\delta(x)$	$[0,\infty)$	Laguerre type
$(1-x)^{\alpha}+\frac{1}{A}\delta(x)$, $(\alpha>-1)$	$[0,1]$	Jacobi type
$1+\frac{1}{A}\delta(x+1)+\frac{1}{A}\delta(x-1)$	$[-1,1]$	Legendre type

Note that the Jacobi polynomials are usually defined on the interval $[-1,1]$. However, in this case, the fourth-order Jacobi type differential equation is simpler on $[0,1]$. The differential equation could be considered on $[-1,0]$ with the weight $(1+x)^{\beta}+\frac{1}{B}\delta(x)$.

In the third case, the Legendre type polynomials, we note that if the jumps at the endpoints $x=\pm 1$ are different, then the order of the differential equation jumps from four to six, which yields the Krall sixth-order differential equation.

2.2 The Sixth-Order Krall Differential Equation

The Krall sixth-order differential equation is given by

$$\begin{aligned}\ell_6[y](x) =& \left(x^2-1\right)^3 y^{(6)} + 18\left(x^2-1\right)^2 y^{(5)} \\ & + \left(x^2-1\right)\left((3A+3B+96)\,x^2 - 3A - 3B - 36\right) y^{(4)} \\ & + (24A+24B+168)\,x\left(x^2-1\right) y^{(3)} \\ & + \Big(\,(12AB+42A+42B+72)\,x^2 + (12B-12A)\,x \\ & \quad - (12AB+30A+30B+72)\,\Big) y'' \\ & + ((24AB+12A+12B)\,x + (12B-12A))\,y' \\ =& \lambda_n y,\end{aligned} \tag{2.1}$$

where $A, B > 0$ and

$$\lambda_n = n(n-1)\left(n^4+2n^3+(3A+3B-1)\,n^2+(3A+3B-2)\,n+12AB\right). \tag{2.2}$$

In Lagrangian symmetric form, the differential equation becomes

$$\begin{aligned}\ell_6[y](x) =& -\left(\left(1-x^2\right)^3 y^{(3)}\right)^{(3)} + \left(\left(1-x^2\right)\left(12+(3A+3B+6)\left(1-x^2\right)\right) y''\right)'' \\ & - \big(\big((-6A-6B-12AB)\,x^2 \\ & \quad + (12A-12B)\,x + (12AB+18A+18B+24)\big)\,y'\big)'.\end{aligned}$$

One solution to $\ell_6[y]=\lambda_n y$ is $y(x)=K_n(x)$ $(n\in\mathbb{N}_0)$, where

$$K_n(x) = \sum_{j=0}^{n} \frac{(-1)^{\lfloor \frac{j}{2} \rfloor}(2n-j)!Q(n,j)x^{n-j}}{2^{n+1}(n-\lfloor\frac{j+1}{2}\rfloor)!\,\lfloor\frac{j}{2}\rfloor!(n-j)!(n^2+n+A+B)}, \tag{2.3}$$

is the n^{th} Krall polynomial. Here

$$\begin{aligned} Q(n,j) =& \frac{2+(-1)^j}{2}\left((n^4+(2A+2B-1)n^2+4AB)+2j(n^2+n+A+B)\right) \\ &+\frac{1-(-1)^j}{2}(4B-4A). \end{aligned}$$

The Krall polynomials were first studied by Littlejohn [23]. The Krall polynomials $\{K_n\}$ form a complete orthogonal set in $L^2([-1,1];W(x))$ where

$$W(x)=\frac{1}{A}\delta(x+1)+\frac{1}{B}\delta(x-1)+1.$$

The maximal domain for $\ell_6[\cdot]$ in $L^2(-1,1)$ is

$$\begin{aligned} \Delta = \big\{ f:(-1,1)\to\mathbb{C} \mid f^{(j)} \in AC_{\text{loc}}(-1,1)\ (j=0,1,2,3,4,5); \\ f, \ell_6[f] \in L^2(-1,1)\big\}. \end{aligned} \tag{2.4}$$

On Δ, we have Green's Formula

$$\int_{-1}^{1} \ell_6[f](x)\overline{g}(x)dx - \int_{-1}^{1} f(x)\overline{\ell_6[g]}(x)dx = [f,g](x)\Big|_{-1}^{1}.$$

Here $[f,g](x)$ is the skew-symmetric sesquilinear form defined by

$$\begin{aligned} [f,g](x) = & \Big(-\left(\left(1-x^2\right)^3 f^{(3)}(x)\right)'' + \left(\left(1-x^2\right)\left(12+\alpha\left(1-x^2\right)\right)f''(x)\right)' \\ & - \pi(x)f'(x)\Big)\overline{g}(x) \\ & - \Big(-\left(\left(1-x^2\right)^3 \overline{g}^{(3)}(x)\right)'' + \left(\left(1-x^2\right)\left(12+\alpha\left(1-x^2\right)\right)\overline{g}''(x)\right)' \\ & - \pi(x)\overline{g}'(x)\Big)f(x) \\ & - \Big(-\left(\left(1-x^2\right)^3 f^{(3)}(x)\right)' + \left(1-x^2\right)\left(12+\alpha\left(1-x^2\right)\right)f''(x)\Big)\overline{g}'(x) \\ & + \Big(-\left(\left(1-x^2\right)^3 \overline{g}^{(3)}(x)\right)' + \left(1-x^2\right)\left(12+\alpha\left(1-x^2\right)\right)\overline{g}''(x)\Big)f'(x) \\ & - \left(1-x^2\right)^3\left(f^{(3)}(x)\overline{g}''(x) - \overline{g}^{(3)}(x)f''(x)\right), \end{aligned} \tag{2.5}$$

where

$$\alpha = 3A+3B+6$$

and

$$\pi(x) = (-6A - 6B - 12AB)\, x^2 + (12A - 12B)\, x + (12AB + 18A + 18B + 24)\,. \tag{2.6}$$

We wish to study $\ell_6[y]$ not in $L^2(-1,1)$ but in $L^2([-1,1]; W(x))$.

We note that $L^2([-1,1]; W(x)) = L^2_\mu[-1,1]$, where

$$L^2_\mu[-1,1] = \left\{ f : [-1,1] \to \mathbb{C} \mid \int_{-1}^{1} |f|\, d\mu < \infty, f \text{ is Lebesgue measurable} \right\},$$

where

$$d\mu = \left(\frac{1}{A}\delta(x+1) + \frac{1}{B}\delta(x-1) + 1 \right) dx$$

and μ is the non-decreasing function

$$\mu(x) = \begin{cases} -1 - \frac{1}{A}, & x \le -1 \\ x, & -1 < x < 1 \\ 1 + \frac{1}{B}, & x \ge 1. \end{cases}$$

Then $L^2_\mu[-1,1]$ is a Hilbert space with inner product

$$\begin{aligned} (f,g)_\mu :=& \int_{[-1,1]} f(t)\overline{g}(t) d\mu(t) \\ =& \frac{f(-1)\overline{g}(-1)}{A} + \int_{-1}^{1} f(t)\overline{g}(t) dt + \frac{f(1)\overline{g}(1)}{B}. \end{aligned} \tag{2.7}$$

Note that $L^2_\mu[-1,1] \approx L^2(-1,1) \oplus \mathbb{C}^2$, which will be important in Chapter Five.

The endpoints $x = \pm 1$ are regular singular endpoints of $\ell_6[y]$. At both endpoints $x = \pm 1$, the indicial equation is given by

$$(r-3)(r-2)(r-1)^2 r(r+1) = 0,$$

and $\ell_6[y]$ is limit-5 at $x = \pm 1$. The six Frobenius solutions to $\ell_6[y] = 0$ at $x = 1$ are

$$\begin{aligned} y_3(x) =& (x-1) \sum_{n=0}^{\infty} a_n (x-1)^n, & a_0 \neq 0, \\ y_2(x) =& (x-1)^2 \left(\log|x-1|\right) \sum_{n=0}^{\infty} h_n (x-1)^n + (x-1)^2 \sum_{n=0}^{\infty} b_n (x-1)^n, & b_0 \neq 0, \end{aligned}$$

$$
\begin{aligned}
y_1(x) =&(x-1)\left(\log|x-1|\right)^2\sum_{n=0}^{\infty} t_n(x-1)^n \\
&+(x-1)\left(\log|x-1|\right)\sum_{n=0}^{\infty} j_n(x-1)^n+(x-1)\sum_{n=0}^{\infty} c_n(x-1)^n, \qquad c_0\neq 0,\\
\widehat{y}_1(x) =&(x-1)\left(\log|x-1|\right)^3\sum_{n=0}^{\infty} t_n(x-1)^n \\
&+3(x-1)\left(\log|x-1|\right)^2\sum_{n=0}^{\infty} j_n(x-1)^n \\
&+3(x-1)\left(\log|x-1|\right)\sum_{n=0}^{\infty} c_n(x-1)^n+(x-1)\sum_{n=0}^{\infty} d_n(x-1)^n, \quad d_0\neq 0,\\
y_0(x) =& \left(\log|x-1|\right)^4\sum_{n=0}^{\infty} k_n(x-1)^n+\left(\log|x-1|\right)^3\sum_{n=0}^{\infty} l_n(x-1)^n \\
&+\left(\log|x-1|\right)^2\sum_{n=0}^{\infty} m_n(x-1)^n+\left(\log|x-1|\right)\sum_{n=0}^{\infty} f_n(x-1)^n \\
&+\sum_{n=0}^{\infty} e_n(x-1)^n, \qquad e_0\neq 0,\\
y_{-1}(x) =&(x-1)^{-1}\left(\log|x-1|\right)^5\sum_{n=0}^{\infty} p_n(x-1)^n \\
&+(x-1)^{-1}\left(\log|x-1|\right)^4\sum_{n=0}^{\infty} q_n(x-1)^n \\
&+(x-1)^{-1}\left(\log|x-1|\right)^3\sum_{n=0}^{\infty} r_n(x-1)^n \\
&+(x-1)^{-1}\left(\log|x-1|\right)^2\sum_{n=0}^{\infty} s_n(x-1)^n \\
&+(x-1)^{-1}\left(\log|x-1|\right)\sum_{n=0}^{\infty} g_n(x-1)^n \\
&+(x-1)^{-1}\sum_{n=0}^{\infty} h_n(x-1)^n, \qquad h_0\neq 0.
\end{aligned}
$$

These six solutions can be simplified to the forms

$$
\begin{aligned}
y_3(x) =&(x-1)^3\sum_{n=0}^{\infty} a_n(x-1)^n, \qquad a_0\neq 0, \qquad (2.8)\\
y_2(x) =&(x-1)^2\sum_{n=0}^{\infty} b_n(x-1)^n
\end{aligned}
$$

$$+ (x-1)^2 \log|x-1| \sum_{n=1}^{\infty} c_n (x-1)^n, \qquad b_0, c_1 \neq 0, \tag{2.9}$$

$$y_1(x) = (x-1) \sum_{n=0}^{\infty} d_n (x-1)^n, \qquad d_n \neq 0, \tag{2.10}$$

$$\widehat{y}_1(x) = (x-1) \sum_{n=0}^{\infty} e_n (x-1)^n$$

$$+ 3(x-1) \log|x-1| \sum_{n=0}^{\infty} f_n (x-1)^n, \qquad e_0, f_0 \neq 0, \tag{2.11}$$

$$y_0(x) = \sum_{n=0}^{\infty} g_n (x-1)^n + 4 \log|x-1| \sum_{n=1}^{\infty} h_n (x-1)^n, \qquad g_0, h_1 \neq 0, \tag{2.12}$$

$$y_{-1}(x) = (x-1)^{-1} \sum_{n=0}^{\infty} j_n (x-1)^n$$

$$+ 5(x-1)^{-1} \log|x-1| \sum_{n=1}^{\infty} k_n (x-1)^n, \qquad j_0, k_1 \neq 0. \tag{2.13}$$

All the series converge for $|x-1| < 2$. The five solutions that belong to Δ are $y_3, y_2, y_1, \widehat{y}_1, y_0$. Furthermore, four of these solutions, y_3, y_2, y_1, y_0, satisfy

$$y'' \in L^2(-1,1),$$

but

$$\widehat{y}_1 \notin L^2(-1,1),$$

and so $\widehat{y}_1$ will cause a problem in $[\cdot,\cdot]$, as defined in (2.5). Therefore, $\widehat{y}_1$ must be eliminated from the domain of the self-adjoint operator we will construct.

We will analogously denote the five solutions of $\ell_6[\cdot]$ at $x=-1$ which are in $L^2(-1,1)$ by $z_3, z_2, z_1, \widehat{z}_1$ and z_0. Denote the solution of $\ell_6[\cdot]$ that is not in $L^2(-1,1)$ at the endpoint $x=-1$ by z_{-1}.

2.3 Basic Properties of the Maximal Domain

In [25], Loveland proved properties of the maximal domain Δ of $\ell_6[\cdot]$. In order to state this theorem, we must first define an important function:

$$\Lambda[f](x) = -\left(\left(1-x^2\right)^3 f^{(3)}(x)\right)' + \left(1-x^2\right)\left(12 + \alpha\left(1-x^2\right)\right) f''(x). \tag{2.14}$$

Theorem 2.1. *Let* $f, g \in \Delta$. *Then*

(i) $f' \in L^2(-1,1)$;

(ii) $f \in AC[-1,1]$;

(iii) $\lim_{x\to\pm1}(1-x^2)^s f^{(s)}(x) = 0$ *for* $s = 1,2,3$;

(iv) $1 \in \Delta$ *and* $\lim_{x\to\pm1}[f,1](x) = \lim_{x\to\pm1}[\Lambda'[f](x) - \pi(x)f'(x)]$;

(v) $(1-x^2) \in \Delta$ *and* $\lim_{x\to1}[f,1-x^2](x) = 2\Lambda[f](1) - 48(A+2)f(1)$,
$\lim_{x\to-1}[f,1-x^2](x) = -2\Lambda[f](-1) + 48(B+2)f(-1)$;

(vi) $(1-x^2)^2 \in \Delta$ *and* $\lim_{x\to\pm1}[f,(1-x^2)^2](x) = \pm192 f(\pm1)$;

(vii) $h_\pm \in \Delta$ *where* $h_\pm \in \mathbb{C}^6(-1,1)$ *are defined by*

$$h_+(x) = \begin{cases} 0 & x \text{ near} -1 \\ \frac{1}{8}(A+2)(1-x^2)^2\ln(1-x^2) + \frac{1}{2}(1-x^2)\ln(1-x^2) & x \text{ near } 1, \end{cases}$$

$$h_-(x) = \begin{cases} \frac{1}{8}(B+2)(1-x^2)^2\ln(1-x^2) + \frac{1}{2}(1-x^2)\ln(1-x^2) & x \text{ near } -1 \\ 0 & x \text{ near } 1. \end{cases}$$

and

$$\begin{aligned} \lim_{x\to+1}[f,h_+](x) = &+4[8A+3B-4]f(1) + \lim_{x\to+1}\Big(-\Lambda[f](x)h_+'(x) + 32f'(x) \\ &- (1-x^2)^3\left(f^{(3)}(x)h_+''(x) - h_+^{(3)}(x)f''(x)\right)\Big), \\ \lim_{x\to-1}[f,h_-](x) = &-4[8B+3A-4]f(-1) + \lim_{x\to-1}\Big(-\Lambda[f](x)h_-'(x) \\ &+ 32f'(x) - (1-x^2)^3\left(f^{(3)}(x)h_-''(x) - h_-^{(3)}(x)f''(x)\right)\Big); \end{aligned}$$

(viii) $(1-x^2)^3 \in \Delta$ *and* $\lim_{x\to\pm1}[f,(1-x^2)^3](x) = 0$; *hence,* $(1-x^2)^3$ *is in the domain of the minimal operator in* $L^2(-1,1)$;

(ix)

$$\begin{aligned}\lim_{x\to\pm1}[f,g](x) = [f,1](\pm1)\overline{g}(\pm1) - [\overline{g},1](\pm1)f(\pm1)\\ + \lim_{x\to\pm1}\Big(-\Lambda[f](x)\overline{g}'(x) + \Lambda[\overline{g}](x)f'(x)\\ - \left(1-x^2\right)^3\left(f^{(3)}(x)\overline{g}''(x) - \overline{g}^{(3)}(x)f''(x)\right)\Big).\end{aligned}$$

The proof of this result is omitted, but the proofs of parts (i)-(iii) depend on a result by R. S. Chisholm and Everitt [3].

Since $\widehat{y}_1 \in \Delta$ but $\widehat{y}_1'' \notin L^2(-1,1)$, we see that Theorem 2.1 is as strong as possible. It was expected that there would be a restriction δ of Δ such that $f^{(3)} \in L^2(-1,1)$ whenever $f \in \delta$. To construct the space δ, we first define $e_\pm \in C^6[-1,1]$ by

$$e_+(x) = \begin{cases} 0 & x \text{ near } -1 \\ \frac{1}{2}(1-x^2) + \frac{1}{8}(A+2)(1-x^2)^2 & x \text{ near } 1 \end{cases} \tag{2.15}$$

and

$$e_-(x) = \begin{cases} -\frac{1}{2}(1-x^2) - \frac{1}{8}(B+2)(1-x^2)^2 & x \text{ near } -1 \\ 0 & x \text{ near } 1. \end{cases} \tag{2.16}$$

Then $e_\pm \in \Delta$. We can now define δ, a restriction of Δ, by

$$\delta := \{f \in \Delta \mid [f,e_+](1) = [f,e_-](-1) = 0\}. \tag{2.17}$$

In [25], the following result is shown:

Lemma 2.2. *Let $f \in \Delta$. Then $f \in \delta$ if and only if $\Lambda[f](\pm1) = 0$, where Λ is as defined in (2.14).*

The following properties of δ are given in [25]:

Theorem 2.3. *Let $f, g \in \delta$. Then* *(i)* $f^{(3)} \in L^2(-1,1)$*;*

(ii) $f^{(s)} \in AC[-1,1]$*, for* $s = 0, 1, 2$*;*

(iii) $\lim_{x\to\pm 1} \left[(1-x^2)^3 f^{(3)}(x)\right]^{(s)} = 0$*, for* $s = 1, 2$*;*

(iv) $\lim_{x\to\pm 1} (1-x^2) f^{(3)}(x) = 0$*;*

(v) $\lim_{x\to\pm 1} (1-x^2)^3 f^{(3)}(x)\overline{g}''(x) = 0$*;*

(vi) $1 \in \delta$*,* $\lim_{x\to+1} [f,1](x) = -24f''(1) - 24(A+1)f'(1)$*, and* $\lim_{x\to-1} [f,1](x) = 24f''(-1) - 24(B+1)f'(-1)$*;*

(vii) $(1-x^2) \in \delta$*,* $\lim_{x\to+1} \left[f, 1-x^2\right](x) = -48(A+2)f(1)$*, and* $\lim_{x\to-1} \left[f, 1-x^2\right](x) = 48(B+2)f(-1)$*;*

(viii) $(1-x^2)^2 \in \delta$ *and* $\lim_{x\to\pm 1} \left[f, (1-x^2)^2\right](x) = \pm 192 f(\pm 1)$*;*

(ix) $(1-x^2)^3 \in \delta$ *and* $\lim_{x\to\pm 1} \left[f, (1-x^2)^3\right](x) = 0$*;*

(x)

$$\begin{aligned}
\lim_{x\to+1} [f,g](x) = &- 24\left[f''(1)\overline{g}(1) - \overline{g}''(1)f(1)\right] \\
&- 24(A+1)\left[f'(1)\overline{g}(1) - \overline{g}'(1)f(1)\right]; \\
\lim_{x\to-1} [f,g](x) = &+ 24\left[f''(-1)\overline{g}(-1) - \overline{g}''(-1)f(-1)\right] \\
&- 24(B+1)\left[f'(-1)\overline{g}(-1) - \overline{g}'(-1)f(-1)\right].
\end{aligned}$$

Then, by part (vi) of Theorem 2.3, for $f \in \delta$, we have

$$[f,1](1) = -24f''(1) - 24(A+1)f'(1)$$

and

$$[f,1](-1) = 24f''(-1) - 24(B+1)f'(-1).$$

2.4 Constructing a Self-Adjoint Operator

The Everitt-Littlejohn-Loveland approach to studying $\ell_6[\cdot]$ and, in particular, for finding the self-adjoint operator T generated by $\ell_6[\cdot]$ in $L^2_\mu[-1,1]$, is to first study $\ell_6[\cdot]$ in $L^2(-1,1)$, not $L^2_\mu[-1,1]$. In fact, we study $\ell_6[\cdot]$ on Δ and check Δ for "smoothness" properties. This approach worked in previous work by Everitt and Littlejohn in their study of H. L. Krall's three 4^{th}-order equations. From this analysis of Δ, the second step is to define T in $L^2_\mu[-1,1]$ on a (dense) subspace $\mathcal{D}(T)$ by using what we learned about smoothness of functions in Δ.

Define the operator T in $L^2_\mu[-1,1]$ by

$$T[f](x) := \begin{cases} 24A\Big(f''(-1) - (B+1)f'(-1)\Big) & x = -1 \\ \ell_6[f](x) & -1 < x < 1 \\ 24B\Big(f''(1) + (A+1)f'(1)\Big) & x = 1 \end{cases}$$

$$\mathcal{D}(T) = \delta.$$

In [25], the author proves the following result:

Theorem 2.4. *T is symmetric in $L^2_\mu[-1,1]$ and $T \geq 0$.*

Proof. Let $f, g \in \mathcal{D}(T)$. Then, by Theorem 2.3, we can write Green's formula as

$$\begin{aligned}\int_{-1}^{1} \ell_6[f](t)\overline{g}(t)dt =& [f,1](1)\overline{g}(1) - [\overline{g},1](1)f(1) - [f,1](-1)\overline{g}(-1) \\ &+ [\overline{g},1](-1)f(-1) + \int_{-1}^{1} \ell_6[\overline{g}](t)f(t)dt.\end{aligned}$$

Then, from the definition of the inner product in $L^2_\mu[-1,1]$ given in (2.7), we have

$$\begin{aligned}(T[f],g)_\mu =& \frac{T[f](1)\overline{g}(1)}{B} + \int_{-1}^{1} \ell_6[f](t)\overline{g}(t)dt + \frac{T[f](-1)\overline{g}(-1)}{A} \\ =& -24\left(f''(1) + (A+1)f'(1)\right)\overline{g}(1) + 24\left(f''(1) + (A+1)f'(1)\right)\overline{g}(1) \\ & -24\left(\overline{g}''(1) + (A+1)\overline{g}'(1)\right)f(1) - 24\left(f''(-1) - (B+1)f'(-1)\right)\overline{g}(-1)\end{aligned}$$

$$
\begin{aligned}
&+ 24\left(\overline{g}''(-1) - (B+1)\overline{g}'(-1)\right) f(-1) \\
&+ \int_{-1}^{1} \ell_6[\overline{g}(t) f(t) dt + 24\left(f''(-1) - (B+1)f'(-1)\right)\overline{g}(-1) \\
=& \frac{T[\overline{g}](1) f(1)}{B} + \int_{-1}^{1} \ell_6[\overline{g}](t) f(t) dt + \frac{T[\overline{g}](-1) f(-1)}{A} \\
=& \left(f, T[g]\right)_\mu .
\end{aligned}
\tag{2.18}
$$

Therefore, the operator T is Hermitian. Since $C_0^\infty[-1,1] \subset \mathcal{D}(T)$ and $C_0^\infty[-1,1]$ is dense in $L_\mu^2[-1,1]$, we see that $\mathcal{D}(T)$ is dense in $L_\mu^2[-1,1]$. Hence, T is symmetric in $L_\mu^2[-1,1]$. Using properties of δ given in Theorem 2.3, Dirichlet's formula

$$
\begin{aligned}
&\int_a^b \Big((1-x^2)^3 f^{(3)}(x)\overline{g}^{(3)}(x) + (1-x^2)(12+\alpha(1-x^2)) f''(x)\overline{g}''(x) \\
&\qquad + \pi(x) f'(x)\overline{g}'(x)\Big) dx \\
=& -\Big(-[(1-x^2)^3 f^{(3)}(x)]'' + \big[(1-x^2)(12+\alpha(1-x^2)) f''(x)\big]' \\
&\qquad - \pi(x) f'(x)\Big)\overline{g}(x)\Big|_a^b \\
&+ \Big(-[(1-x^2)^3 f^{(3)}(x)]' + (1-x^2)(12+\alpha(1-x^2)) f''(x)\Big)\overline{g}'(x)\Big|_a^b \\
&+ (1-x^2)^3 f^{(3)}(x)\overline{g}''(x)\Big|_a^b + \int_a^b \ell_6[f](x)\overline{g}(x) dx
\end{aligned}
\tag{2.19}
$$

becomes, for $f, g \in \delta$,

$$
\begin{aligned}
\int_{-1}^{1} \ell_6[\overline{g}](t) f(t) dt = &\int_{-1}^{1} \Big\{ (1-t^2)^3 f^{(3)}(t)\overline{g}^{(3)}(t) \\
&\qquad + (1-t^2)(12+\alpha(1-t^2)) f''(t)\overline{g}''(t) + \pi(t) f(t)\overline{g}'(t)\Big\} dt \\
&+ 24\left(g''(1) + (A+1)g'(1)\right) f(1) \\
&- 24\left(g''(-1) - (B+1)g'(-1)\right) f(-1).
\end{aligned}
$$

Combining this with (2.18), we have

$$
\begin{aligned}
(T[f], g)_\mu = &\int_{-1}^{1} \Big\{ (1-t^2)^3 f^{(3)}(t)\overline{g}^{(3)}(t) \\
&\qquad + (1-t^2)(12+\alpha(1-t^2)) f''(t)\overline{g}''(t) + \pi(t) f'(t)\overline{g}'(t)\Big\} dt.
\end{aligned}
$$

Since, on the interval $(-1,1)$,

$$(1-x^2)^3 \left|f^{(3)}(x)\right|^2 + (1-x^2)(12+\alpha(1-x^2)) \left|f''(x)\right|^2 + \pi(x) \left|f'(x)\right|^2 \geq 0,$$

we have

$$\begin{aligned}(T[f],f)_\mu = &\int_{-1}^{1} \Big\{(1-t^2)^3 \left|f^{(3)}(t)\right|^2 + (1-t^2)(12+\alpha(1-t^2)) \left|f''(t)\right|^2 \\ &\qquad + \pi(t) \left|f'(t)\right|^2 \Big\} dt \\ \geq &\ 0.\end{aligned}$$

Thus, T is bounded below by 0. □

Now define a related operator A in $L^2(-1,1)$ by

$$\begin{aligned}A[f] =& \ell_6[f] \\ \mathcal{D}(A) =& \{f \in \Delta \mid [f,e_-](-1) = [f,e_+](1) = [f,1_-](-1) = [f,1_+](1) = 0\},\end{aligned}$$

where $1_\pm \in C^6[-1,1]$ are defined by

$$1_+ := \begin{cases} 1 & \text{for } x \text{ near } 1 \\ 0 & \text{for } x \text{ near } -1 \end{cases}$$

and

$$1_- := \begin{cases} 0 & \text{for } x \text{ near } 1 \\ 1 & \text{for } x \text{ near } -1. \end{cases}$$

In [25], the following result is proven about the operator A:

Theorem 2.5. *A is self-adjoint in $L^2(-1,1)$. Furthermore, $A \geq 0$.*

Proof. Since $\ell_6[\cdot]$ is limit-5 at both $x = \pm 1$, the Naimark theory for constructing self-adjoint operators requires, in the separated case, two boundary conditions at each endpoint. Since the four functions $1_\pm$ and $e_\pm$ are linearly independent modulo the minimal domain of $\ell_6[\cdot]$ in $L^2(-1,1)$ and satisfy the Naimark symmetry conditions

$$[e_\pm, e_\pm](\pm 1) = [1_\pm, 1_\pm](\pm 1) = [e_\pm, 1_\pm](\pm 1) = 0,$$

we see that A is self-adjoint.

Since $\mathcal{D}(A) \subset \delta$, it follows from Dirichlet's formula (2.19) and Theorem 2.3 that

$$\begin{aligned}(A[f], f) &= \int_{-1}^{1} \ell_6[f](t)\overline{f}(t)dt \\ &= \int_{-1}^{1} \Big\{(1-t^2)^3 \left|f^{(3)}(t)\right|^2 + (1-t^2)(12+\alpha(1-t^2)) \left|f''(t)\right|^2 \\ &\qquad + \pi(t) \left|f'(t)\right|^2 \Big\} dt \\ &\geq 0.\end{aligned}$$

□

In order to prove that the operator T is self-adjoint, we first introduce yet another related operator, B:

$$B[f] := \begin{cases} 24A\left(f''(-1) - (B+1)f'(-1)\right) & x = -1 \\ \ell_6[f](x) & -1 < x < 1 \\ -24B\left(f''(1) + (A+1)f'(1)\right) & x = 1 \end{cases}$$

$$\mathcal{D}(B) := \delta.$$

Theorem 2.6. *B is self-adjoint in $L^2_\mu[-1,1]$.*

The proof of this result relies on properties of the operator A and of the solutions of $\ell_6[\cdot]$. Recall that the solutions at $x = 1$, $y_3, y_2, y_1, \widehat{y}_1, y_1$, and y_{-1}, were given by (2.8) through (2.13) and the solutions at $x = -1$ are $z_3, z_2, z_1, \widehat{z}_1, z_0$, and z_{-1}. Note that $y_3, y_2, y_1, \widehat{y}_1, y_1$ are linearly dependent on $z_3, z_2, z_1, \widehat{z}_1, z_0$. The proof constructs the unique linearly independent functions $\varphi_+, \varphi_- \in \delta$ that are linear combinations of $z_3, z_2, z_1, \widehat{z}_1, z_0$. These two functions, and the resolvent operator of A, which exists, is bounded, and maps $L^2(-1,1)$ onto $\mathcal{D}(A)$, are critical components of the proof of the above result.

With the above results, we can now state this chapter's main result.

Theorem 2.7. *T is self-adjoint in $L^2_\mu[-1,1]$.*

The proof of this theorem relies on the properties of the operators A and B. We can now use this result to give a characterization of the spectrum of the operator T, as stated in [25]:

Theorem 2.8. *(i) The Krall polynomials $\{K_n\}_{n=0}^{\infty}$, as defined in (2.3), form a complete set of orthogonal eigenfunctions for the self-adjoint operator T in $L^2_\mu[-1,1]$.*

(ii) The spectrum of T in $L^2_\mu[-1,1]$ is given by

$$\sigma(T) = \{\lambda_n \mid n = 0, 1, 2, \ldots\},$$

where λ_n is as defined in (2.2). So, T has a discrete spectrum that is bounded below and all eigenvalues of T are simple.

The proof is omitted, but can be found in [25], but the proof of part (ii) follows from the fact that T is self-adjoint (and so the residual spectrum is empty) and that $\{\lambda_n\}$ has no finite accumulation points (and so the continuous spectrum is empty [28]).

CHAPTER THREE

Extended GKN Theory

The Glazman-Krein-Naimark Theorem, Theorem 1.33, can be generalized to an arbitrary closed symmetric operator with equal and finite deficiency indices in an arbitrary Hilbert space. This generalization is stated in the GKN-EM Theorem, Theorem 3.22, which will be discussed in this chapter, and is an application of general complex symplectic theory developed by Everitt and Markus. This theory and its applications to linear ordinary differential equations and partial differential equations was developed by W. N. Everitt and L. Markus in the papers [10], [11], [12], and [13].

The motivation and framework of this extended theory is nicely summed up in this quote from Everitt, Littlejohn, and Wellman in [9]:

> The GKN theory provides a recipe, in theory, for determining all self-adjoint extensions in the Hilbert space $L^2(I;w)$ of formally symmetric differential expressions of the form
>
> $$\ell_{2r}[y](u) = \frac{1}{w(u)} \sum_{j=0}^{r} (-1)^j (q_j(u) y^{(j)}(u))^{(j)} \quad (u \in I) \tag{3.1}$$
>
> on some open interval $I = (a, b)$; we assume here that $w > 0$ and each coefficient q_j is sufficiently differentiable on I. This theory works well in developing the spectral theory for the second-order classical differential equations of Jacobi, Laguerre, and Hermite.
>
> However, for nonclassical symmetric differential equations (3.1) with orthogonal polynomial solutions, the appropriate right-definite setting is a Hilbert-Sobolev space S with orthogonalizing Sobolev inner product
>
> $$\langle f, g\rangle = \int_a^b f(u)\overline{g}(u)w(u)du + \sum_{j=0}^{p} \left(\alpha_j f^{(j)}(a)\overline{g}^{(j)}(a) + \beta_j f^{(j)}(b)\overline{g}^{(j)}(b)\right). \tag{3.2}$$
>
> The Sobolev space S has the form $L^2(I;w) \oplus \mathbb{C}^k$ for some $k \leq 2p$. Develop a general GKN-type theory for this setting; in particular,

provide a 'recipe' for determining the self-adjoint operator having the orthogonal polynomials as eigenfunctions.

Bochner's 1929 Classification Theorem [2] states that the only orthogonal polynomials, with respect to a positive measure on the real line, that satisfy second-order differential equations are the Laguerre, Jacobi, and Hermite. In [19, 20], H. L. Krall asked if there were other orthogonal polynomials that satisfy higher order differential equations. As mentioned in Chapter Two, Krall found three fourth-order differential equations, found in Table 2.1, that admit non-classical orthogonal polynomials. However, classical GKN theory does not apply directly to these equations because of the jumps of the orthogonalizing weight distribution at one or both endpoints in the interval or orthogonality. This requires a new theory, which culminates in the GKN-EM Theorem 3.22.

In [24], the authors build up to and state the GKN-EM Theorem in an extended Hilbert space (see theorem 5.4 at the end of this chapter). The results in the rest of this chapter, especially sections 3.2 and 3.3, are from [24], though many of the proofs are provided in this chapter. Motivation for the authors of [24] came from the work H. L. Krall [19, 20] and A. M. Krall [18], which concerned orthogonal polynomials being eigenfunctions of symmetric differential expressions. Some of their contributions resulted in [7] and [9]. The Legendre type self-adjoint operator was constructed in [5] and [6] and was more motivation for [24].

For the rest of this chapter, $(H, \langle\cdot,\cdot\rangle)_H$ will be a Hilbert space with its associated inner product, $T_0 : \mathcal{D}(T_0) \subseteq H \to H$ will be an arbitrary closed, symmetric, linear operator in H, and $T_1 : \mathcal{D}(T_1) \subseteq H \to H$ will be a linear operator such that

$$T_1^* = T_0 \subseteq T_0^* = T_1.$$

We call T_1 the 'maximal' operator and T_0 the 'minimal' operator.

3.1 Complex Symplectic Geometry

Definition 3.1. A *complex symplectic space* S is a complex vector space with a conjugate bilinear complex-valued function $[\cdot : \cdot] : \mathsf{S} \times \mathsf{S} \to \mathbb{C}$ satisfying the following properties:

(i) $[c_1 x_1 + c_2 x_2 : y] = c_1[x_1 : y] + c_2[x_2 : y]$,

(ii) $[x : y] = -\overline{[y : x]}$,

(iii) $[x : \mathsf{S}] = 0 \implies x = 0$ (non-degenerate condition).

We call $[\cdot : \cdot]$ a (non-degenerate) *symplectic form*.

Complex symplectic spaces are generalizations of classical real symplectic spaces of Lagrangian and Hamiltonian mechanics, see [15] for more. We note that while real symplectic spaces cannot be odd-dimensional, complex symplectic spaces can. Indeed, for every $n \in \mathbb{N}$, there exists complex symplectic spaces of dimension n.

Both real and complex symplectic spaces have the notion of Lagrangian subspaces.

Definition 3.2. A subspace L of a complex symplectic space S is called *Lagrangian* if $[\mathsf{L} : \mathsf{L}] = 0$; that is to say, when

$$[x : y] = 0 \text{ for all } x, y \in \mathsf{L}.$$

A Lagrangian $\mathsf{L} \subseteq \mathsf{S}$ is called a *complete Lagrangian* when

$$x \in \mathsf{S} \text{ and } [x : \mathsf{L}] = 0 \implies x \in \mathsf{L}.$$

The next result gives a characterization of complete Lagrangian subspaces.

Lemma 3.3. *A Lagrangian subspace* $\mathsf{L} \subseteq \mathsf{S}$ *is a complete Lagrangian subspace if and only if*

$$\mathsf{L} = \{x \in \mathsf{S} \mid [x : y] = 0,\ y \in \mathsf{L}\}.$$

Proof. Let S be a complex symplectic space and suppose $\mathsf{L} \subseteq \mathsf{S}$ is a complete Lagrangian subspace of S. Then, by the definition of complete Lagrangian, it can be seen that

$$\{x \in \mathsf{S} \mid [x : y] = 0,\, y \in \mathsf{L}\} \subseteq \mathsf{L}.$$

However, since L is Lagrangian, if $x \in \mathsf{L}$, then $[x : y] = 0$ for all $y \in \mathsf{L}$. Therefore,

$$\mathsf{L} \subseteq \{x \in \mathsf{S} \mid [x : y] = 0,\, y \in \mathsf{L}\},$$

and thus

$$\mathsf{L} = \{x \in \mathsf{S} \mid [x : y] = 0,\, y \in \mathsf{L}\}.$$

Conversely, if L is a Lagrangian subspace given by

$$\mathsf{L} \subseteq \{x \in \mathsf{S} \mid [x : y] = 0,\, y \in \mathsf{L}\},$$

then is is clear that L is complete. □

An important step in moving forward with the work of Everitt and Markus is to generalize the skew-symmetric bilinear form $[\cdot,\cdot]|_\alpha^\beta$ given by Green's formula in (1.3):

Definition 3.4. $[x, y]_H := \langle T_1 x, y\rangle_H - \langle x, T_1 y\rangle_H$ for $x, y \in \mathcal{D}(T_1)$.

In [13], Everitt and Markus show that the quotient space

$$\mathsf{S}' := \mathcal{D}(T_1)/\mathcal{D}(T_0) \tag{3.3}$$

with the zero element $0 = \mathcal{D}(T_0)$ and endowed with the form $[\cdot,\cdot]_H$, is a complex symplectic space. Since

$$\operatorname{def}(T_0) = \dim(\mathcal{D}_+) = \dim(\mathcal{D}_-)$$

and

$$T_1 = \mathcal{D}(T_0) + \mathcal{D}_+ + \mathcal{D}_-,$$

we see that

$$\operatorname{def}(\mathsf{S}') = 2\operatorname{def}(T_0).$$

In fact, the space S' can be viewed, by von Neumann's Theorem 1.21, as an isomorphic copy of the orthogonal sum of $\mathcal{D}_+$ and $\mathcal{D}_-$, the deficiency spaces of T_0. Everitt and Markus call the quotient space S' the *boundary space* of T_0. The elements of the boundary space are cosets

$$\mathsf{x} = \{x + \mathcal{D}(T_0)\}$$

for $x \in \mathcal{D}(T_1)$. The element $x \in \mathcal{D}(T_1)$ is called the *representative vector* of the coset $\{x + \mathcal{D}(T_0)\}$. We next consider the projection from $\phi : \mathcal{D}(T_1) \to \mathsf{S}'$ given by

$$\phi(x) = \{x + \mathcal{D}(T_0)\}.$$

Lemma 3.5. *A collection of cosets $\{\phi t_j\}_{j=1}^d \subset \mathcal{D}(T_1)$, is a basis for a subspace of dimension d of the boundary space S' if and only if the representative vectors $\{t_j\}_{j=1}^d$ satisfy*

$$\sum_{j=1}^{d} \alpha_j t_j \in \mathcal{D}(T_0) \implies \alpha_j = 0$$

for $j = 1, 2, \ldots, d$; which is to say, $\{t_j\}_{j=1}^d$ is linearly independent modulo $\mathcal{D}(T_0)$.

Proof. The proof follows from the fact that the condition $\sum_{j=1}^d \alpha_j t_j \in \mathcal{D}(T_0)$ is equivalent to the equation $\sum_{j=1}^d \alpha_j \phi t_j = 0$. $\square$

The next result will generalize the characterization of the domain of the minimal operator T_0.

Lemma 3.6. $\mathcal{D}(T_0) = \{x \in \mathcal{D}(T_1) \mid [x, y]_H = 0 \;\forall y \in \mathcal{D}(T_1)\}.$

Proof. Let $x \in \mathcal{D}(T_1)$ and suppose that

$$[x, y]_H = 0$$

for all $y \in \mathcal{D}(T_1)$. Then

$$\langle T_1 y, x\rangle_H = \langle y, T_1 x\rangle_H$$

since $[x,y]_H = -\overline{[y,x]}_H$. Therefore, $x \in \mathcal{D}(T_1^*) = \mathcal{D}(T_0)$. To prove the converse, let $x \in \mathcal{D}(T_0)$. Since $T_0^* = T_1$ and $T_0 x = T_1 x$, we observe that

$$\langle T_1 x, y\rangle_H = \langle T_0 x, y\rangle_H = \langle x, T_1 y\rangle_H$$

for all $y \in \mathcal{D}(T_1)$. Hence, for every $y \in \mathcal{D}(T_1)$, we have

$$[x,y]_H = \langle T_1 x, y\rangle_H - \langle x, T_1 y\rangle_H = 0,$$

and the proof is complete. □

This lemma allows us to equip the boundary space S' with a complex symplectic form.

Definition 3.7. The *boundary space symplectic form* is given by

$$[\phi x : \phi y]_{\mathsf{S}'} := [x,y]_H \tag{3.4}$$

for all $x, y \in \mathcal{D}(T_1)$.

The next result extends Lemma 3.5.

Proposition 3.8. *A collection of cosets $\{\phi t_j\}_{j=1}^d$ form a basis for a d-dimensional Lagrangian subspace of the boundary space S' if and only if the representative vectors $\{t_j\}_{j=1}^d$ satisfy*

(a)

$$\sum_{j=1}^d \alpha_j t_j \in \mathcal{D}(T_0) \implies \alpha_j = 0$$

for $j = 1, \ldots, d$; and

(b)

$$[t_i, t_j]_H = 0$$

for all $i, j = 1, \ldots, d$.

These two conditions are especially important when $d = \operatorname{def}(T_0)$, and so we will define sets with these properties as GKN sets:

Definition 3.9. A collection of vectors $\{t_j \mid j = 1, \ldots, \operatorname{def}(T_0)\} \subseteq \mathcal{D}(T_1)$ is called a *GKN set* for T_0 if

(i) the set $\{t_j \mid j = 1, \ldots, \operatorname{def}(T_0)\}$ is linearly independent modulo the minimal domain $\mathcal{D}(T_0)$, which is to say

$$\text{if } \sum_{j=1}^{\operatorname{def}(T_0)} \alpha_j t_j \in \mathcal{D}(T_0) \text{ then } \alpha_j = 0 \text{ for } j = 1, \ldots, \operatorname{def}(T_0);$$

and

(ii) the set $\{t_j \mid j = 1, \ldots, \operatorname{def}(T_0)\}$ satisfies the symmetry conditions

$$[t_i, t_j]_H = 0$$

for all $i, j = 1, \ldots, \operatorname{def}(T_0)$.

If $G \subseteq \mathcal{D}(T_1)$ is a GKN set for T_0, then a non-empty, proper subset $P \subset G$ is called a *partial GKN set.*

We now focus on characterizing complete Lagrangians. Everitt and Markus showed in [13] that complete Lagrangians L exist and that

$$\dim \mathsf{L} = \operatorname{def}(T_0). \tag{3.5}$$

Indeed, we have the following result.

Lemma 3.10. *With* $\operatorname{def}(T_0) < \infty$, *a Lagrangian subspace* $\mathsf{L} \subseteq \mathsf{S}'$ *is complete if and only if each of the two conditions hold:*

(i) $\dim \mathsf{L} = \operatorname{def}(T_0)$*;*

(ii) $\mathsf{L} = \{\phi x \mid [\phi x : \phi t_j]_{\mathsf{S}'} = 0 \ (j = 1, 2, \ldots, \operatorname{def}(T_0))\}$ *for some GKN set* $\{t_j \mid j = 1, \ldots, \operatorname{def}(T_0)\}$.

Moreover, in this case,

$$\phi^{-1}\mathsf{L} = \{x \in \mathcal{D}(T_1) \mid [x, t_j]_H = 0 \ \forall j = 1, 2, \ldots, \operatorname{def}(T_0)\}. \tag{3.6}$$

Proof. First suppose that $\mathsf{L} \subseteq \mathsf{S}$ is a complete Lagrangian. Then, by (3.5), we have condition (i). Now we will establish condition (ii).

By Lemma 3.3, we have that

$$\mathsf{L} = \{\phi x \mid [\phi x : \phi y]_{\mathsf{S}'} = 0 \ (\phi y \in \mathsf{L})\}. \tag{3.7}$$

Now let $\{\phi t_j \mid j = 1, 2, \ldots, \operatorname{def}(T_0)\}$ be a basis for L.

Then the set $\{t_j \mid j = 1, 2, \ldots, \operatorname{def}(T_0)\}$ is a GKN set for T_0 by Proposition 3.8. So, by (3.7), it follows that

$$\mathsf{L} = \{\phi x \mid [\phi x : \phi t_j]_{\mathsf{S}'} = 0 \ (j = 1, 2, \ldots, \operatorname{def}(T_0))\},$$

which establishes condition (ii). Finally, using Definition 3.4 and the above equality, we see that

$$\phi^{-1}\mathsf{L} = \{x \in \mathcal{D}(T_1) \mid [x, t_j]_H = 0 \ (j = 1, 2, \ldots, \operatorname{def}(T_0))\}.$$

To show that converse is true, suppose conditions (i) and (ii) hold. Then we see that L is a subspace of S. Also, (3.6) follows from condition (ii), and

$$[\phi t_i : \phi t_j]_{\mathsf{S}'} = [t_i, t_j]_H = 0$$

since $\{t_j \mid j = 1, 2, \ldots, \operatorname{def}(T_0)\}$ is a GKN set for T_0.

It can be seen that L is Lagrangian by taking linear combinations of elements of $\{t_j \mid j = 1, 2, \ldots, \operatorname{def}(T_0)\}$. Lastly, by (3.5), we see that L is a complete Lagrangian, completing the proof. □

In [13], Everitt and Markus state an important characterization of self-adjoint extensions of T_0 in connection with complete Lagrangian subspaces L of S' in the following result.

Theorem 3.11 (The Finite-Dimensional GKN-EM Theorem). *Let T_0 and T_1 be, respectively, the minimal and maximal operators as defined at the beginning of Chapter Three and let S' be given by (3.3). There exists a one-to-one correspondence between the set $\{T\}$ of all self-adjoint extensions of T_0 and the set $\{\mathsf{L}\}$ of all complete Lagrangians $\mathsf{L} \subseteq \mathsf{S}'$. More specifically,*

(a) if T is a self-adjoint operator with $T_0 \subseteq T \subseteq T_1$, then

$$\mathsf{L} := \{\phi x \in \mathsf{S}' \mid x \in \mathcal{D}(T)\}$$

is a complete Lagrangian subspace of S' of dimension $\operatorname{def}(T_0)$. Moreover, $\phi^{-1}\mathsf{L} = \mathcal{D}(T)$.

(b) If L is a complete Lagrangian subspace of S', then L has dimension $\operatorname{def}(T_0)$. Define

$$\mathcal{D}(T) = \{x \in \mathcal{D}(T_1) \mid \phi x \in \mathsf{L}\}.$$

Then the operator $T : \mathcal{D}(T) \subseteq H \to H$ given by

$$Tx = T_1 x$$
$$x \in \mathcal{D}(T)$$

is a self-adjoint operator satisfying $T_0 \subseteq T \subseteq T_1$. Moreover, $\phi^{-1}\mathsf{L} = \mathcal{D}(T)$.

We can now state and prove an important consequence of this result by combining Theorem 3.11 with Lemma 3.3 and Lemma 3.10. In fact, the next result is an exact generalization of the classical GKN Theorem stated in Theorem 1.33.

Theorem 3.12 (The Finite-Dimensional Symplectic GKN-EM Theorem). *Suppose T_0 and T_1 are linear operators satisfying the conditions set forth in Chapter Two and $[\cdot,\cdot]_H$ is the symplectic form defined in Definition 3.4. In particualr, we assume T_0 has equal and finite deficiency indices denoted by $\operatorname{def}(T_0)$.*

(i) If the operator $T : \mathcal{D}(T) \subseteq H \to H$ is self-adjoint and satisfies

$$T_0 \subseteq T \subseteq T_1,$$

then there exists a GKN set $\{t_j \mid j = 1, \ldots, \operatorname{def}(T_0)\} \subseteq \mathcal{D}(T_1)$ of T_0 such that

$$\mathcal{D}(T) = \{x \in \mathcal{D}(T_1) \mid [x, t_j]_H = 0 \ (j = 1, \ldots, \ \operatorname{def}(T_0))\}. \tag{3.8}$$

(ii) If $\{t_j \mid j = 1, \ldots, \operatorname{def}(T_0)\} \subseteq \mathcal{D}(T_1)$ is a GKN set for T_0, then the operator $T : \mathcal{D}(T) \subseteq H \to H$ given by

$$Tx = T_1 x \tag{3.9}$$

$$x \in \mathcal{D}(T) = \{x \in \mathcal{D}(T_1) \mid [x, t_j]_H = 0 \ (j = 1, \ldots, \operatorname{def}(T_0))\} \tag{3.10}$$

is self-adjoint and satisfies

$$T_0 \subseteq T \subseteq T_1.$$

Proof. (i): If the operator $T : \mathcal{D}(T) \subseteq H \to H$ is self-adjoint and T is such that $T_0 \subseteq T \subseteq T_1$, then

$$\mathsf{L} = \{\phi x \in \mathsf{S}' \mid x \in \mathcal{D}(T)\}$$

is a complete Lagrangian subspace of S' and has dimension $\operatorname{def}(T_0)$ by Theorem 3.11. Then

$$\phi^{-1}\mathsf{L} = \mathcal{D}(T). \tag{3.11}$$

So, there must exist a GKN set $\{t_j \mid j = 1, 2, \ldots \operatorname{def}(T_0)\}$ for T_0, by Lemma 3.10, such that

$$\mathsf{L} = \{\phi x \mid [\phi x : \phi t_j]_{\mathsf{S}'} = 0 \ (j = 1, 2, \ldots, \ \operatorname{def}(T_0))\}$$

and

$$\phi^{-1}\mathsf{L} = \{x \in \mathcal{D}(T_1) \mid [x, t_j]_H = 0 \ (j = 1, 2, \ldots, \ \operatorname{def}(T_0))\}. \tag{3.12}$$

We can now obtain (3.8) by comparing (3.11) and (3.12).

(ii): Now suppose that we have a GKN set $\{t_j \mid j = 1, 2, \ldots, \mathrm{def}(T_0)\}$ for T_0 and let

$$\mathsf{L} = \{\phi x \mid [\phi x : \phi t_j]_{\mathsf{S}'} = 0 \ (j = 1, 2, \ldots, \mathrm{def}(T_0))\} . \tag{3.13}$$

Then, by Lemma 3.10, we see that L is a complete Lagrangian subspace of S' and has dimension $\mathrm{def}(T_0)$.

Now define the operator $T : \mathcal{D}(T_0) \subseteq H \to H$ as in (3.9) and (3.10). Then

$$\mathsf{L} = \{\phi x \mid x \in \mathcal{D}(T_0)\}$$

by (3.10) and (3.13). Therefore,

$$\mathcal{D}(T) = \phi^{-1}\mathsf{L} = \{x \in \mathcal{D}(T_1) \mid \phi x \in \mathsf{L}\} .$$

Hence, by Theorem 3.11, T is self-adjoint and $T_0 \subseteq T \subseteq T_1$. □

There are a few noteworthy remarks to be made about this theorem. First, note that if $H = L^2(I; w)$, T_0 is the minimal operator, and T_1 is the maximal operator generated by the differential expression given in (1.4), then Theorem 3.12 is exactly Theorem 1.33, the classical GKN theorem. The Finite-Dimensional Symplectic GKN-EM theorem extends the classical GKN theorem to a general Hilbert space for any closed symmetric operator with equal deficiency indices. As with the classical GKN setting, the conditions

$$[x, t_j]_H = 0 \text{ for } j = 1, \ldots, \mathrm{def}(T_0)$$

are called 'boundary conditions.' If the deficiency indices of T_0 are 0, then no such boundary conditions exist and thus the only self-adjoint extension of T_0 is T_1, the maximal operator (which is equal to T_0 in this case).

3.2 The Extended Space $H \oplus W$

Recall that the operator T_0 is a closed and symmetric operator such that $T_0 : \mathcal{D}(T_0) \subseteq H \to H$ with equal and finite deficiency indices $\mathrm{def}(T_0)$. Its adjoint is the operator T_1, which satisfies $T_1^* = T_0 \subseteq T_0^* = T_1$.

Let $(W, \langle\cdot,\cdot\rangle_W)$ be a finite-dimensional complex inner product space. Recall that $(H, \langle\cdot,\cdot\rangle_H)$ is a complex Hilbert space. We will call H the *base space* and W the *extension space*. Now define $H \oplus W$, the direct sum of the base space and the extension space, as the Hilbert space defined by

$$H \oplus W = \{(x, a) \mid x \in H, a \in W\}$$

with inner product

$$\langle (x, a), (y, b)\rangle_{H\oplus W} := \langle x, y\rangle_H + \langle a, b\rangle_W$$

and associated norm

$$\|(x, a)\|^2_{H\oplus W} = \|x\|^2_H + \|a\|^2_W.$$

We will call $H \oplus W$ the *extended space*. A critical assumption that we will maintain throughout this dissertation is that

$$\dim(W) \leq \operatorname{def}(T_0). \tag{3.14}$$

Now we will build a continuum of maximal and minimal operators in the extended space $H \oplus W$.

First, fix a partial GKN set

$$\{t_j \mid j = 1, \ldots, \dim(W)\} \subseteq \mathcal{D}(T_1).$$

Then T_1, the maximal operator in the base space H, is symmetric on the space

$$\Delta_0 := \mathcal{D}(T_0) + \operatorname{span}\{t_j \mid j = 1, \ldots, \dim(W)\} \subseteq \mathcal{D}(T_1).$$

Note that T_1 in general is not symmetric on $\mathcal{D}(T_1)$. Let

$$\{\xi_j \mid j = 1, \ldots, \dim(W)\} \subseteq W$$

be an orthonormal basis of the base space W. Now define the operator $\Psi : \Delta_0 \to W$ by

$$\Psi(t_j) = \xi_j \quad (j = 1, \ldots, \dim(W))$$

$$\Psi(s) = 0 \quad (s \in \mathcal{D}(T_0)).$$

We can now extend Ψ to Δ_0 as follows:

$$\Psi\left(s + \sum_{j=1}^{\dim(W)} \alpha_j t_j\right) = \sum_{j=1}^{\dim(W)} \alpha_j \xi_j.$$

Note that Ψ maps the GKN set $\{t_j \mid j = 1, \ldots, \dim(W)\} \subseteq \mathcal{D}(T_1)$ onto W. Also define the linear transformation $\Omega : \mathcal{D}(T_1) \to W$ by

$$\Omega x := \sum_{j=1}^{\dim(W)} [x, t_j]_H \xi_j \quad (x \in \mathcal{D}(T_1)). \tag{3.15}$$

Finally, fix an arbitrary self-adjoint operator $\mathcal{B} : W \to W$. We are now in a position to define the maximal and minimal operators in the extended space.

Definition 3.13. The *maximal operator in the extended space* $H \oplus W$, $\widehat{T}_1 : \mathcal{D}(\widehat{T}_1) \subseteq H \oplus W \to H \oplus W$, is defined by

$$\widehat{T}_1(x, a) = (T_1 x, \mathcal{B}a - \Omega x) \tag{3.16}$$

$$(x, a) \in \mathcal{D}(\widehat{T}_1) := \{(x, a) \mid x \in \mathcal{D}(T_1),\ a \in W\}. \tag{3.17}$$

Definition 3.14. The *minimal operator in the extended space*, $\widehat{T}_0 : \mathcal{D}(\widehat{T}_0) \subseteq H \oplus W \to H \oplus W$, is defined by

$$\widehat{T}_0(x, a) = (T_1 x, \mathcal{B}a) \tag{3.18}$$

$$(x, a) \in \mathcal{D}(\widehat{T}_0) := \{(x, \Psi x) \mid x \in \Delta_0\}. \tag{3.19}$$

Note that if $x \in \Delta_0$, then $\Omega x = 0$. Also note that if $(x, \Psi x) \in \mathcal{D}(\widehat{T}_0)$, then $(x, \Psi x) \in \mathcal{D}(\widehat{T}_1)$. In this case, $\Omega x = 0$ and so

$$\widehat{T}_1(x, \Psi x) = (T_1 x, \mathcal{B}\Psi x - \Omega x) = (T_1 x, \mathcal{B}\Psi x) = \widehat{T}_0(x, \Psi x).$$

Hence, we see that

$$\widehat{T}_0 \subseteq \widehat{T}_1.$$

The terms "maximal" and "minimal" are indeed appropriate for the operators $\widehat{T}_1$ and $\widehat{T}_0$. $\mathcal{D}(\widehat{T}_1)$ is the largest linear manifold in the extended Hilbert space $H \oplus W$ on which an operator representation of T_1 makes sense. Additionally, it will later be shown in Theorem 3.17 that $\left(\widehat{T}_0\right)^* = \widehat{T}_1$, and so the term "minimal" makes sense for $\widehat{T}_0$.

Proposition 3.15. *The extension $J : \mathcal{D}(J) \subseteq H \to H$ of the minimal operator T_0 defined by*

$$Jx := T_1 x$$
$$x \in \mathcal{D}(J) := \Delta_0$$

is a closed symmetric operator.

The proof of this result can be found in [24]. This proposition shows that T_1 is a closed, symmetric operator on Δ_0.

Theorem 3.16. *The operator $\widehat{T}_0$ is a closed, densely defined symmetric operator in $H \oplus W$.*

The result, as proved in [24], leads to the next critical result.

Theorem 3.17. $\left(\widehat{T}_0\right)^* = \widehat{T}_1$.

For the proof of this theorem, see [24]. By combining the previous two theorems, we obtain the following fundamental operator relationship between $\widehat{T}_0$ and $\widehat{T}_1$.

Theorem 3.18. $\widehat{T}_0 = \overline{\widehat{T}_0} \subseteq \left(\widehat{T}_0\right)^* = \widehat{T}_1$.

The proof of this can be found in [24]. With this result, we can now apply the Stone-von Neumann theory to the minimal operator in the extended space $\widehat{T}_0$. So, we can now define the positive and negative deficiency spaces of $\widehat{T}_0$ in $H \oplus W$.

Definition 3.19 (*Deficiency Spaces in the Extended Space $H \oplus W$*).

$$Y_{\pm} := \left\{(x,a) \in \mathcal{D}(\widehat{T}_1) \mid \widehat{T}_1(x,a) = \pm i(x,a)\right\}.$$

Lemma 3.20. *$(x,a) \in Y_{\pm}$ if and only if $x \in X_{\pm}$ and $a = (\mathcal{B} \mp iI)^{-1}\,\Omega x$. Moreover, the deficiency indices of $\widehat{T}_0$ are equal and finite and satisfy* $\operatorname{def}(\widehat{T}_0) = \operatorname{def}(T_0)$.

Proof. Let $(x,a) \in Y_{\pm}$. Then, by definition, $T_1 x = \pm ix$ and $\mathcal{B}a - \Omega x = \pm ia$. Then $x \in X_{\pm}$ and $a = (\mathcal{B} \mp iI)^{-1}\,\Omega x$. Conversely, let $x \in X_{\pm}$ and $a = (\mathcal{B} \mp iI)^{-1}$. Then $\mathcal{B}a - \Omega x = \pm ia$ and thus $\widehat{T}_1(x,a) = \pm i(x,a)$. Therefore, the mappings

$$\begin{aligned} X_{\pm} &\to Y_{\pm} \\ x &\to \left(x, (\mathcal{B} \mp iI)^{-1}\,\Omega x\right) \end{aligned}$$

are vector space isomorphisms and $\dim(X_{\pm}) = \dim(Y_{\pm})$. Hence, the deficiency indices of $\widehat{T}_0$, the minimal operator in the extended space, are finite and equal with

$$\dim(Y_+) = \dim(Y_-) < \infty, \tag{3.20}$$

which completes the proof. □

Equation 3.20 guarantees that the GKN-EM theorem will apply to $\widehat{T}_0$.

Definition 3.21 (*General Symplectic Form).*

$$[(x,a),(y,b)]_{H\oplus W} := [x,y]_H - \langle \Omega x, b\rangle_W + \langle a, \Omega y\rangle_W \tag{3.21}$$

for $(x,a),(y,b) \in \mathcal{D}(\widehat{T}_1)$, where $[\cdot,\cdot]_H$ is the symplectic form defined in (3.4) and the mapping Ω is defined in (3.15).

3.3 The GKN-EM Theorem in the Extended Space $H \oplus W$

We can now state the GKN-EM Theorem in $H \oplus W$, which applies the GKN-EM Theorem to the minimal operator $\widehat{T}_0$ in the extended space $H \oplus W$ and characterizes all self-adjoint extensions of $\widehat{T}_0$ in $H \oplus W$.

Theorem 3.22 (*GKN-EM Theorem in* $H \oplus W$). *We have the following assumptions and definitions:*

(i) T_0 *and* T_1 *are, respectively, the minimal and maximal operators in the Hilbert space* $(H, \langle\cdot,\cdot\rangle_H)$*, called the base (complex) Hilbert space, with domains* $\mathcal{D}(T_0)$ *and* $\mathcal{D}(T_1)$*;* T_0 *is a closed, symmetric operator satisfying* $T_0 \subseteq T_1$ *with* $T_0^* = T_1$ *and* $T_1^* = T_0$*;*

(ii) The deficiency indices of T_0 *are assumed to be equal and finite and denoted by* $\operatorname{def}(T_0)$*;*

(iii) $[\cdot,\cdot]_H$ *is the symplectic form given by*

$$[x, y]_H = \langle T_1 x, y\rangle_H - \langle x, T_1 y\rangle_H \quad (x, y \in \mathcal{D}(T_1));$$

(iv) $(W, \langle\cdot,\cdot\rangle_W)$*, the extension space, is a finite-dimensional complex Hilbert space with* $\dim W \le \operatorname{def}(T_0)$ *and orthonormal basis* $\{\xi_j \mid j = 1, \ldots, \dim W\}$*;*

(v) $\mathcal{B} : W \to W$ *is a fixed self-adjoint operator;*

(vi) $H \oplus W$*, the extended space, is the Hilbert space defined by* $H \oplus W = \{(x, a) \mid x \in H, a \in W\}$ *with inner product*

$$\langle (x, a), (y, b)\rangle_{H\oplus W} := \langle x, y\rangle_H + \langle a, b\rangle_W;$$

(vii) $P = \{t_j \mid j = 1, \ldots, \dim W\}$ *is a partial GKN set for* T_0*;*

(viii) $\Delta_0 = \mathcal{D}(T_0) + \operatorname{span}\{t_j \mid j = 1, \ldots, \dim W\}$*;*

(ix) $\Psi : \Delta_0 \to W$ *is defined to be*

$$\Psi\left(x_0 + \sum_{j=1}^{\dim W} \alpha_j t_j\right) = \sum_{j=1}^{\dim W} \alpha_j \xi_j \quad (x_0 \in \mathcal{D}(T_0));$$

(x) $\Omega : \mathcal{D}(T_1) \to W$ *is given by*

$$\Omega x = \sum_{j=1}^{\dim W} [x, t_j]_H \xi_j;$$

(xi) $\widehat{T}_0 : \mathcal{D}(T_0) \subseteq H \oplus W \to H \oplus W$ *is the minimal operator in* $H \oplus W$ *defined by*

$$\widehat{T}_0(x,a) = (T_1 x, \mathcal{B}a)$$
$$(x,a) \in \mathcal{D}(\widehat{T}_0) = \{(x, \Psi x) \mid x \in \Delta_0\};$$

(xii) $\widehat{T}_1 : \mathcal{D}(\widehat{T}_1) \subseteq H \oplus W \to H \oplus W$ *is the maximal operator in* $H \oplus W$ *defined by*

$$\widehat{T}_1(x,a) = (T_1 x, \mathcal{B}a - \Omega x)$$
$$\mathcal{D}(\widehat{T}_1) = \{(x,a) \mid x \in \mathcal{D}(T_1);\ a \in W\};$$

(xiii) $[\cdot,\cdot]_{H\oplus W}$ *is the symplectic form given by*

$$[(x,a),(y,b)]_{H\oplus W} := [x,y]_H - \langle \Omega x, b\rangle_W + \langle a, \Omega y\rangle_W,$$

for $((x,a),(y,b)) \in \mathcal{D}(\widehat{T}_1)$.

With the above definitions and assumptions, we obtain the following result:

(a) $\widehat{T}_0$ *is a closed, symmetric operator satisfying* $\widehat{T}_0 \subseteq \widehat{T}_1$ *with* $(\widehat{T}_0)^* = \widehat{T}_1$ *and* $(\widehat{T}_1)^* = \widehat{T}_0$*;*

(b) The deficiency indices of $\widehat{T}_0$ *are equal and finite and* $\operatorname{def}(\widehat{T}_0) = \operatorname{def}(T_0)$*;*

(c) Suppose $\widehat{T}$ *is a self-adjoint extension of* $\widehat{T}_0$ *satisfying* $\widehat{T}_0 \subseteq \widehat{T} \subseteq \widehat{T}_1$. *Then there exists a GKN set* $\{(x_j, a_j) \mid j = 1, \ldots, \operatorname{def}(T_0)\} \subseteq \mathcal{D}(\widehat{T}_1)$ *such that*

$$\widehat{T}(x,a) = (T_1 x, \mathcal{B}a - \Omega x)$$
$$\mathcal{D}(\widehat{T}) = \left\{(x,a) \in \mathcal{D}(\widehat{T}_1) \mid [(x,a),(x_j,a_j)]_{H\oplus W} = 0\ (j, 1, \ldots, \operatorname{def}(T_0))\right\}.$$

(d) If $\widehat{T}$ *is defined as above where* $\{(x_j, a_j) \mid j = 1, \ldots, \operatorname{def}(T_0)\} \subseteq \mathcal{D}(\widehat{T}_1)$ *is a GKN set, then* $\widehat{T}$ *is a self-adjoint extension of* $\widehat{T}_0$ *in* $H \oplus W$.

CHAPTER FOUR

Variations on the Fourier Self-Adjoint Operator in an Extended Hilbert Space

In this chapter, we illustrate several examples of the application of the GKN-EM Theorem in $H \oplus W$ to construct self-adjoint operators in extended Hilbert spaces.

Consider the Fourier differential expression

$$\ell_F[y](u) = -y''(u) \quad (u \in [a,b])$$

where $[a,b]$ is a compact interval. The base space is $H = L^2[a,b]$ and we will work with either $\mathbb{C}$ or $\mathbb{C}^2$ as the extension space.

The maximal operator $T_1 : \mathcal{D}(T_1) \subset H \to H$ generated by ℓ_F is defined by

$$T_1 x = \ell_F[x]$$
$$\mathcal{D}(T_1) = \left\{ x : [a,b] \to \mathbb{C} \mid x, x' \in AC[a,b]; x'' \in L^2[a,b] \right\}$$

and the minimal operator $T_0 : \mathcal{D}(T_0) \subset H \to H$ is given by

$$T_0 x = \ell_F[x]$$
$$\mathcal{D}(T_0) = \{ x \in \mathcal{D}(T_1) \mid x(a) = x'(a) = x(b) = x'(b) = 0 \}.$$

The symplectic form in H associated with T_1 is

$$[x,y]_H = x(b)\overline{y}'(b) - x'(b)\overline{y}(b) + x'(a)\overline{y}(a) - x(a)\overline{y}'(a)$$

for $x, y \in \mathcal{D}(T_1)$.

4.1 One-Dimensional Extension Spaces

We will first consider the one-dimensional extension space $W = \mathbb{C}$ with inner product $\langle z_1, z_2 \rangle_W = z_1 \overline{z}_2$ for $z_1, z_2 \in W$. We will use $\{\xi_1 = 1\}$ as a basis for W.

Every self-adjoint operator $\mathcal{B} : W \to W$ has the form αz, where $\alpha \in \mathbb{R}$. For the following one-dimensional extension space examples, fix such a $\mathcal{B}$.

Example 4.1. Our first example with a one-dimensional extension space will incorporate continuity at b as a result of the boundary conditions.

Define $t_1 \in \mathcal{D}(T_1)$ by

$$t_1(u) = \begin{cases} 0 & u \text{ near } a \\ 1 & u \text{ near } b. \end{cases}$$

Claim 4.2. *$\{t_1\}$ is a partial GKN set for T_0.*

Proof. Since $t_1(b) = 1 \neq 0$, $t_1 \notin \mathcal{D}(T_0)$. So, if $ct_1 \in \mathcal{D}(T_0)$, then $c = 0$. Also,

$$[t_1, t_1]_H = t_1'(a)\bar{t}_1(a) - t_1'(b)\bar{t}_1(b) + t_1(b)\bar{t}_1'(b) - t_1(a)\bar{t}_1'(a) = 0$$

because $t_1(a) = t_1'(b) = 0$. Therefore, $\{t_1\}$ is a partial GKN set for T_0. □

With the partial GKN set above, we have $\Delta_0 = \mathcal{D}(T_0) + \text{span}\{t_1\}$ and so $\Psi : \Delta_0 \to W$ is defined by $\Psi(t_0 + ct_1) = c$. Also, for $x \in \mathcal{D}(T_1)$,

$$\begin{aligned} \Omega x &= [x, t_1]_H \xi_1 \\ &= \Big(x'(a)\bar{t}_1(a) - x'(b)\bar{t}_1(b) + x(b)\bar{t}_1'(b) - x(a)\bar{t}_1'(a)\Big) \cdot 1 \\ &= 0 - x'(b) + 0 - 0 \\ &= -x'(b). \end{aligned}$$

The symplectic form $[\cdot, \cdot]_{H\oplus W}$ in $H \oplus W = L^2[a, b] \oplus \mathbb{C}$ is given by

$$\begin{aligned} [(x, c_1), (y, c_2)]_{H\oplus W} &= [x, y]_H - \langle \Omega x, c_2 \rangle_W + \langle c_1, \Omega y \rangle_W \\ &= x'(a)\overline{y}(a) - x'(b)\overline{y}(b) + x(b)\overline{y}'(b) - x(a)\overline{y}'(a) \\ &+ x'(b)\bar{c}_2 - c_1\overline{y}'(b) \end{aligned}$$

for $x, y \in H$ and $c_1, c_2 \in W$.

Therefore, the minimal operator $\widehat{T}_0 : \mathcal{D}(\widehat{T}_0) \subseteq H \oplus W \to H \oplus W$ is defined by

$$\widehat{T}_0(x, z) = (T_1 x, \mathcal{B}z) = (-x'', \alpha z),$$
$$\mathcal{D}(\widehat{T}_0) = \{(x, \Psi x) \mid x \in \Delta_0\}$$

and so

$$\widehat{T}_0(x, \Psi x) = (T_1 x, \mathcal{B}\Psi x) = (-x'', \alpha \Psi x).$$

The maximal operator $\widehat{T}_1 : \mathcal{D}(\widehat{T}_1) \subseteq H \oplus W \to H \oplus W$ is defined by

$$\begin{aligned}\widehat{T}_1(x, c) &= (T_1 x, \mathcal{B}c - \Omega x) \\ &= (-x'', \alpha c + x'(b)), \\ \mathcal{D}(\widehat{T}_1) &= \{(x, c) \mid x \in \mathcal{D}(T_1), c \in \mathbb{C}\}.\end{aligned}$$

Now define $x_1, x_2 \in \mathcal{D}(T_1)$ by

$$x_1(u) = \begin{cases} 0 & u \text{ near } a \\ u - b & u \text{ near } b, \end{cases}$$
$$x_2(u) = \begin{cases} u - a & u \text{ near } a \\ 0 & u \text{ near } b. \end{cases}$$

Claim 4.3. *$\{(x_1, 0), (x_2, 0)\}$ is a GKN set for $\widehat{T}_0$ in $H \oplus W$.*

Proof. It will first be shown that $\{(x_1, 0), (x_2, 0)\}$ is linearly independent modulo $\mathcal{D}(\widehat{T}_0)$. To do so, define $f_1, f_2 \in \mathcal{D}(T_1) \backslash \mathcal{D}(T_0)$ by $f_1(u) = u$ and $f_2(u) = u^2$. Let $\alpha, \beta \in \mathbb{C}$. If $\alpha(x_1, 0) + \beta(x_2, 0) \in \mathcal{D}(\widehat{T}_0)$, then

$$[(f_1, 0), (\alpha x_1 + \beta x_2, 0)]_{H \oplus W} = [(f_2, 0), (\alpha x_1 + \beta x_2, 0)]_{H \oplus W} = 0.$$

Since,

$$[(f_1, 0), (\alpha x_1 + \beta x_2, 0)]_{H \oplus W} = [f_1, \alpha x_1 + \beta x_2]_H$$

$$
\begin{aligned}
=& f_1'(a)\overline{(\alpha x_1(a) + \beta x_2(a))} - f_1'(b)\overline{(\alpha x_1(b) + \beta x_2(b))} \\
&+ f_1(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_1(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \\
=& b\overline{\alpha} - a\overline{\beta}
\end{aligned}
$$

and

$$
\begin{aligned}
[(f_2, 0), (\alpha x_1 + \beta x_2, 0)]_{H\oplus W} =& [f_2, \alpha x_1 + \beta x_2]_H \\
=& f_2'(a)\overline{(\alpha x_1(a) + \beta x_2(a))} - f_2'(b)\overline{(\alpha x_1(b) + \beta x_2(b))} \\
&+ f_2(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_2(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \\
=& b^2\overline{\alpha} - a^2\overline{\beta},
\end{aligned}
$$

we have $0 = b\overline{\alpha} - a\overline{\beta} = b^2\overline{\alpha} - a^2\overline{\beta}$. Therefore, $\alpha = \beta = 0$. Hence, $\{(x_1, 0), (x_2, 0)\}$ is linearly independent modulo $\mathcal{D}(\widehat{T}_0)$.

Now we will show that $[(x_i, 0), (x_j, 0)]_{H\oplus W} = 0$ for $i, j = 0$. We have

$$
\begin{aligned}
[(x_1, 0), (x_1, 0)]_{H\oplus W} =& x_1'(a)\overline{x}_1(a) - x_1'(b)\overline{x}_1(b) + x_1(b)\overline{x}_1'(b) - x_1(a)\overline{x}_1'(a) \\
&+ x_1'(b)\overline{(0)} - (0)\overline{x}_1'(b) \\
=& 0, \\
[(x_1, 0), (x_2, 0)]_{H\oplus W} =& x_1'(a)\overline{x}_2(a) - x_1'(b)\overline{x}_2(b) + x_1(b)\overline{x}_2'(b) - x_1(a)\overline{x}_2'(a) \\
&+ x_1'(b)\overline{(0)} - (0)\overline{x}_2'(b) \\
=& 0,
\end{aligned}
$$

and

$$
\begin{aligned}
[(x_2, 0), (x_2, 0)]_{H\oplus W} =& x_2'(a)\overline{x}_2(a) - x_2'(b)\overline{x}_2(b) + x_2(b)\overline{x}_2'(b) - x_2(a)\overline{x}_2'(a) \\
&+ x_2'(b)\overline{(0)} - (0)\overline{x}_2'(b) \\
=& 0.
\end{aligned}
$$

Therefore, $\{(x_1, 0), (x_2, 0)\}$ is a GKN set for $\widehat{T}_0$. □

Note that, for $x \in H$ and $c \in W$,

$$
\begin{aligned}
[(x, c), (x_1, 0)]_{H\oplus W} =& x'(a)\overline{x}_1(a) - x'(b)\overline{x}_1(b) + x(b)\overline{x}_1'(b) - x(a)\overline{x}_1'(a) \\
& + x'(b)(0) - c\overline{x}_1'(b) \\
=& x(b) - c
\end{aligned}
$$

and

$$
\begin{aligned}
[(x, c), (x_2, 0)]_{H\oplus W} =& x'(a)\overline{x}_2(a) - x'(b)\overline{x}_2(b) + x(b)\overline{x}_2'(b) - x(a)\overline{x}_2'(a) \\
& + x'(b)(0) - c\overline{x}_2'(b) \\
=& - x(a).
\end{aligned}
$$

Therefore, if $[(x, c), (x_1, 0)]_{H\oplus W} = 0$, then $x(b) = c$ and if $[(x, c), (x_2, 0)]_{H\oplus W} = 0$, then $x(a) = 0$.

Now we can define a self-adjoint operator $\widehat{T} : \mathcal{D}(\widehat{T}) \subseteq H \oplus W \to H \oplus W$ by

$$
\begin{aligned}
\widehat{T}(x, c) =& (T_1 x, \mathcal{B}c - \Omega x) \\
=& (-x'', \alpha c + x'(b)), \\
\mathcal{D}(\widehat{T}) =& \left\{ (x, c) \in \mathcal{D}(\widehat{T}_1) \mid x(b) - c = -x(a) = 0 \right\}.
\end{aligned}
$$

However, since $x(b) = c$, we can write this as

$$
\begin{aligned}
\widehat{T}(x, (x(b)) =& (-x'', \alpha x(b) + x'(b)), \\
\mathcal{D}(\widehat{T}) =& \{(x, x(b)) \mid x(a) = 0,\ x \in \mathcal{D}(T_1)\}.
\end{aligned}
$$

As a result, we can see that continuity at the endpoint $x = b$ is forced by the boundary conditions. The operator $\widehat{T}$ in $H \oplus W$ is self-adjoint in $H \oplus W$ by Theorem 3.22.

Example 4.4. This example is similar to the previous one, but incorporates $x(a)$ in the boundary conditions instead of $x(b)$.

Define $t_1(u) \in \mathcal{D}(T_1)$ by

$$t_1(u) = \begin{cases} 1 & u \text{ near } a \\ 0 & u \text{ near } b. \end{cases}$$

Then $\{t_1\}$ is a partial GKN set for T_0. The proof is analogous to the proof of Claim 4.2. With this partial GKN set, we can define $\Delta_0 = \mathcal{D}(T_0) + \operatorname{span}\{t_1\}$ and $\Psi : \Delta_0 \to W$ by $\Psi(t_0 + ct_1) = c$ for $t_0 \in \mathcal{D}(T_0)$. We also define, for $x \in \mathcal{D}(T_1)$,

$$\begin{aligned} \Omega x =& [x, t_1]_H \xi_1 \\ =& x'(a)\bar{t}_1(a) - x'(b)\bar{t}_1(b) + x(b)\bar{t}_1'(b) - x(a)\bar{t}_1'(a) \\ =& x'(a). \end{aligned}$$

Therefore, we define $[\cdot,\cdot]_{H\oplus W}$ by

$$\begin{aligned} [(x, c_1), (y, c_2)]_{H\oplus W} =& [x, y]_H - \langle \Omega x, c_2\rangle_W + \langle c_1, \Omega y\rangle_W \\ =& x'(a)\overline{y}(a) - x'(b)\overline{y}(b) + x(b)\overline{y}'(b) - x(a)\overline{y}'(a) \\ & - x'(a)\overline{c}_2 + c_1\overline{y}'(a). \end{aligned}$$

We define the minimal operator $\widehat{S}_0 : \mathcal{D}(\widehat{S}_0) \subseteq H \oplus W \to H \oplus W$ by

$$\begin{aligned} \widehat{S}_0(x, c) =& (T_1 x, \mathcal{B}c) = (-x'', \alpha x) \\ \mathcal{D}(\widehat{S}_0) =& \{(x, \Psi x) \mid x \in \Delta_0\}. \end{aligned}$$

Thus, $\widehat{S}_0(x, \Psi x) = (-x'', \alpha\Psi x)$.

We also define the maximal operator $\widehat{S}_1 : \mathcal{D}(\widehat{S}_1) \subseteq H \oplus W \to H \oplus W$ by

$$\begin{aligned} \widehat{S}_1(x, c) =& (T_1 x, \mathcal{B}c - \Omega x) = (-x'', \alpha c - x'(a)), \\ \mathcal{D}(\widehat{S}_1) =& \{(x, c) \mid x \in \mathcal{D}(T_1),\ c \in \mathbb{C}\}. \end{aligned}$$

Now we need a GKN set for $\widehat{S}_0$. Define $y_1, y_2 \in \mathcal{D}(T_1)$ by

$$y_1(u) = \begin{cases} u - a & u \text{ near a} \\ 0 & u \text{ near } b, \end{cases}$$

$$y_2(u) = \begin{cases} 0 & u \text{ near } a \\ u - b & u \text{ near } b. \end{cases}$$

Claim 4.5. $\{(y_1, 0), (y_2, 0)\}$ *is a GKN set for* $\widehat{S}_0$.

Proof. Define $f_1, f_2 \in \mathcal{D}(T_1)\backslash\mathcal{D}(S_0)$ by

$$\begin{aligned} f_1(u) =& u \\ f_2(u) =& u^2, \end{aligned}$$

and let $\alpha, \beta \in \mathbb{C}$. Then, if $\alpha(y_1, 0) + \beta(y_2, 0) \in \mathcal{D}(\widehat{S}_0)$, we have

$$[(f_1, 0), (\alpha y_1 + \beta y_2, 0)]_{H \oplus W} = [(f_2, 0), (\alpha y_1 + \beta y_2, 0)]_{H \oplus W} = 0$$

for $c_1, c_2 \in \mathbb{C}$. Note that

$$\begin{aligned} [(f_1, 0), (\alpha y_1 + \beta y_2, 0)]_{H \oplus W} =& [f_1, \alpha y_1 + \beta y_2]_H \\ =& f_1'(a)\overline{(\alpha y_1(a) + \beta y_2(a))} - f_1'(b)\overline{(\alpha y_1(b) + \beta y_2(b))} \\ &+ f_1(b)\overline{(\alpha y_1'(b) + \beta y_2'(b))} - f_1(a)\overline{(\alpha y_1'(a) + \beta y_2'(a))} \\ =& b\overline{\beta} - a\overline{\alpha}, \end{aligned}$$

and

$$\begin{aligned} [(f_2, 0), (\alpha y_1 + \beta y_2, 0)]_{H \oplus W} =& [f_2, \alpha y_1 + \beta y_2]_H \\ =& f_2'(a)\overline{(\alpha y_1(a) + \beta y_2(a))} - f_2'(b)\overline{(\alpha y_1(b) + \beta y_2(b))} \\ &+ f_2(b)\overline{(\alpha y_1'(b) + \beta y_2'(b))} - f_2(a)\overline{(\alpha y_1'(a) + \beta y_2'(a))} \\ =& b^2\overline{\beta} - a^2\overline{\alpha}. \end{aligned}$$

Hence, $0 = b\overline{\beta} - a\overline{\alpha}$ and $0 = b^2\overline{\beta} - a^2\overline{\alpha}$, and solving this system of equations yields $\alpha = \beta = 0$. Therefore, $\{(y_1, 0), (y_2, 0)\}$ is linearly independent modulo $\mathcal{D}(\widehat{S}_0)$. It remains to be shown that $[(y_i, 0), (y_j, 0)]_{H \oplus W} = 0$ for $i, j = 1, 2$. However, since

$[(y_i,0),(y_j,0)]_{H\oplus W} = [y_i,y_j]_H$, it suffices to show that $[y_i,y_j]_H = 0$ for $i,j = 1,2$:

$$\begin{aligned}
[y_1,y_1]_H =& y_1(b)\overline{y}_1'(b) - y_1'(b)\overline{y}_1(b) + y_1'(a)\overline{y}_1(a) - y_1(a)\overline{y}_1'(a) \\
=& 0, \\
[y_1,y_2]_H =& y_1(b)\overline{y}_2'(b) - y_1'(b)\overline{y}_2(b) + y_1'(a)\overline{y}_2(a) - y_1(a)\overline{y}_2'(a) \\
=& 0, \\
[y_2,y_2]_H =& y_2(b)\overline{y}_2'(b) - y_2'(b)\overline{y}_2(b) + y_2'(a)\overline{y}_2(a) - y_2(a)\overline{y}_2'(a) \\
=& 0.
\end{aligned}$$

Thus, $[(y_i,0),(y_j,0)]_{H\oplus W}$ is a GKN set for $\widehat{S}_0$. □

For any $(x,c) \in \mathcal{D}(\widehat{S}_0)$, we have

$$\begin{aligned}
[(x,c),(y_1,0)]_{H\oplus W} =& x'(a)\overline{y}_1(a) - x'(b)\overline{y}_1(b) + x(b)\overline{y}_1'(b) - x(a)\overline{y}_1'(a) \\
& - x'(a)(0) + c\overline{y}_1'(a) \\
=& -x(a) + c
\end{aligned}$$

and

$$\begin{aligned}
[(x,c),(y_2,0)]_{H\oplus W} =& x'(a)\overline{y}_2(a) - x'(b)\overline{y}_2(b) + x(b)\overline{y}_2'(b) - x(a)\overline{y}_2'(a) \\
& - x'(a)(0) + c\overline{y}_2'(a) \\
=& x(b).
\end{aligned}$$

So, if $0 = [(x,c),(y_1,0)]_{H\oplus W} = 0$, then $c = x(a)$. Likewise, if $[(x,c),(y_2,0)]_{H\oplus W} = 0$, then $x(b) = 0$.

Now define $\widehat{S} : \mathcal{D}(\widehat{S}) \subseteq H \oplus W \to H \oplus W$ by

$$\begin{aligned}
\widehat{S}(x,c) =& (T_1 x, \mathcal{B}c - \Omega x) \\
=& (-x'', \alpha c - x'(a)), \\
\mathcal{D}(\widehat{S}) =& \left\{ (x,c) \in \mathcal{D}(\widehat{T}_1) \mid -x(a) + c = x(b) = 0 \right\}.
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\widehat{S}(x, x(a)) =&(-x'', -\alpha x(a) - x'(a)),\\ \mathcal{D}(\widehat{S}) =& \{(x, x(a)) \mid x(b) = 0, x \in \mathcal{D}(T_1)\}.\end{aligned}$$

The operator $\widehat{S}$ is self-adjoint in $H \oplus W$ by Theorem 3.22 the GKN-EM Theorem in $H \oplus W$.

For our last example with a one-dimensional extensions space, we pick a more complex function for our partial GKN set in H by switching the roles of y_1 and t_1 in the previous example.

Example 4.6. Define $t_1 \in \mathcal{D}(T_1)$ by

$$t_1(u) = \begin{cases} u - a & u \text{ near } a \\ 0 & u \text{ near } b. \end{cases}$$

Claim 4.7. $\{t_1\}$ *is a partial GKN set for* T_0.

Proof. Since $t_1'(a) = 1 \neq 0$, we have that $t_1 \in \mathcal{D}(T_1) \backslash \mathcal{D}(T_0)$. So, if $\alpha t_1 \in \mathcal{D}(T_0)$, then we obtain $\alpha = 0$. Hence, $\{t_1\}$ is linearly independent modulo $\mathcal{D}(T_0)$. Also note that

$$\begin{aligned}[t_1, t_1]_H =& t_1'(a)\bar{t}_1(a) - t_1'(b)\bar{t}_1(b) + t_1(b)\bar{t}_1'(b) - t_1(a)\bar{t}_1'(a)\\ =&0.\end{aligned}$$

Therefore, $\{t_1\}$ is a partial GKN set for T_0. □

With the above partial GKN set, we have $\Delta_0 = \mathcal{D}(T_0) + \text{span}\{t_1\}$ and $\Psi : \Delta_0 \to W$ defined by

$$\Psi(t_0 + ct_1) = c$$

for $t_0 \in \mathcal{D}(T_0)$ and $c \in W$. For $x \in \mathcal{D}(T_1)$, define Ω by

$$\Omega x =[x, t_1]_H(\xi_1)$$

$$
\begin{aligned}
&= x'(a)\bar{t}_1(a) - x'(b)\bar{t}_1(b) + x(b)\bar{t}_1'(b) - x(a)\bar{t}_1'(a) \\
&= -x(a).
\end{aligned}
$$

Note that in the previous two examples, Ω involved x' at one of the endpoints, but here, Ω involves x instead.

The symplectic form $[\cdot,\cdot]_{H\oplus W}$ can be defined, in this case, by

$$
\begin{aligned}
[(x,c_1),(y,c_2)]_{H\oplus W} =& [x,y]_H - \langle \Omega x, c_2\rangle_W + \langle c_1, \Omega y\rangle_W \\
=& x'(a)\overline{y}(a) - x'(b)\overline{y}(b) + x(b)\overline{y}'(b) - x(a)\overline{y}'(a) \\
& + x(a)\bar{c}_2 - c_1\overline{y}(a).
\end{aligned}
$$

The minimal operator in $\widehat{R}_0 : \mathcal{D}(\widehat{R}_0) \subseteq H \oplus W \to H \oplus W$ is defined by

$$
\begin{aligned}
\widehat{R}_0(x,c) =& (T_1 x, \mathcal{B}c) = (-x'', \alpha c), \\
\mathcal{D}(\widehat{R}_0) =& \{(x, \Psi x) \mid x \in \Delta_0\}.
\end{aligned}
$$

So, $\widehat{R}_0(x,c) = (-x'', \alpha\Psi x)$.

The maximal operator $\widehat{R}_1 : \mathcal{D}(\widehat{R}_1) \subseteq H \oplus W \to H \oplus W$ is defined by

$$
\begin{aligned}
\widehat{R}_1(x,c) =& (T_1 x, \mathcal{B}c - \Omega x) \\
=& (-x'', \alpha c + x(a)), \\
\mathcal{D}(\widehat{R}_1) =& \{(x,c) \mid x \in \mathcal{D}(T_1),\ c \in \mathbb{C}\}.
\end{aligned}
$$

Now define $x_1, x_2 \in \mathcal{D}(T_1)$ by

$$
x_1(u) = \begin{cases} 1 & u \text{ near } a \\ 0 & u \text{ near } b, \end{cases}
$$

$$
x_2(u) = \begin{cases} 0 & u \text{ near } a \\ u-b & u \text{ near } b. \end{cases}
$$

Claim 4.8. *$\{(x_1,0),(x_2,0)\}$ is a GKN set for $\widehat{R}_0$.*

Proof. First, note that since $x_1(a) = 1 \neq 0$ and $x_2'(b) = 1 \neq 0$, we have that $x_1, x_2 \in \mathcal{D}(T_1)\backslash\mathcal{D}(T_0)$. Now define $f_1, f_2 \in \mathcal{D}(T_1)\backslash\mathcal{D}(T_0)$ by

$$\begin{aligned} f_1(u) =& u \\ f_2(u) =& u^2. \end{aligned}$$

So, if $\alpha(x_1, 0) + \beta(x_2, 0) \in \mathcal{D}(\widehat{R}_0)$, then

$$[(f_1, 0), (\alpha x_1 + \beta x_2, 0)]_{H\oplus W} = [(f_2, 0), (\alpha x_1 + \beta x_2, 0)]_{H\oplus W} = 0.$$

So,

$$\begin{aligned} [(f_1, 0), (\alpha x_1 + \beta x_2, 0)]_{H\oplus W} =& [f_1, \alpha x_1 + \beta x_2]_H \\ =& f_1'(a)\overline{(\alpha x_1(a) + \beta x_2(a))} - f_1'(b)\overline{(\alpha x_1(b) + \beta x_2(b))} \\ & + f_1(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_1(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \\ =& f_1'(a)(\overline{\alpha}) + f_1(b)(\overline{\beta}) \\ =& \overline{\alpha} + b\overline{\beta} \end{aligned}$$

and

$$\begin{aligned} [(f_2, 0), (\alpha x_1 + \beta x_2, 0)]_{H\oplus W} =& [f_2, \alpha x_1 + \beta x_2]_H \\ =& f_2'(a)\overline{(\alpha x_1(a) + \beta x_2(a))} - f_2'(b)\overline{(\alpha x_1(b) + \beta x_2(b))} \\ & + f_2(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_2(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \\ =& f_2'(a)(\overline{\alpha}) + f_2(b)(\overline{\beta}) \\ =& 2a\overline{\alpha} + b^2\overline{\beta}. \end{aligned}$$

This implies that $0 = \overline{\alpha} + b\overline{\beta}$ and $0 = 2a\overline{\alpha} + b^2\overline{\beta}$. Thus, $\alpha = \beta = 0$, and so $\{(x_1, 0), (x_2, 0)\}$ is linearly independent modulo $\mathcal{D}(\widehat{R}_0)$.

It remains to be shown that $0 = [(x_i, 0), (x_j, 0)]_{H\oplus W} = [x_i, x_j]_H$ for $i, j = 1, 2$:

$$[x_1, x_1]_H = x_1'(a)\overline{x}_1(a) - x_1'(b)\overline{x}_1(b) + x_1(b)\overline{x}_1'(b) - x_1(a)\overline{x}_1'(a)$$

$$
\begin{aligned}
&=0,\\
[x_1, x_2]_H &= x_1'(a)\overline{x}_2(a) - x_1'(b)\overline{x}_2(b) + x_1(b)\overline{x}_2'(b) - x_1(a)\overline{x}_2'(a)\\
&=0,\\
[x_2, x_2]_H &= x_2'(a)\overline{x}_2(a) - x_2'(b)\overline{x}_2'(b) + x_2(b)\overline{x}_2'(b) - x_2(a)\overline{x}_2'(a)\\
&=0.
\end{aligned}
$$

Therefore, $\{(x_1, 0), (x_2, 0)\}$ is a GKN set for $\widehat{R}_0$. □

For $(x, c) \in \mathcal{D}(\widehat{R}_1)$, we have

$$
\begin{aligned}
[(x, c), (x_1, 0)]_{H\oplus W} &= x'(a)\overline{x}_1(a) - x'(b)\overline{x}_1(b) + x(b)\overline{x}_1'(b) - x(a)\overline{x}_1'(a)\\
&\quad + x(a)(0) - c\overline{x}_1(a)\\
&= x'(a) - c
\end{aligned}
$$

and

$$
\begin{aligned}
[(x, c), (x_2, 0)]_{H\oplus W} &= x'(a)\overline{x}_2(a) - x'(b)\overline{x}_2(b) + x(b)\overline{x}_2'(b) - x(a)\overline{x}_2'(a)\\
&\quad + x(a)(0) - c\overline{x}_2(a)\\
&= x(b).
\end{aligned}
$$

So, if $[(x, c), (x_1, 0)]_{H\oplus W} = 0$ and $[(x, c), (x_2, 0)]_{H\oplus W} = 0$, then $x'(a) = c$ and $x(b) = 0$.

We can now define $\widehat{R} : \mathcal{D}(\widehat{R}) \subseteq H \oplus W \to H \oplus W$ by

$$
\begin{aligned}
\widehat{R}(x, c) &= (T_1 x, \mathcal{B}c - \Omega x)\\
&= (-x'', \alpha c + x(a)),\\
\mathcal{D}(\widehat{R}) &= \left\{(x, c) \in \mathcal{D}(\widehat{R}_1) \mid x'(a) - c = x(b) = 0\right\}.
\end{aligned}
$$

Since $x'(a) = c$, this can be written as

$$
\widehat{R}(x, x'(a)) = (-x'', \alpha x'(a) + x(a)),
$$

$$\mathcal{D}(\widehat{R}) = \{(x, x'(a)) \mid x \in \mathcal{D}(T_1),\ x(b) = 0\}.$$

This operator $\widehat{R}$ is self-adjoint by Theorem 3.22.

4.2 Two-Dimensional Extension Spaces

We now turn our attention to a few examples with a two-dimensional extension extension space. Let $W = \mathbb{C}^2$ with inner product

$$\langle (z_1, z_2), (z_1', z_2') \rangle_W = \frac{z_1 \overline{z}_1'}{M} + \frac{z_2 \overline{z}_2'}{N},$$

where $M, N > 0$. Define $\xi_1 = (\sqrt{M}, 0)$ and $\xi_2 = (0, \sqrt{N})$. Then $\{\xi_1, \xi_2\}$ is an orthonormal basis for W. Using this inner product, every self-adjoint operator $\mathcal{B} : W \to W$ has the form

$$\mathcal{B} = \begin{pmatrix} \alpha & \beta \\ \overline{\beta}\frac{N}{M} & \gamma \end{pmatrix},$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$. For the following examples, fix such a matrix $\mathcal{B}$.

Example 4.9. In this example, we will construct a self-adjoint operator in $H \oplus W = L^2[a, b] \oplus \mathbb{C}^2$ with continuity at both endpoints $x = a$ and $x = b$.

Define $t_1, t_2 \in \mathcal{D}(T_1)$ by

$$t_1(u) = \begin{cases} \sqrt{M} & u \text{ near } a \\ 0 & u \text{ near } b, \end{cases}$$

$$t_2(u) = \begin{cases} 0 & u \text{ near } a \\ \sqrt{N} & u \text{ near } b. \end{cases}$$

Claim 4.10. $\{t_1, t_2\}$ *is a GKN set for* T_0 *in* H.

Proof. Note that since $t_1(a) = \sqrt{M} \neq 0$ and $t_2(b) = \sqrt{N} \neq 0$, we see that $t_1, t_2 \in \mathcal{D}(T_1) \backslash \mathcal{D}(T_0)$. So, if $c_1 t_1 + c_2 t_2 \in \mathcal{D}(T_0)$, then $c_1 t_1(a) + c_2 t_2(a) = 0$. However, $c_1 t_1(a) + c_2 t_2(a) = c_1 \sqrt{M}$, so we must have $c_1 = 0$. Likewise, we must have $c_1 t_1(b) +$

$c_2t_2(b) = 0$, but $c_1t_1(b) + c_2t_2(b) = c_2\sqrt{N}$, so $c_2 = 0$. Therefore, $\{t_1, t_2\}$ is linearly independent modulo $\mathcal{D}(T_0)$. Then, since

$$\begin{aligned}
[t_1, t_1]_H =& t_1'(a)\bar{t}_1(a) - t_1'(b)\bar{t}_1(b) + t_1(b)\bar{t}_1'(b) - t_1(a)\bar{t}_1'(a) \\
=& 0, \\
[t_1, t_2]_H =& t_1'(a)\bar{t}_2(a) - t_1'(b)\bar{t}_2(b) + t_1(b)\bar{t}_2'(b) - t_1(a)\bar{t}_2(a) \\
=& 0, \\
[t_2, t_2]_H =& t_2'(a)\bar{t}_2(a) - t_2'(b)\bar{t}_2(b) + t_2(b)\bar{t}_2'(b) - t_2(a)\bar{t}_2'(a) \\
=& 0,
\end{aligned}$$

$\{t_1, t_2\}$ is a GKN set for T_0 in H. □

Now we can define $\Delta_0 = \mathcal{D}(T_0) + \text{span}\{t_1, t_2\}$ and define $\Psi : \Delta_0 \to W$ by

$$\begin{aligned}
\Psi(t_0 + c_1t_1 + c_2t_2) =& c_1\xi_1 + c_2\xi_2 \\
=& c_1\left(\sqrt{M}, 0\right) + c_2\left(0, \sqrt{N}\right) \\
=& \left(c_1\sqrt{M}, c_2\sqrt{N}\right),
\end{aligned}$$

for $t_0 \in \mathcal{D}(T_0)$. Note that for $x \in \mathcal{D}(T_1)$, we have

$$\begin{aligned}
[x, t_1]_H =& x'(a)\bar{t}_1(a) - x'(b)\bar{t}_1(b) + x(b)\bar{t}_1'(b) - x(a)\bar{t}_1'(a) \\
=& \sqrt{M}x'(a)
\end{aligned}$$

and

$$\begin{aligned}
[x, t_2]_H =& x'(a)\bar{t}_2(a) - x'(b)\bar{t}_2(b) + x(b)\bar{t}_2'(b) - x(a)\bar{t}_2'(a) \\
=& -\sqrt{N}x'(b).
\end{aligned}$$

So, we define Ω, for $x \in \mathcal{D}(T)$, by

$$\Omega x = [x, t_1]\xi_1 + [x, t_2]_H\xi_2$$

$$
\begin{aligned}
&=\sqrt{M}x'(a)\left(\sqrt{M},0\right)-\sqrt{N}x'(b)\left(0,\sqrt{N}\right)\\
&=(Mx'(a),-Nx'(b)).
\end{aligned}
$$

The symplectic form in $H\oplus W$ with $\{t_1,t_2\}$ as the GKN set for T_0 is given by

$$
\begin{aligned}
[(x,(c_1,c_2)),(y,(c_1',c_2'))]_{H\oplus W} =&[x,y]_H-\langle\Omega x,(c_1',c_2')\rangle_W+\langle(c_1,c_2),\Omega y\rangle_W\\
=&x'(a)\overline{y}(a)-x'(b)\overline{y}(b)+x(b)\overline{y}'(b)-x(a)\overline{y}'(a)\\
&-\langle(Mx'(a),-Nx'(b)),(c_1',c_2')\rangle_W\\
&+\langle(c_1,c_2),(My'(a),-Ny'(b))\rangle_W\\
=&x'(a)\overline{y}(a)-x'(b)\overline{y}(b)+x(b)\overline{y}'(b)-x(a)\overline{y}'(a)\\
&-\left(\frac{Mx'(a)\overline{c}_1'}{M}-\frac{Nx'(b)\overline{c}_2'}{N}\right)\\
&+\left(\frac{c_1M\overline{y}'(a)}{M}-\frac{c_2N\overline{y}'(b)}{N}\right)\\
=&x'(a)\overline{y}(a)-x'(b)\overline{y}(b)+x(b)\overline{y}'(b)-x(a)\overline{y}'(a)\\
&-x'(a)\overline{c}_1'+x'(b)\overline{c}_2'+c_1\overline{y}'(a)-c_2\overline{y}'(b),
\end{aligned}
$$

for $(x,(c_1,c_2)),(y,(c_1',c_2'))\in H\oplus W$.

We are now ready to define the minimal and maximal operators in $H\oplus W$. Define $\widehat{U}_0:\mathcal{D}(\widehat{U}_0)\subseteq H\oplus W\to H\oplus W$ by

$$
\begin{aligned}
\widehat{U}_0(x,(c_1,c_2)) =&(T_1x,\mathcal{B}(c_1,c_2))\\
=&(-x'',\mathcal{B}(c_1,c_2)),\\
\mathcal{D}(\widehat{U}_0) =&\{(x,\Psi x)\mid x\in\Delta_0\},
\end{aligned}
$$

and so

$$
\widehat{U}_0(x,\Psi x)=\left(-x'',\mathcal{B}\left(c_1\sqrt{M},c_2\sqrt{N}\right)\right).
$$

Define $\widehat{U}_1:\mathcal{D}(\widehat{U}_1)\subseteq H\oplus W\to H\oplus W$ by

$$
\widehat{U}_1(x,(c_1,c_2))=(-x'',\mathcal{B}(c_1,c_2)-\Omega x)
$$

$$=\big(-x'', \mathcal{B}(c_1,c_2) - (Mx'(a), -Nx'(b))\big),$$

$$\mathcal{D}(\widehat{U}_1) = \left\{(x,(c_1,c_2)) \mid x \in \mathcal{D}(T_1),\ (c_1,c_2) \in \mathbb{C}^2\right\}.$$

Now define $x_1, x_2 \in \mathcal{D}(\widehat{U}_1)$ by

$$x_1(u) = \begin{cases} \sqrt{M}(u-a) & u \text{ near } a \\ 0 & u \text{ near } b, \end{cases}$$

$$x_2(u) = \begin{cases} 0 & u \text{ near } a \\ \sqrt{N}(u-b) & u \text{ near } b. \end{cases}$$

Claim 4.11. *$\{(x_1,(0,0)),(x_2,(0,0))\}$ is a GKN set for $\widehat{U}_0$ in $H \oplus W$.*

Proof. If $\alpha(x_1,(0,0))+\beta(x_2,(0,0)) \in \mathcal{D}(\widehat{U}_0)$, where $\alpha,\beta \in \mathbb{C}$, then $(\alpha x_1+\beta x_2,(0,0)) \in \mathcal{D}(\widehat{U}_0)$. So, by the definition of $\mathcal{D}(\widehat{U}_0)$, we have that $\Psi(\alpha x_1+\beta x_2) = (0,0)$. Therefore, $\alpha x_1 + \beta x_2 \in \mathcal{D}(T_0)$. Thus,

$$[f, \alpha x_1 + \beta x_2]_H = 0$$

for every $f \in \mathcal{D}(T_1)$.

Now define $f_1, f_2 \in \mathcal{D}(T_1)\backslash\mathcal{D}(T_0)$ by

$$f_1(u) = u,$$
$$f_2(u) = u^2.$$

Then we have

$$\begin{aligned}[f_1, \alpha x_1 + \beta x_2]_H =& f_1'(a)\overline{(\alpha x_1(a)+\beta x_2(a))} - f_1'(b)\overline{(\alpha x_1(b)+\beta x_2(b))} \\ &+ f_1(b)\overline{(\alpha x_1'(b)+\beta x_2'(b))} - f_1(a)\overline{(\alpha x_1'(a)+\beta x_2'(a))} \\ =& f_1(b)\left(\overline{\beta}\sqrt{N}\right) - f_1(a)\left(\overline{\alpha}\sqrt{M}\right) \\ =& b\overline{\beta}\sqrt{N} - a\overline{\alpha}\sqrt{M}\end{aligned}$$

and

$$[f_2, \alpha x_1 + \beta x_2]_H = f_2'(a)\overline{(\alpha x_1(a)+\beta x_2(a))} - f_2'(b)\overline{(\alpha x_1(b)+\beta x_2(b))}$$

$$
\begin{aligned}
&+ f_2(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_2(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \\
=&f_2(b)\left(\overline{\beta}\sqrt{N}\right) - f_2(a)\left(\overline{\alpha}\sqrt{M}\right) \\
=&b^2\overline{\beta}\sqrt{N} - a^2\overline{\alpha}\sqrt{M}.
\end{aligned}
$$

Solving the system of equations

$$
\begin{aligned}
0 =&b\overline{\beta}\sqrt{N} - a\overline{\alpha}\sqrt{M}, \\
0 =&b^2\overline{\beta}\sqrt{N} - a^2\overline{\alpha}\sqrt{M}
\end{aligned}
$$

yields $\alpha = 0$ and $\beta = 0$. Therefore, $\{(x_1, (0,0)), (x_2, (0,0))\}$ is linearly independent modulo $\mathcal{D}(\widehat{U}_0)$.

It remains to be shown that $0 = [(x_i, (0,0)), (x_j, (0,0))]_{H\oplus W}$ for $i, j = 1, 2$:

$$
\begin{aligned}
[(x_1, (0,0)), (x_1, (0,0))]_{H\oplus W} =&[x_1, x_1]_H \\
=&x_1'(a)\overline{x}_1(a) - x_1'(b)\overline{x}_1(b) + x_1(b)\overline{x}_1'(b) - x_1(a)\overline{x}_1'(a) \\
=&0, \\
[(x_1, (0,0)), (x_2, (0,0))]_{H\oplus W} =&[x_1, x_2]_H \\
=&x_1'(a)\overline{x}_2(a) - x_1'(b)\overline{x}_2(b) + x_1(b)\overline{x}_2'(b) - x_1(a)\overline{x}_2'(a) \\
=&0, \\
[(x_2, (0,0)), (x_2, (0,0))]_{H\oplus W} =&[x_2, x_2]_H \\
=&x_2'(a)\overline{x}_2(a) - x_2'(b)\overline{x}_2(b) + x_2(b)\overline{x}_2'(b) - x_2(a)\overline{x}_2'(a) \\
=&0.
\end{aligned}
$$

Hence, $\{(x_1, (0,0)), (x_2, (0,0))\}$ is a GNK set for $\widehat{U}_0$ in $H \oplus W$. □

Note that

$$
\begin{aligned}
\Omega x_1 &= (Mx_1'(a), -Nx_1'(b)) \\
&= \left(M^{3/2}, 0\right),
\end{aligned}
$$

and

$$\begin{aligned}\Omega x_2 &= (Mx_2'(a), -Nx_2'(b)) \\ &= \left(0, -N^{3/2}\right).\end{aligned}$$

We also have, for $x \in \mathcal{D}(T_1)$ and $(c_1, c_2) \in \mathbb{C}^2$,

$$\begin{aligned}[(x,(c_1,c_2)),(x_1,(0,0))]_{H\oplus W} =& x'(a)\overline{x}_1(a) - x'(b)\overline{x}_1(b) + x(b)\overline{x}_1'(b) - x(a)\overline{x}_1'(a) \\ & - x'(a)(0) + x'(b)(0) + c_1\overline{x}_1'(a) - c_2\overline{x}_1'(b) \\ =& - x(a)\sqrt{M} + c_1\sqrt{M}\end{aligned}$$

and

$$\begin{aligned}[(x,(c_1,c_2)),(x_2,(0,0))]_{H\oplus W} =& x'(a)\overline{x}_2(a) - x'(b)\overline{x}_2(b) + x(b)\overline{x}_2'(b) - x(a)\overline{x}_2'(a) \\ & - x'(a)(0) + x'(b)(0) + c_1\overline{x}_2'(a) - c_2\overline{x}_2'(b) \\ =& x(b)\sqrt{N} - c_2\sqrt{N}.\end{aligned}$$

So, if $0 = [(x,(c_1,c_2)),(x_1,(0,0))]_{H\oplus W}$, then $c_1 = x(a)$. Likewise, $c_2 = x(b)$ if $0 = [(x,(c_1,c_2)),(x_2,(0,0))]_{H\oplus W}$.

Define $\widehat{U} : \mathcal{D}(\widehat{U}) \subseteq H \oplus W \to H \oplus W$ by

$$\begin{aligned}\widehat{U}(x,(c_1,c_2)) &= (T_1 x, \mathcal{B}(c_1,c_2) - \Omega x) \\ &= \big(- x'', \mathcal{B}(c_1,c_2) - (Mx'(a), -Nx'(b))\big), \\ \mathcal{D}(\widehat{U}) &= \Big\{(x,(c_1,c_2)) \in \mathcal{D}(\widehat{U}_1) \mid -x(a)\sqrt{M} + c_1\sqrt{M} = 0, \\ & \qquad x(b)\sqrt{N} - c_2\sqrt{N} = 0\Big\}.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\widehat{U}(x,(x(a),x(b))) &= \big(- x'', \mathcal{B}(x(a),x(b)) - (Mx'(a), -Nx'(b))\big), \\ \mathcal{D}(\widehat{U}) &= \{(x,(x(a),x(b)) \mid x \in \mathcal{D}(T_1)\}.\end{aligned}$$

Then $\widehat{U}$ is self-adjoint in $H \oplus W$ by Theorem 3.22.

This next example will switch the roles of $\{t_1, t_2\}$ and $\{x_1, x_2\}$ from the previous example.

Example 4.12. Define $t_1, t_2 \in \mathcal{D}(T_0)$ by

$$t_1(u) = \begin{cases} \sqrt{M}(u-a) & u \text{ near } a \\ 0 & u \text{ near } b, \end{cases}$$
$$t_2(u) = \begin{cases} 0 & u \text{ near } a \\ \sqrt{N}(u-b) & \text{ near } b. \end{cases}$$

Claim 4.13. *$\{t_1, t_2\}$ is a GKN set for T_0.*

Proof. This claim follows directly from the proof of Claim 4.11. Since $t_1'(a) = \sqrt{M} \neq 0$ and $t_2'(b) = \sqrt{N} \neq 0$, we have that $t_1, t_2 \notin \mathcal{D}(T_0)$. If $\alpha t_1 + \beta t_2 \in \mathcal{D}(T_0)$, then $[f, \alpha t_1 + \beta t_2]_H = 0$ for all $f \in \mathcal{D}(T_1)$. Define $f_1, f_2 \in \mathcal{D}(T_1) \backslash \mathcal{D}(T_0)$ by

$$f_1(u) = u,$$
$$f_2(u) = u^2.$$

Then,

$$\begin{aligned} [f_1, \alpha t_1 + \beta t_2]_H =& f_1'(a)\overline{(\alpha t_1(a) + \beta t_2(a))} - f_1'(b)\overline{(\alpha t_1(b) + \beta t_2(b))} \\ &+ f_1(b)\overline{(\alpha t_1'(b) + \beta t_2'(b))} - f_1(a)\overline{(\alpha t_1'(a) + \beta t_2'(a))} \\ =& f_1(b)\left(\overline{\beta}\sqrt{N}\right) - f_1(a)\left(\overline{\alpha}\sqrt{M}\right) \\ =& b\overline{\beta}\sqrt{N} - a\overline{\alpha}\sqrt{M} \end{aligned}$$

and

$$\begin{aligned} [f_2, \alpha t_1 + \beta t_2]_H =& f_2'(a)\overline{(\alpha t_1(a) + \beta t_2(a))} - f_2'(b)\overline{(\alpha t_1(b) + \beta t_2(b))} \\ &+ f_2(b)\overline{(\alpha t_1'(b) + \beta t_2'(b))} - f_2(a)\overline{(\alpha t_1'(a) + \beta t_2'(a))} \\ =& f_2(b)\left(\overline{\beta}\sqrt{N}\right) - f_2(a)\left(\overline{\alpha}\sqrt{M}\right) \end{aligned}$$

$$=b^2\overline{\beta}\sqrt{N} - a^2\overline{\alpha}\sqrt{M}.$$

By solving the system of equations

$$\begin{aligned}0 =& b\overline{\beta}\sqrt{N} - a\overline{\alpha}\sqrt{M},\\ 0 =& b^2\overline{\beta}\sqrt{N} - a^2\overline{\alpha}\sqrt{M},\end{aligned}$$

we must have $\alpha = 0$ and $\beta = 0$. Thus, $\{t_1, t_2\}$ is linearly independent modulo $\mathcal{D}(T_0)$.

It now will be shown that $0 = [t_i, t_j]_H$ for $i, j = 1, 2$:

$$\begin{aligned}[t_1, t_1]_H =& t_1'(a)\overline{t}_1(a) - t_1'(b)\overline{t}_1(b) + t_1(b)\overline{t}_1'(b) - t_1(a)\overline{t}_1'(a)\\ =& 0,\\ [t_1, t_2]_H =& t_1'(a)\overline{t}_2(a) - t_1'(b)\overline{t}_2(b) + t_1(b)\overline{t}_2'(b) - t_1(a)\overline{t}_2'(a)\\ =& 0,\\ [t_2, t_2]_H =& t_2'(a)\overline{t}_2(a) - t_2'(b)\overline{t}_2(b) + t_2(b)\overline{t}_2'(b) - t_2(a)\overline{t}_2'(a)\\ =& 0\end{aligned}$$

since $t_1(a) = t_1(b) = t_1'(b) = t_2(a) = t_2(b) = t_2'(a) = 0$. Therefore, $\{t_1, t_2\}$ is a GNK set for T_0 in H. $\square$

Now define Δ_0 by $\Delta_0 = \mathcal{D}(T_0) + \text{span}\{t_1, t_2\}$. Then $\Psi : \Delta_0 \to W$ is defined by

$$\Psi(t_0 + c_1 t_1 + c_2 t_2) = c_1\xi_1 + c_2\xi_2 = \left(c_1\sqrt{M}, c_2\sqrt{N}\right).$$

Since , for $x \in \mathcal{D}(T_1)$,

$$\begin{aligned}[x, t_1]_H =& x'(a)\overline{t}_1(a) - x'(b)\overline{t}_1(b) + x(b)\overline{t}_1'(b) - x(a)\overline{t}_1'(a)\\ =& -\sqrt{M}x(a)\end{aligned}$$

and

$$[x, t_2]_H = x'(a)\overline{t}_2(a) + x'(b)\overline{t}_2(b) + x(b)\overline{t}_2'(b) - x(a)\overline{t}_2'(a)$$

$$=\sqrt{N}x(b),$$

we can now define Ω by

$$\begin{aligned}\Omega x =&[x,t_1]_H\xi_1+[x,t_2]_H\xi_2\\ =&-\sqrt{M}x(a)\left(\sqrt{M},0\right)+\sqrt{N}x(b)\left(0,\sqrt{N}\right)\\ =&\left(-Mx(a),Nx(b)\right),\end{aligned}$$

for $x\in\mathcal{D}(T_1)$.

The symplectic form in $H\oplus W$ is given by

$$\begin{aligned}[(x,(c_1,c_2)),(y,(c_1',c_2'))]_{H\oplus W} =&[x,y]_H-\langle\Omega x,(c_1',c_2')\rangle_W+\langle(c_1,c_2),\Omega y\rangle_W\\ =&x'(a)\overline{y}(a)-x'(b)\overline{y}(b)+x(b)\overline{y}'(b)-x(a)\overline{y}'(a)\\ &+x(a)\overline{c}_1'-x(b)\overline{c}_2'-c_1\overline{y}(a)-c_2\overline{y}(b),\end{aligned}$$

for $(x,(c_1,c_2)),(y,(c_1',c_2'))\in H\oplus W$.

The minimal operator $\widehat{V}_0:\mathcal{D}(\widehat{V}_0)\subseteq H\oplus W\to H\oplus W$ is defined by

$$\begin{aligned}\widehat{V}_0(x,(c_1,c_2)) =&(T_1x,\mathcal{B}(c_1,c_2))\\ =&(-x'',\mathcal{B}(c_1,c_2)),\\ \mathcal{D}(\widehat{V}_0)=&\{(x,\Psi x)\mid x\in\Delta_0\}.\end{aligned}$$

Rewriting this we have

$$\widehat{V}_0(x,\Psi x)=\left(-x'',\mathcal{B}\left(c_1\sqrt{M},c_2\sqrt{N}\right)\right).$$

The maximal operator $\widehat{V}_1:\mathcal{D}(\widehat{V}_1)\subseteq H\oplus W\to H\oplus W$ is defined by

$$\begin{aligned}\widehat{V}_1(x,(c_1,c_2)) =&(T_1x,\mathcal{B}(c_1,c_2)-\Omega x)\\ =&\big(-x'',\mathcal{B}(c_1,c_2)-(-Mx(a),Nx(b))\big),\\ \mathcal{D}(\widehat{T}_1)=&\left\{(x,(c_1,c_2))\mid x_1\in\mathcal{D}(T_1),\ (c_1,c_2)\in\mathbb{C}^2\right\}.\end{aligned}$$

Now define $x_1, x_2 \in \mathcal{D}(T_1)$ by

$$x_1(u) = \begin{cases} \sqrt{M} & u \text{ near } a \\ 0 & u \text{ near } b, \end{cases}$$

$$x_2(u) = \begin{cases} 0 & u \text{ near } a \\ \sqrt{N} & u \text{ near } b. \end{cases}$$

Claim 4.14. *$\{(x_1,(0,0)),(x_2,(0,0))\}$ is a GKN set for $\widehat{V}_0$ in $H \oplus W$.*

Proof. This claim mostly follows from the proof of Claim 4.10. If $\alpha(x_1,(0,0)) + \beta(x_2,(0,0)) = (\alpha x_1 \beta x_2,(0,0)) \in \mathcal{D}(\widehat{V}_0)$, then $\alpha x_1 + \beta x_2 \in \mathcal{D}(T_0)$ since $\Psi(\alpha x_1 + \beta x_2) = (0,0)$. So, for all $f \in \mathcal{D}(T_1)$,

$$[f, \alpha x_1 + \beta x_2]_H = 0.$$

Define $f_1, f_2 \in \mathcal{D}(T_1)\backslash\mathcal{D}(T_0)$ by

$$\begin{aligned} f_1(u) =& u, \\ f_2(y) =& u^2. \end{aligned}$$

Then we have

$$\begin{aligned} [f_1, \alpha x_1 + \beta x_2]_H =& f_1'(a)\overline{(\alpha x_1(a) + \beta x_2(a))} - f_1'(b)\overline{(\alpha x_1(b) + \beta x_2(b))} \\ &+ f_1(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_1(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \\ =& f_1'(a)\left(\overline{\alpha}\sqrt{M}\right) - f_1'(b)\left(\overline{\beta}\sqrt{N}\right) \\ =& \overline{\alpha}\sqrt{M} - \overline{\beta}\sqrt{N} \end{aligned}$$

and

$$\begin{aligned} [f_2, \alpha x_1 + \beta x_2]_H =& f_2'(a)\overline{(\alpha x_1(a) + \beta x_2(a))} - f_2'(b)\overline{(\alpha x_1(b) + \beta x_2(b))} \\ &+ f_2(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_2(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \end{aligned}$$

$$
\begin{aligned}
&=f_2'(a)\left(\overline{\alpha}\sqrt{M}\right)-f_2'(b)\left(\overline{\beta}\sqrt{N}\right)\\
&=2a\overline{\alpha}\sqrt{M}-2b\overline{\beta}\sqrt{N}.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
0=&\overline{\alpha}\sqrt{M}-\overline{\beta}\sqrt{N},\\
0=&2a\overline{\alpha}\sqrt{M}-2b\overline{\beta}\sqrt{N},
\end{aligned}
$$

and thus $\alpha=0$ and $\beta=0$. Hence, $\{(x_1(0,0)),(x_2,(0,0))\}$ is linearly independent modulo $\mathcal{D}(\widehat{V}_0)$.

It now remains to be shown that $0=[(x_i,(0,0)),(x_j,(0,0))]_{H\oplus W}=[x_i,x_j]_H$ for $i,j=1,2$:

$$
\begin{aligned}
[x_1,x_1]_H=&x_1'(a)\overline{x}_1(a)-x_1'(b)\overline{x}_1(b)+x_1(b)\overline{x}_1'(b)-x_1(a)\overline{x}_1'(a)\\
=&0,\\
[x_1,x_2]_H=&x_1'(a)\overline{x}_2(a)-x_1'(b)\overline{x}_2(b)+x_1(b)\overline{x}_2'(b)-x_1(a)\overline{x}_2(a)\\
=&0,\\
[x_2,x_2]_H=&x_2'(a)\overline{x}_2(a)-x_2'(b)\overline{x}_2(b)+x_2(b)\overline{x}_2'(b)-x_2(a)\overline{x}_2'(a)\\
=&0
\end{aligned}
$$

since $x_1'(a)=x_2'(a)=x_1'(b)=x_2'(b)=0$. So $\{(x_1,(0,0)),(x_2,(0,0))\}$ is a GKN set for $\widehat{V}_0$ in $H\oplus W$. □

For $(x,(c_1,c_2))\in\mathcal{D}(\widehat{V}_1)$, we have

$$
\begin{aligned}
[(x,(c_1,c_2)),(x_1,(0,0))]_{H\oplus W}=&x'(a)\overline{x}_1(a)-x'(b)\overline{x}_1(b)+x(b)\overline{x}_1(b)-x(a)\overline{x}_1'(a)\\
&+x(a)(0)-x(b)(0)-c_1\overline{x}_1(a)+c_2\overline{x}_1(b)\\
=&x'(a)\sqrt{M}-\sqrt{M}c_1
\end{aligned}
$$

and

$$
[(x,(c_1,c_2)),(x_2,(0,0))]_{H\oplus W}=x'(a)\overline{x}_2(a)-x'(b)\overline{x}_2(b)+x(b)\overline{x}_2(b)-x(a)\overline{x}_2'(a)
$$

$$
\begin{aligned}
&+ x(a)(0) - x(b)(0) - c_1\overline{x}_2(a) + c_2\overline{x}_2(b) \\
&= - x'(b)\sqrt{N} + c_2\sqrt{N}.
\end{aligned}
$$

So, if $0 = [(x,(c_1,c_2)),(x_1,(0,0))]_{H\oplus W} = [(x,(c_1,c_2)),(x_2,(0,0))]_{H\oplus W}$, then $c_1 = x'(a)$ and $c_2 = x'(b)$.

Define $\widehat{V} : \mathcal{D}(\widehat{V}) : \mathcal{D}(\widehat{V}) \subseteq H \oplus W \to H \oplus W$ by

$$
\begin{aligned}
\widehat{V}(x,(c_1,c_2)) =& (T_1 x, \mathcal{B}(c_1,c_2) - \Omega x) \\
=& \big(- x'', \mathcal{B}(c_1,c_2) - (Mx(a), Nx(b)) \big) \\
\mathcal{D}(\widehat{V}) =& \Big\{ (x,(c_1,c_2)) \in \mathcal{D}(\widehat{V}_1) \mid x'(a)\sqrt{M} - c_1\sqrt{M} = 0, \\
& \qquad -x'(b)\sqrt{N} + c_2\sqrt{N} = 0 \Big\} \\
=& \left\{ (x,(x'(a),x'(b)) \mid x \in \mathcal{D}(T_1) \right\}.
\end{aligned}
$$

The operator $\widehat{V}$ is self-adjoint in $H \oplus W$ by Theorem 3.22.

Example 4.15. In our final two-dimensional extension space, we will construct an operator that has both $x(a)$ and $x'(b)$ involved in the boundary conditions. This will be done by combining elements of the GKN sets for $\mathcal{D}(T_0)$ and $\mathcal{D}(\widehat{T}_0)$ in the previous example.

Define $t_1, t_2 \in \mathcal{D}(T_1)$ by

$$
t_1(u) = \begin{cases} \sqrt{M} & u \text{ near } a \\ 0 & u \text{ near } b, \end{cases}
$$

$$
t_2(u) = \begin{cases} 0 & u \text{ near } a \\ \sqrt{N}(u-b) & u \text{ near } b. \end{cases}
$$

Claim 4.16. *$\{t_1, t_2\}$ is a GKN set for T_0.*

Proof. Since $t_1(a) = \sqrt{M} \neq 0$ and $t_1'(b) = \sqrt{N} \neq 0$, $t_1, t_2 \notin \mathcal{D}(T_0)$. If $c_1t_1 + c_2t_2 \in \mathcal{D}(T_0)$, then $c_1t_1(a) + c_2t_2(a) = c_1\sqrt{M} = 0$ and $c_1t_1'(b) + c_2t_2'(b) = c_2\sqrt{N} = 0$.

Therefore, $c_1 = c_2 = 0$ and $\{t_1, t_2\}$ is linearly independent modulo $\mathcal{D}(T_0)$. We also have that

$$\begin{aligned}
[t_1, t_1]_H =& t_1'(a)\bar{t}_1(a) - t_1'(b)\bar{t}_1(b) + t_1(b)\bar{t}_1'(b) - t_1(a)\bar{t}_1'(a) = 0, \\
[t_1, t_2]_H =& t_1'(a)\bar{t}_2(a) - t_1'(b)\bar{t}_2(b) + t_1(b)\bar{t}_2'(b) - t_1(a)\bar{t}_2'(a) = 0, \\
[t_2, t_2]_H =& t_2'(a)\bar{t}_2(a) - t_2'(b)\bar{t}_2(b) + t_2(b)\bar{t}_2'(b) - t_2(a)\bar{t}_2'(a) = 0.
\end{aligned}$$

Thus, $\{t_1, t_2\}$ is a GKN set for $\mathcal{D}(T_0)$ in H. □

As before, we can define Δ_0 by $\Delta_0 = \mathcal{D}(T_0) + \text{span}\{t_1, t_2\}$, and $\Psi : \Delta_0 \to W$ is defined by

$$\Psi(t_0 + c_1 t_1 + c_2 t_2) = c_1\xi_1 + c_2\xi_2 = \left(c_1\sqrt{M}, c_2\sqrt{N}\right)$$

for $t_0 \in \mathcal{D}(T_0)$.

Since, for $x \in \mathcal{D}(T_1)$,

$$\begin{aligned}
[x, t_1]_H =& x'(a)\bar{t}_1(a) - x'(b)\bar{t}_1(b) + x(b)\bar{t}_1'(b) - x(a)\bar{t}_1'(a) \\
=& \sqrt{M}x'(a)
\end{aligned}$$

and

$$\begin{aligned}
[x, t_2]_H =& x'(a)\bar{t}_2(a) - x'(b)\bar{t}_2(b) + x(b)\bar{t}_2'(b) - x(a)\bar{t}_2'(a) \\
=& \sqrt{N}x(b),
\end{aligned}$$

we can define Ω by

$$\begin{aligned}
\Omega x =& [x, t_1]_H\xi_1 + [x, t_2]_H\xi_2 \\
=& \sqrt{M}x'(a)\left(\sqrt{M}, 0\right) + \sqrt{N}x(b)\left(0, \sqrt{N}\right) \\
=& \left(Mx'(a), Nx(b)\right),
\end{aligned}$$

for any $x \in \mathcal{D}(T_1)$.

The symplectic form in $H \oplus W$ is given by

$$\begin{aligned}[(x,(c_1,c_2)),(y,(c_1',c_2'))]_{H\oplus W} =& [x,y]_H - \langle \Omega x,(c_1',c_2')\rangle_W + \langle (c_1,c_2),\Omega y\rangle_W \\ & x'(a)\overline{y}(a) - x'(b)\overline{y}(b) + x(b)\overline{y}'(b) - x(a)\overline{y}'(a) \\ & - x'(a)\overline{c}_1' - x(b)\overline{c}_2' + c_1\overline{y}'(a) + c_2\overline{y}(b),\end{aligned}$$

for $(x,(c_1,c_2)),(y,(c_1',c_2')) \in H \oplus W$.

We will now define $\widehat{Q}_0$ and $\widehat{Q}_1$, the minimal and maximal operators in $H \oplus W$. Define the minimal operator $\widehat{Q}_0 : \mathcal{D}(\widehat{Q}_0) \subseteq H \oplus W \to H \oplus W$ by

$$\begin{aligned}\widehat{Q}_0(x,(c_1,c_2)) =& (T_1 x, \mathcal{B}(c_1,c_2)) \\ =& (-x'', \mathcal{B}(c_1,c_2)), \\ \mathcal{D}(\widehat{Q}_0) =& \{(x,\Psi x) \mid x \in \Delta_0\},\end{aligned}$$

and so $\widehat{Q}_0(x,\Psi x) = \left(-x'', \mathcal{B}\left(c_1\sqrt{M}, c_2\sqrt{N}\right)\right)$. Define the maximal operator $\widehat{Q}_1 : \mathcal{D}(\widehat{Q}_1) \subseteq H \oplus W \to H \oplus W$ by

$$\begin{aligned}\widehat{Q}_1(x,(c_1,c_2)) =& (T_1 x, \mathcal{B}(c_1,c_2) - \Omega x) \\ =& \big(-x'', \mathcal{B}(c_1,c_2) - (Mx'(a), Nx(b))\big), \\ \mathcal{D}(\widehat{Q}_1) =& \left\{(x,c_1,c_2)) \mid x \in \mathcal{D}(T_1),\ (c_1,c_2) \in \mathbb{C}^2\right\}.\end{aligned}$$

Now define $x_1, x_2 \in \mathcal{D}(T_1)\backslash\mathcal{D}(T_0)$ by

$$x_1(u) = \begin{cases} 0 & u \text{ near } a \\ \sqrt{N} & u \text{ near } b, \end{cases}$$

$$x_2(u) = \begin{cases} \sqrt{M}(u-a) & u \text{ near } a \\ 0 & u \text{ near } b. \end{cases}$$

Claim 4.17. *$\{(x_1,(0,0)),(x_2,(0,0))\}$ is a GKN set for $\widehat{Q}_0$ in $H \oplus W$.*

Proof. It will first be shown that $\{(x_1,(0,0)),(x_2,(0,0))\}$ is linearly independent modulo $\mathcal{D}(\widehat{Q}_0)$. If $\alpha(x_1,(0,0)) + \beta(x_2,(0,0)) = (\alpha x_1 + \beta x_2,(0,0)) \in \mathcal{D}(\widehat{Q}_0)$, then $\Psi(\alpha x_1 + \beta x_2) = (0,0)$. Hence, $\alpha x_1 + \beta x_2 \in \mathcal{D}(T_0)$. So, for any $f \in \mathcal{D}(\widehat{Q}_0)$,

$$[f, \alpha x_1 + \beta x_2]_H = 0.$$

Define $f_1, f_2 \in \mathcal{D}(T_0)\backslash\mathcal{D}(T_0)$ by

$$f_1(u) = u \quad \text{and} \quad f_2(u) = u^2.$$

Then

$$\begin{aligned}[f_1, \alpha x_1 + \beta x_2]_H =& f_1'(a)\overline{(\alpha x_1(a) + \beta x_2(a))} - f_1'(b)\overline{(\alpha x_1(b) + \beta x_2(b))} \\ &+ f_1(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_1(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \\ =& - f_1'(b)\overline{\left(\alpha\sqrt{N}\right)} - f_1(a)\overline{\left(\beta\sqrt{M}\right)} \\ =& - \overline{\alpha}\sqrt{N} - a\overline{\beta}\sqrt{M}\end{aligned}$$

and

$$\begin{aligned}[f_2, \alpha x_1 + \beta x_2]_H =& f_2'(a)\overline{(\alpha x_1(a) + \beta x_2(a))} - f_2'(b)\overline{(\alpha x_1(b) + \beta x_2(b))} \\ &+ f_2(b)\overline{(\alpha x_1'(b) + \beta x_2'(b))} - f_2(a)\overline{(\alpha x_1'(a) + \beta x_2'(a))} \\ =& - f_2'(b)\overline{\left(\alpha\sqrt{N}\right)} - f_2(a)\overline{\left(\beta\sqrt{M}\right)} \\ =& - 2b\overline{\alpha}\sqrt{N} - a^2\overline{\beta}\sqrt{M},\end{aligned}$$

and so solving the system of equations

$$\begin{aligned}0 =& - \overline{\alpha}\sqrt{N} - a\overline{\beta}\sqrt{M} \\ 0 =& - 2b\overline{\alpha}\sqrt{N} - a^2\overline{\beta}\sqrt{M}\end{aligned}$$

yields $\alpha, \beta = 0$. Hence, $\{(x_1,(0,0)),(x_2,(0,0))\}$ is linearly independent modulo $\mathcal{D}(\widehat{Q}_0)$.

It remains to be shown that $[(x_i,(0,0)),(x_j,(0,0))]_{H\oplus W} = [x_i, x_j]_H = 0$ for $i, j = 1, 2$:

$$[x_1, x_1]_H = x_1'(a)\overline{x}_1(a) - x_1'(b)\overline{x}_1(b) + x_1(b)\overline{x}_1'(b) - x_1(a)\overline{x}_1'(a) = 0$$

$$[x_1, x_2]_H = x_1'(a)\overline{x}_2(a) - x_1'(b)\overline{x}_2(b) + x_1(b)\overline{x}_2'(b) - x_1(a)\overline{x}_2'(a) = 0$$

$$[x_2, x_2]_H = x_2'(a)\overline{x}_2(a) - x_2'(b)\overline{x}_2(b) + x_2(b)\overline{x}_2'(b) - x_2(a)\overline{x}_2'(a) = 0.$$

Therefore, $\{(x_1, (0,0)), (x_2, (0,0))\}$ is a GKN set for $\widehat{Q}_0$. □

Note that, for $(c_1, c_2) \in \mathbb{C}^2$ and $x \in \mathcal{D}(T_1)$, we have

$$\begin{aligned}
[(x, (c_1, c_2)), (x_1, (0,0))]_{H\oplus W} =& x'(a)\overline{x}_1(a) - x'(b)\overline{x}_1(b) + x(b)\overline{x}_1'(b) - x(a)\overline{x}_1'(a) \\
&- x'(a)(0) - x(b)(0) + c_1\overline{x}_1'(a) + c_2\overline{x}_1(b) \\
=& - x'(b)\sqrt{N} + c_2\sqrt{N},
\end{aligned}$$

and

$$\begin{aligned}
[(x, (c_1, c_2)), (x_2, (0,0))]_{H\oplus W} =& x'(a)\overline{x}_2(a) - x'(b)\overline{x}_2(b) + x(b)\overline{x}_2'(b) - x(a)\overline{x}_2'(a) \\
&- x'(a)(0) - x(b)(0) + c_1\overline{x}_2'(a) + c_2\overline{x}_2(b) \\
=& - x(a)\sqrt{M} + c_1\sqrt{M}.
\end{aligned}$$

So, if

$$[(x, (c_1, c_2)), (x_1, (0,0))]_{H\oplus W} = 0$$

and

$$[(x, (c_1, c_2)), (x_2, (0,0))]_{H\oplus W} = 0,$$

then $c_1 = x(a)$ and $c_2 = x'(b)$.

Define the operator $\widehat{Q} : \mathcal{D}(\widehat{Q}) \subseteq H \oplus W \to H \oplus W$ by

$$\begin{aligned}
\widehat{Q}(x, (c_1, c_2)) =& (T_1 x, \mathcal{B}(c_1, c_2) - \Omega x) \\
=& \big(- x'', \mathcal{B}(c_1, c_2) - (Mx'(a), Nx(b))\big), \\
\mathcal{D}(\widehat{Q}) =& \Big\{ (x, (c_1, c_2)) \in \mathcal{D}(\widehat{Q}_1) \mid -x'(b)\sqrt{N} + c_2\sqrt{N} = 0, \\
& -x(a)\sqrt{M} + c_1\sqrt{M} \Big\}.
\end{aligned}$$

Equivalently,

$$\begin{aligned} \widehat{Q}(x,(x(a),x'(b))) &= \big(-x'', \mathcal{B}(x(a),x'(b)) - (Mx'(a),Nx(b))\big), \\ \mathcal{D}(\widehat{Q}) &= \{(x,(x(a),x'(b))) \mid x \in \mathcal{D}(T_1)\}\,. \end{aligned}$$

Then, by Theorem 3.22, $\widehat{Q}$ is self-adjoint in $H \oplus W$.

CHAPTER FIVE

A Self-Adjoint Operator Generated by the Sixth-Order Krall Differential Expression in the Extended Space $L^2[-1,1] \oplus \mathbb{C}^2$ with the Krall Polynomials as Eigenfunctions

In this chapter, we will construct a self-adjoint operator generated by the sixth-order Krall differential expression in the extended space $L^2[-1,1] \oplus \mathbb{C}^2$ having the Krall polynomials as eigenfunctions. We will do so by applying the extended GKN-EM Theorem developed in Chapter Three to the sixth-order Krall differential expression.

5.1 The Extension Space

Let the extension space be $W = \mathbb{C}^2$ and define the inner product $\langle \cdot, \cdot \rangle_W$ by

$$\langle (a,b),(a',b')\rangle_W := \frac{a\overline{a}'}{A} + \frac{b\overline{b}'}{B},$$

where A and B are as in (2.1). Now define $\xi_1 = (\sqrt{A}, 0)$ and $\xi_2 = (0, \sqrt{B})$. Then $\{\xi_1, \xi_2\}$ is an orthonormal basis for W.

Every self-adjoint operator $\mathcal{B} : W \to W$ has the form

$$\mathcal{B} = \begin{pmatrix} \delta & \beta \\ -\overline{\beta}\frac{B}{A} & \gamma \end{pmatrix},$$

where $\delta, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

The inner product in the extended space $H \oplus W$ is now given by

$$\langle (f,(a_1,b_1)),(g,(a_2,b_2))\rangle_{H\oplus W} = \langle f,g\rangle_H + \langle (a_1,b_1),(a_2,b_2)\rangle_W.$$

5.2 A Partial GKN Set for T_0

Define $P = \{t_1, t_2\} \subseteq C^6[-1,1]$ where

$$t_1(x) = \begin{cases} \sqrt{A} & x \text{ near } -1 \\ 0 & x \text{ near } 1 \end{cases}$$

and

$$t_2(x) = \begin{cases} 0 & x \text{ near } -1 \\ \sqrt{B} & x \text{ near } 1 \end{cases}$$

are each real-valued and in Δ.

Lemma 5.1. *P is a partial GKN set for T_0.*

Proof. It will first be shown that P is linearly independent modulo $\mathcal{D}(T_0)$. Let $f \in \Delta$, where Δ is defined in (2.4), and $c_1, c_2 \in \mathbb{C}$. Then

$$\begin{aligned}
[f, c_1t_1 + c_2t_2]_H =& \overline{c}_1[f, t_1]_H + \overline{c}_2[f, t_2]_H \\
=& -\overline{c}_1\sqrt{A}\left(\Lambda'[f](-1) - \pi(-1)f'(-1)\right) \\
&+ \overline{c}_2\sqrt{B}\left(\Lambda'[f](1) - \pi(1)f'(1)\right),
\end{aligned}$$

where $\pi(x)$ is defined as in (2.6), $[\cdot,\cdot]_H := [\cdot,\cdot]$ as given in (2.5), and $\Lambda[\cdot]$ is given in (2.14). Now define $f_1, f_2 \in \Delta$ by, $f_1, f_2 \subset C^6[-1, 1]$

$$f_1(x) = \begin{cases} 0 & x \text{ near } -1 \\ x & \text{near } 1 \end{cases}$$

and

$$f_2(x) = \begin{cases} x & \text{near } -1 \\ 0 & \text{near } 1 \end{cases}$$

and both are real-valued.

Then we have

$$\begin{aligned}
[f_1, c_1t_1 + c_2t_2]_H =& -\overline{c}_1\sqrt{A}\left(\Lambda'[f_1](-1) - \pi(-1)f_1'(-1)\right) \\
&+ \overline{c}_2\sqrt{B}\left(\Lambda'[f_1](1) - \pi(1)f_1'(1)\right) \\
=& \overline{c}_2\sqrt{B}\pi(1) \\
=& \overline{c}_2\sqrt{B}\left(24A + 24AB + 24\right)
\end{aligned}$$

and

$$
\begin{aligned}
[f_2, c_1t_1 + c_2t_2]_H =& -\overline{c}_1\sqrt{A}\left(\Lambda'[f_2](-1) - \pi(-1)f_2'(-1)\right) \\
& + \overline{c}_2\sqrt{B}\left(\Lambda'[f_2](1) - \pi(1)f_2'(1)\right) \\
=& \overline{c}_1\sqrt{A}\pi(-1) \\
=& \overline{c}_1\sqrt{A}\left(24B + 24\right).
\end{aligned}
$$

Solving the system of equations

$$
\begin{aligned}
[f_1, c_1t_1 + c_2t_2]_H =& 0 \\
[f_2, c_1t_1 + c_2t_2]_H =& 0
\end{aligned}
$$

yields $c_1 = 0$ and $c_2 = 0$. Therefore, $P = \{t_1, t_2\}$ is linearly independent modulo $\mathcal{D}(T_0)$.

It remains to be shown that $[t_1, t_1]_H = [t_1, t_2]_H = [t_2, t_2]_H = 0$:

$$
\begin{aligned}
[t_1, t_1]_H =& [t_1, t_1](1) - [t_1, t_1](-1) \\
=& \sqrt{A}[t_1, 1](1) - \sqrt{A}[t_1, 1](-1) \\
=& 0 - \sqrt{A}\left(\Lambda'[t_1](-1) - \pi(-1)t_1'(-1)\right) \\
=& 0; \\
[t_1, t_2]_H =& [t_1, t_2](1) - [t_1, t_2](-1) \\
=& \sqrt{B}[t_1, 1](1) - \sqrt{B}[t_1, 1](-1) \\
=& 0 - \sqrt{B}\left(\Lambda'[t_1](-1) - \pi(-1)t_1'(-1)\right) \\
=& 0; \\
[t_2, t_2]_H =& [t_2, t_2](1) - [t_2, t_2](-1) \\
=& \sqrt{B}[t_2, 1](1) - \sqrt{B}[t_2, 1](-1) \\
=& \sqrt{B}\left(\Lambda'[t_2](1) - \pi(1)t_2'(1)\right) - 0 \\
=& 0
\end{aligned}
$$

by the properties of Δ stated in Theorem 2.1. Thus, $P = \{t_1, t_2\}$ is a partial GKN set for T_0. $\square$

Now that we have a partial GKN set for T_0, we can define Δ_0, Ψ, and Ω as follows: define Δ_0 by $\Delta_0 := \mathcal{D}(T_0) + \text{span}\{t_1, t_2\}$, define $\Psi : \Delta_0 \to W$ by

$$\begin{aligned}\Psi(f_0 + \alpha_1 t_1 + \alpha_2 t_2) &= \alpha_1 \xi_1 + \alpha_2 \xi_2 \\ &= \left(\alpha_1 \sqrt{A}, \alpha_2 \sqrt{B}\right),\end{aligned}$$

where $f_0 \in \mathcal{D}(T_0)$, and define $\Omega : \Delta \to W$ by

$$\begin{aligned}\Omega f &= [f, t_1]_H \xi_1 + [f, t_2]_H \xi_2 \\ &= -\sqrt{A}[f, 1](-1)\xi_1 + \sqrt{B}[f, 1](1)\xi_2 \\ &= (-A[f, 1](-1), B[f, 1](1)).\end{aligned}$$

5.3 The Maximal and Minimal Operators in the Extended Space $H \oplus W$

Note that Theorem 3.22 gives a one-parameter family of self-adjoint operators. By fixing the operator $\mathcal{B}$, we will be working with one specific self-adjoint operator, namely the self-adjoint operator that has the Krall polynomials as eigenfunctions.

A computation involving the Krall polynomials show that $\mathcal{B} = 0$. Then the minimal operator in $H \oplus W$, $\widehat{T}_0 : \mathcal{D}(\widehat{T}_0) \subseteq H \oplus W \to H \oplus W$, is defined by

$$\begin{aligned}\widehat{T}_0 \left(f, (a, b)\right) &= (T_1 f, 0) \\ \mathcal{D}(\widehat{T}_0) &= \{(f, \Psi f) \mid f \in \Delta_0\}.\end{aligned}$$

So, we can write $\widehat{T}_0$ as $\widehat{T}_0(f, \Psi f) = (T_1 f, 0)$.

The maximal operator in $H \oplus W$, $\widehat{T}_1 : \mathcal{D}(\widehat{T}_1) \subseteq H \oplus W \to H \oplus W$, is defined by

$$\begin{aligned}\widehat{T}_1 \left(f, (a, b)\right) &= (T_1 f, -\Omega f) \\ \mathcal{D}(\widehat{T}_1) &= \{(f, (a, b)) \mid f \in \Delta, (a, b) \in W\}.\end{aligned}$$

Therefore, $\widehat{T}_1 \left(f, (a, b)\right) = \big(T_1 f, (-A[f, 1](-1), -B[f, 1](1))\big)$.

The symplectic form $[\cdot,\cdot]_{H\oplus W}$ is given by

$$[(f,(a_1,b_1)),(g,(a_2,b_2))]_{H\oplus W} = [f,g]_H - \langle \Omega f,(a_2,b_2)\rangle_W + \langle (a_1,b_1),\Omega g\rangle_W$$

for $(f,(a_1,b_1)),(g,(a_2,b_2)) \in \mathcal{D}\left(\widehat{T}_1\right)$.

5.4 A GKN Set for $\widehat{T}_0$ in $H\oplus W$

Define $y_1,y_2,y_3,y_4 \subseteq C^6[-1,1]$ to be real-valued such that

$$\begin{aligned}
y_1(x) &= \begin{cases} 0 & x \text{ near } -1 \\ (1-x^2)^2 & x \text{ near } 1, \end{cases} \\
y_2(x) &= \begin{cases} (1-x^2)^2 & x \text{ near } -1 \\ 0 & x \text{ near } 1, \end{cases} \\
y_3(x) &= \begin{cases} 0 & x \text{ near } -1 \\ 1-x^2 & x \text{ near } 1, \end{cases} \\
y_4(x) &= \begin{cases} 1-x^2 & x \text{ near } -1 \\ 0 & x \text{ near } 1. \end{cases}
\end{aligned} \tag{5.1}$$

Then each $y_i \in \Delta$.

Remark 5.2. Note that, by parts (v) and (vi) of Theorem 2.1, we have, for $f \in \Delta$,

$$[f,y_1]_H = [f,(1-x^2)^2](1) = 192f(1)$$

$$[f,y_2]_H = -[f,(1-x^2)^2](-1) = 192f(-1)$$

$$[f,y_3]_H = [f,1-x^2](1) = 2\Lambda[f](1) - 48(A+2)f(1)$$

$$[f,y_4]_H = -[f,1-x^2](-1) = 2\Lambda[f](-1) - 48(B+2)f(-1).$$

Lemma 5.3. $\{y_i,(0,0))\}_{i=1}^4$ *is a GKN set for* $\widehat{T}_0$ *in* $H\oplus W$.

Proof. It will first be shown that $\{y_i,(0,0))\}_{i=1}^4$ is linearly independent modulo $\mathcal{D}(\widehat{T}_0)$. Since $[(y_i,(0,0)),(y_j,(0,0))]_{H\oplus W} = [y_i,y_j]_H$ for $i,j = 1,2,3,4$, it is sufficient to show that $\{y_i\}_{i=1}^4$ is linearly independent modulo $\mathcal{D}(T_0)$.

Let $f \in \Delta$ and $c_1, c_2, c_3, c_4 \in \mathbb{C}$. Then

$$\begin{aligned}[f, c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4]_H =& \overline{c}_1[f, y_1]_H + \overline{c}_2[f, y_2]_H + \overline{c}_3[f, y_3]_H + \overline{c}_4[f, y_4]_H \\ =& 192\overline{c}_1 f(1) + 192\overline{c}_2 f(-1) \\ &+ \overline{c}_3\left(2\Lambda[f](1) - 48(A+2)f(1)\right) \\ &+ \overline{c}_4\left(2\Lambda[f](-1) - 48(B+2)f(-1)\right).\end{aligned}$$

Define $f_1, f_2 \subset C^6[-1,1]$ to be real-valued such that

$$f_1(x) = \begin{cases} 1 & x \text{ near } -1 \\ 0 & x \text{ near } 1, \end{cases}$$

$$f_2(x) = \begin{cases} 0 & x \text{ near } -1 \\ 1 & x \text{ near } 1. \end{cases}$$

Then each $f_i \in \Delta$ and

$$[f_1, c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4]_H = 192\overline{c}_2 - 48(B+2)\overline{c}_4$$

and

$$[f_2, c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4]_H = 192\overline{c}_1 - 48(A+2)\overline{c}_3.$$

Now define $h_\pm(x)$ as in Theorem 2.1 (vii) by

$$h_+(x) = \begin{cases} 0 & x \text{ near } -1 \\ \frac{1}{8}(A+2)(1-x^2)^2\ln(1-x^2) + \frac{1}{2}(1-x^2)\ln(1-x^2) & x \text{ near } 1, \end{cases}$$

$$h_-(x) = \begin{cases} \frac{1}{8}(B+2)(1-x^2)^2\ln(1-x^2) + \frac{1}{2}(1-x^2)\ln(1-x^2) & x \text{ near } -1 \\ 0 & x \text{ near } 1. \end{cases}$$

By Theorem 2.1 part (vii), $h_\pm \in \Delta$. Note that near $x = 1$,

$$h'_+(x) = \ln(1-x^2)\left(-\frac{1}{2}(A+2)x(1-x^2) - x\right) - \frac{1}{4}(A+2)x(1-x^2) - x$$

and

$$\begin{aligned}h_+''(x) =&(A+2)x^2 + \frac{2x^2}{1-x^2} + \ln(1-x^2)\left(-\frac{1}{2}(A+2)(1-3x^2)-1\right)\\&-\frac{1}{4}(A+2)(1-3x^2)-1.\end{aligned}$$

So, near $x=1$, we have

$$\begin{aligned}\Lambda[h_+](x) =& -\left(\left(1-x^2\right)^3 h_+^{(3)}(x)\right)' + \left(1-x^2\right)\left(12+\alpha\left(1-x^2\right)\right)h_+''(x)\\ =&(12+\alpha(1-x^2))\Bigg((1-x^2)(A+2)x^2+2x^2\\ &+(1-x^2)\ln(1-x^2)\left(-\frac{1}{2}(A+2)(1-3x^2)-1\right)\\ &-\frac{1}{4}(1-x^2)(A+2)(1-3x^2)-(1-x^2)\Bigg),\end{aligned}$$

so $\Lambda[h_+](1)=24$ and

$$[h_+, c_1y_1+c_2y_2+c_3y_3+c_4y_4]_H = 48\overline{c}_3.$$

Likewise, near $x=-1$, we have

$$h_-'(x) = \ln(1-x^2)\left(-\frac{1}{2}(B+2)x(1-x^2)-x\right) - \frac{1}{4}(B+2)x(1-x^2)-x$$

and

$$\begin{aligned}h_-''(x) =&(B+2)x^2 + \frac{2x^2}{1-x^2} + \ln(1-x^2)\left(-\frac{1}{2}(B+2)(1-3x^2)-1\right)\\&-\frac{1}{4}(B+2)(1-3x^2)-1.\end{aligned}$$

So,

$$\begin{aligned}\Lambda[h_-](x) =& -\left(\left(1-x^2\right)^3 h_-^{(3)}(x)\right)' + \left(1-x^2\right)\left(12+\alpha\left(1-x^2\right)\right)h_-''(x)\\ =&(12+\alpha(1-x^2))\Bigg((1-x^2)(B+2)x^2+2x^2\\ &+(1-x^2)\ln(1-x^2)\left(-\frac{1}{2}(B+2)(1-3x^2)-1\right)\end{aligned}$$

$$-\frac{1}{4}(1-x^2)(B+2)(1-3x^2)-(1-x^2)\Big)$$

near $x=-1$, so $\Lambda[h_-](-1)=24$, and

$$[h_-, c_1y_1+c_2y_2+c_3y_3+c_4y_4]_H = 48\overline{c}_4.$$

By solving the system of equations

$$[f_1, c_1y_1+c_2y_2+c_3y_3+c_4y_4]_H = 0$$
$$[f_2, c_1y_1+c_2y_2+c_3y_3+c_4y_4]_H = 0$$
$$[h_+, c_1y_1+c_2y_2+c_3y_3+c_4y_4]_H = 0$$
$$[h_-, c_1y_1+c_2y_2+c_3y_3+c_4y_4]_H = 0,$$

or equivalently,

$$0 = 192\overline{c}_2 - 48(B+2)\overline{c}_4$$
$$0 = 192\overline{c}_1 - 48(A+2)\overline{c}_3$$
$$0 = 48\overline{c}_3$$
$$0 = 48\overline{c}_4,$$

we have $\overline{c}_1=\overline{c}_2=\overline{c}_3=\overline{c}_4=0$. Therefore, $\{y_i\}_{i=1}^4$ is linearly independent modulo $\mathcal{D}(T_0)$ and $\{y_i,(0,0))\}_{i=1}^4$ is linearly independent modulo $\mathcal{D}(\widehat{T}_0)$.

Now it will be shown that $[(y_i,(0,0)),(y_j,(0,0))]_{H\oplus W} = [y_i,y_j]_H = 0$ for $i,j = 1,2,3,4$. Using Remark 5.2, we have

$$[y_1,y_1]_H = 192y_1(1)$$
$$= 0,$$
$$[y_1,y_2]_H = 192y_1(-1)$$
$$= 0,$$
$$[y_1,y_3]_H = -[y_3,y_1]_H$$

$$
\begin{aligned}
&= -192 y_3(1) \\
&= 0, \\
[y_1, y_4]_H &= -[y_4, y_1]_H \\
&= -192 y_4(1) \\
&= 0, \\
[y_2, y_2]_H &= 192 y_2(-1) \\
&= 0, \\
[y_2, y_3]_H &= -[y_3, y_2]_H \\
&= -192 y_3(-1) \\
&= 0, \\
[y_2, y_4]_H &= -[y_4, y_2]_H \\
&= -192 y_4(-1) \\
&= 0, \\
[y_3, y_3]_H &= 2\Lambda[y_3](1) - 48(A+2) y_3(1) \\
&= 0, \\
[y_3, y_4]_H &= 2\Lambda[y_3](-1) - 48(B+2) y_3(-1) \\
&= 0, \\
[y_4, y_4]_H &= 2\Lambda[y_4](-1) - 48(B+2) y_4(-1) \\
&= 0.
\end{aligned}
$$

Thus, $\{(y_i, (0,0))\}_{i=1}^4$ is a GKN set for $\widehat{T}_0$. □

Now recall the functions $e_\pm$ defined in (2.15) and (2.16) as

$$
e_+(x) = \begin{cases} 0 & x \text{ near } -1 \\ \frac{1}{2}(1-x^2) + \frac{1}{8}(A+2)(1-x^2)^2 & x \text{ near } 1 \end{cases}
$$

and

$$e_-(x) = \begin{cases} -\frac{1}{2}(1-x^2) - \frac{1}{8}(B+2)(1-x^2)^2 & x \text{ near } -1 \\ 0 & x \text{ near } 1. \end{cases}$$

Note that e_+ and e_- are linear combinations of y_1, y_2, y_3, and y_4, so $e_+, e_- \in \Delta$ and $[f, y_i]_H = 0 \iff [f, e_+]_H = [f, e_-]_H = 0$ for $f \in \Delta$.

In fact, we define

$$\delta := \{f \in \Delta \mid [f, e_+](1) = [f, e_-](-1) = 0\}. \tag{5.2}$$

Then, by part (vi) of Theorem 2.3, for $f \in \delta$, we have

$$[f, 1](1) = -24f''(1) - 24(A+1)f'(1)$$

and

$$[f, 1](-1) = 24f''(-1) - 24(B+1)f'(-1).$$

So, for $f \in \delta$,

$$\begin{aligned} \Omega f =& (-A[f,1](-1), B[f,1](1)) \\ =& \big(-24Af''(-1) + 24(AB+A)f'(-1), -24Bf''(1) - 24(AB+B)f'(1)\big). \end{aligned}$$

Also note that

$$\begin{aligned} \Omega e_+ =& [e_+, t_1]_H \xi_1 + [e_+, t_2]_H \xi_2 \\ =& \frac{1}{2}[1-x^2, t_1](1)\xi_1 + \frac{1}{8}(A+2)[(1-x^2)^2, t_1](1)\xi_1 \\ & + \frac{1}{2}[1-x^2, t_2](1)\xi_2 + \frac{1}{8}(A+2)[(1-x^2)^2, t_2](1)\xi_2 \\ =& \bigg(\frac{1}{2}A[1-x^2, 1](1) + \frac{1}{8}(AB+2B)[(1-x^2)^2, 1](1), \\ & \frac{1}{2}B[1-x^2, 1](1) + \frac{1}{8}(AB+2B)[(1-x^2)^2, 1](1)\bigg) \\ =& \bigg(24A + 24A^2 + 24A + \frac{1}{8}(A^2+2A)(-192), \end{aligned}$$

$$24B + 24(AB + B) + \frac{1}{8}(AB + 2B)(-192)\Big)$$

$$=(0, 0).$$

Likewise,

$$\begin{aligned}
\Omega e_- =&[e_-, t_1]_H \xi_1 + [e_-, t_2]_H \xi_2 \\
=&\frac{1}{2}[1 - x^2, t_1](-1)\xi_1 + \frac{1}{8}(B + 2)[(1 - x^2)^2, t_1](-1)\xi_1 \\
&+ \frac{1}{2}[1 - x^2, t_2](-1)\xi_2 + \frac{1}{8}(B + 2)[(1 - x^2)^2, t_2](-1)\xi_2 \\
=&\Big(\frac{1}{2}A[1 - x^2, 1](-1) + \frac{1}{8}A(B + 2)[(1 - x^2)^2, 1](-1), \\
&\frac{1}{2}B[1 - x^2, 1](-1) + \frac{1}{8}B(B + 2)[(1 - x^2)^2, 1](-1)\Big) \\
=&(-24AB - 48A + 24AB + 48A, -24AB - 48B + 24AB + 48B) \\
=&(0, 0).
\end{aligned}$$

Note that by Remark 5.2, $\Omega y_1 = (0, -192B)$ and $\Omega y_2 = (-192A, 0)$. So, if $f \in \delta$, or equivalently $[f, (a, b)), (y_i, (0, 0))]_{H \oplus W} = [f, y_i]_H = 0$ for $i = 1, 2, 3, 4$, then

$$\begin{aligned}
0 =&[(f, (a, b)), (y_1, (0, 0))]_{H \oplus W} \\
=&[f, y_1]_H - \langle \Omega f, (0, 0)\rangle_W + \langle (a, b), \Omega y_1\rangle_W \\
=&192f(1) - \frac{192Bb}{B} \\
=&192f(1) - 192b,
\end{aligned}$$

and $b = f(1)$.

Similarly, if $f \in \delta$, then

$$\begin{aligned}
0 =&[(f, (a, b)), (y_2, (0, 0))]_{H \oplus W} \\
=&[f, y_2]_H - \langle \Omega f, (0, 0)\rangle_W + \langle (a, b), \Omega y_2\rangle_W \\
=&192f(-1) - \frac{192Aa}{A} \\
=&192f(-1) - 192a,
\end{aligned}$$

and $a = f(-1)$.

5.5 A Self-Adjoint Operator in $H \oplus W$

We are now in a position to define a self-adjoint operator in the extended space $H \oplus W = L^2(-1,1) \oplus \mathbb{C}^2$. Define $\widehat{T} : \mathcal{D}(\widehat{T}) \subseteq H \oplus W \to H \oplus W$ by

$$\begin{aligned}\widehat{T}\,(f,(a,b)) &= (T_1 f, -\Omega f)\\ &= \Big(T_1 f, \Big(24Af''(-1) - 24(AB+A)f'(-1),\\ &\qquad 24Bf''(1) + 24(AB+B)f'(1)\Big)\Big),\\ \mathcal{D}(\widehat{T}) &= \Big\{(f,(a,b)) \in \mathcal{D}\left(\widehat{T_1}\right) \mid [(f,(a,b)),(y_j,(0,0))]_{H\oplus W} = 0,\\ &\qquad (j=1,2,3,4)\Big\}.\end{aligned}$$

Then $\widehat{T}$ is self-adjoint in $H \oplus W$ by Theorem 3.22.

From the above calculations, namely $a = f(-1)$ and $b = f(1)$, the form and domain of $\widehat{T}$ simplify to:

$$\begin{aligned}\widehat{T}\big(f,(f(-1),f(1))\big) &= \Big(T_1 f, \Big(24Af''(-1) - 24(AB+A)f'(-1),\\ &\qquad 24Bf''(1) + 24(AB+B)f'(1)\Big)\Big)\\ \mathcal{D}(\widehat{T}) &= \big\{\big(f,(f(-1),f(1))\big) \mid f \in \delta\big\}.\end{aligned}$$

Note that the definition of the domain of $\widehat{T}$ shows that the boundary conditions force continuity at the endpoints $x = \pm 1$.

5.6 The Krall Polynomials as Eigenfunctions

Recall that the Krall polynomials $\{P_n\}_{n=0}^{\infty}$, as defined in (2.3) are given by

$$K_n(x) = \sum_{j=0}^{n} \frac{(-1)^{\lfloor \frac{j}{2} \rfloor}(2n-j)!\,Q(n,j)\,x^{n-j}}{2^{n+1}(n-\lfloor\frac{j+1}{2}\rfloor)!\,\lfloor\frac{j}{2}\rfloor!\,(n-j)!\,(n^2+n+A+B)},$$

where

$$\begin{aligned}Q(n,j) =& \frac{2+(-1)^j}{2}\left((n^4 + (2A+2B-1)n^2 + 4AB) + 2j(n^2+n+A+B)\right)\\ &+ \frac{1-(-1)^j}{2}(4B-4A).\end{aligned}$$

We will show that the Krall polynomials are actually eigenfunctions of $\widehat{T}$. Note that $\{K_n\}_{n=0}^{\infty}$ satisfy $\ell[K_n] = \lambda K_n$. Also,

$$\ell[K_n](1) = 24BK_n''(1) + (24AB + 24B)K_n'(1)$$

and

$$\ell[K_n](-1) = 24AK_n''(-1) + (24AB - 24A)K_n'(-1).$$

So,

$$\begin{aligned}[K_n, 1](1) =& \Lambda'[K_n](1) - \pi(1)K_n'(1) \\ =& -24K''(1) - (24A + 24)K'(1),\end{aligned}$$

and

$$\begin{aligned}[K_n, 1](-1) =& \Lambda'[K_n](-1) - \pi(-1)K_n'(-1) \\ =& 24K_n''(-1) - (24B + 24)K_n'(-1).\end{aligned}$$

Then we have

$$-B[K_n, 1](1) = \ell[K_n](1)$$

and

$$A[K_n, 1](-1) = \ell[K_n](-1).$$

Also note that

$$\begin{aligned}[K_n, e_+]_H =& [K_n, e_+](1) \\ =& \frac{1}{2}[K_n, 1 - x^2](1) + \frac{1}{8}(A + 2)[K_n, (1 - x^2)^2](1) \\ =& \frac{1}{2}\left(2\Lambda[K_n](1) - 48(A + 2)K_n(1)\right) + \frac{1}{8}(A + 2)(192)K_n(1) \\ =& -24(A + 2)K(1) + 24(A + 2)K_n(1) \\ =& 0\end{aligned}$$

and

$$[K_n, e_-]_H = -[K_n, e_-](-1)$$

$$
\begin{aligned}
=&\frac{1}{2}[K_n, 1-x^2](-1) + \frac{1}{8}(B+2)[K_n, (1-x^2)^2](-1)\\
=&\frac{1}{2}\left(2\Lambda[K_n](-1) - 48(B+2)K_n(-1)\right) + \frac{1}{8}(B+2)(192)K_n(-1)\\
=&-24(B+2)K_n(-1) + 24(B+2)K_n(-1)\\
=&0.
\end{aligned}
$$

Thus, $\{K_n\}_{n=0}^{\infty} \subset \delta$ and $\left\{\left(K_n, (K_n(-1), K_n(1))\right)\right\}_{n=0}^{\infty} \subset \mathcal{D}(\widehat{T})$. Then

$$
\begin{aligned}
\widehat{T}\left(K_n, (K_n(-1), K_n(1))\right) =&\big(T_1 K_n, (24AK_n''(-1) - 24(AB+A)K_n'(-1),\\
&24BK_n''(1) + 24(AB+B)K_n'(1))\big)\\
=&\big(\ell[K_n], (\ell[K_n](-1), \ell[K_n](1))\big)\\
=&\lambda\big(K_n, (K_n(-1), K_n(1))\big).
\end{aligned}
$$

Therefore, $\{K_n\}_{n=0}^{\infty}$ are eigenfunctions of $\widehat{T}$.

5.7 Another Self-Adjoint Operator Generated by the Sixth-Order Krall Differential Expression

We construct another self-adjoint operator generated by the sixth-order Krall differential expression in $H \oplus W = L^2(-1,1) \oplus \mathbb{C}^2$ by selecting an different partial GKN set for T_0. We keep everything independent of the original GKN set the same, especially the GKN set for $\widehat{T}_0$: $\{(y_i, (0,0))\}_{i=1}^{4}$. This allows us to still use the properties of δ provided in Theorem 2.3.

Define $Q = \{t_1, t_2\}$ where

$$
t_1(x) = \begin{cases} 0 & x \text{ near } -1 \\ x-1 & x \text{ near } 1 \end{cases}
$$

and

$$
t_2(x) = \begin{cases} x+1 & x \text{ near } -1 \\ 0 & x \text{ near } 1. \end{cases}
$$

Note that for $f \in \Delta$,

$$\begin{aligned}
[f, t_1]_H =& [f, t_1](1) \\
=& [f, x-1](1) \\
=& [f, x](1) - [f, 1](1) \\
=& -[\overline{x}, 1](1) f(1) - \Lambda[f](1) \\
=& -\left(\Lambda'[\overline{x}](1) - \pi(1)\right) f(1) - \Lambda[f](1) \\
=& \pi(1) f(1) - \Lambda[f](1) \\
=& \left(-6A - 6B - 12AB + 12A - 12B + 12AB + 18A + 18B + 24\right) f(1) \\
& - \Lambda[f](1) \\
=& 24(A+1) f(1) - \Lambda[f](1)
\end{aligned}$$

and

$$\begin{aligned}
[f, t_2]_H =& -[f, t_2](-1) \\
=& -[f, x+1](-1) \\
=& -[f, x](-1) - [f, 1](-1) \\
=& [\overline{x}, 1](-1) f(-1) + \Lambda[f](-1) \\
=& \left(\Lambda'[\overline{x}](-1) - \pi(-1)\right) f(-1) + \Lambda[f](-1) \\
=& -\pi(-1) f(-1) + \Lambda[f](-1) \\
=& -\left(-6A - 6B - 12AB - 12A + 12B + 12AB + 18A + 18B + 24\right) f(-1) \\
& + \Lambda[f](-1) \\
=& -24(B+1) + \Lambda[f](-1).
\end{aligned}$$

Claim 5.4. *$Q = \{t_1, t_2\}$ is a partial GKN set for T_0.*

Proof. Let $f \in \Delta$ and $c_1, c_2 \in \mathbb{C}$. Then

$$[f, c_1 t_1 + c_2 t_2]_H = \overline{c}_1 [f, t_1]_H + \overline{c}_2 [f, t_2]_H$$

$$=\overline{c}_1\left(24(A+1)f(1)-\Lambda[f](1)\right)-\overline{c}_2\left(24(B+1)f(-1)+\Lambda[f](-1)\right).$$

Now define $f_1, f_2 \in \Delta$ by

$$f_1(x)=\begin{cases}0 & x \text{ near } -1\\ x & x \text{ near } 1,\end{cases}$$

$$f_2(x)=\begin{cases}x & x \text{ near } -1\\ 0 & x \text{ near } 1.\end{cases}$$

Then $[f_1, c_1t_1+c_2t_2]_H = 24\overline{c}_1(A+1)$ and $[f_2, c_1t_1+c_2t_2]_H = 24\overline{c}_2(B+1)$. Therefore, if $[f_1, c_1t_1+c_2t_2]_H = [f_2, c_1t_1+c_2t_2]_H = 0$, then $c_1 = c_2 = 0$. Thus, Q is linearly independent modulo $\mathcal{D}(T_0)$.

Since

$$\begin{aligned}[t_1,t_1]_H =&24(A+1)t_1(1)-\Lambda[t_1](1)\\ =&0\\ [t_1,t_2]_H =&-24(B+1)t_1(-1)+\Lambda[t_1](-1)\\ =&0\\ [t_2,t_2]_H =&-24(B+1)t_2(-1)+\Lambda[t_2](-1)\\ =&0,\end{aligned}$$

Q is a partial GKN set for T_0. □

Since $\xi_1 = \left(\sqrt{A}, 0\right)$, $\xi_2 = \left(0, \sqrt{B}\right)$, and $\Delta_0 = \mathcal{D}(T_0) + \text{span}\{t_1, t_2\}$, we can define $\Omega : \mathcal{D}(T_1) \to W$ by

$$\begin{aligned}\Omega f =&[f,t_1]_H\xi_1+[f,t_2]_H\xi_2\\ =&\left(\sqrt{A}(24(A+1)f(1)-\Lambda[f](1)), \sqrt{B}(-24(B+1)f(-1)+\Lambda[f](-1))\right).\end{aligned}$$

Once again, we choose the self-adjoint operator in W to be $\mathcal{B} = 0$ and define $\{y_i\}_{i=1}^4$ as in (5.1). Then $\widehat{S}_0 : \mathcal{D}(\widehat{S}_0) \subseteq H \oplus W \to H \oplus W$, the minimal operator in $H \oplus W$, is defined by

$$\begin{aligned}\widehat{S}_0(f,(a,b)) =&(T_1 f, 0)\\ \mathcal{D}(\widehat{S}_0) =& \{(f,\Psi f) \mid f \in \Delta\}\,.\end{aligned}$$

Hence, $\widehat{S}_0(f,\Psi f) = (T_1 f, 0)$.

The maximal operator in $H \oplus W$, $\widehat{S}_1 \subseteq H \oplus W \to H \oplus W$, is defined by

$$\begin{aligned}\widehat{S}_1(f,(a,b)) &= (T_1 f, -\Omega f)\\ &= \Big(T_1 f, \Big(-\sqrt{A}(24(A+1)f(1) - \Lambda[f](1)),\\ &\qquad -\sqrt{B}(-24(B+1)f(-1) + \Lambda[f](-1))\Big)\Big)\\ \mathcal{D}(\widehat{S}_1) &= \{(f,(a,b)) \mid f \in \mathcal{D}(T_1), (a,b) \in W\}\,.\end{aligned}$$

Note that $\Omega y_i = 0$ for $i = 1,2,3,4$ and $\{y_i,(0,0))\}_{i=1}^4$ is a GKN set for $\widehat{S}_0$ by Lemma 5.3 . So, for $f \in \delta$,

$$\begin{aligned}[(f,(a,b)),(y_1,(0,0))]_{H\oplus W} =&[f,y_1]_H - \langle \Omega f,(0,0)\rangle_W + \langle (a,b),\Omega y_1\rangle_W\\ =&[f,y_1]_H\\ =&192 f(1)\end{aligned}$$

by Theorem 2.3. Since $[(f,(a,b)),(y_1,(0,0))]_{H\oplus W} = 0$, we must have $f(1) = 0$. Likewise,

$$\begin{aligned}[(f,(a,b)),(y_2,(0,0))]_{H\oplus W} =&[f,y_2]_H - \langle \Omega f,(0,0)\rangle_W + \langle (a,b),\Omega y_2\rangle_W\\ =&[f,y_2]_H\\ =&192 f(-1).\end{aligned}$$

Therefore, $f(-1) = 0$ since $[(f,(a,b)),(y_2,(0,0))]_{H\oplus W} = 0$.

Then $\widehat{S} : \mathcal{D}(\widehat{S}) \subseteq H \oplus W \to H \oplus W$ defined by

$$\widehat{S}(f,(a,b)) = \Big(T_1 f, \Big(\sqrt{A}\Lambda[f](1), -\sqrt{B}\Lambda[f](-1)\Big)\Big)$$

$$\mathcal{D}(\widehat{S}) = \left\{(f,(a,b)) \in \mathcal{D}\left(\widehat{S}_1\right) \mid f \in \delta,\ f(-1) = f(1) = 0\right\}$$

is self-adjoint in $H \oplus W$ by Theorem 3.22. So, we have found another self-adjoint operator generated by the sixth-order Krall differential expression in the extended Hilbert space $L^2[-1,1] \oplus \mathbb{C}^2$.

CHAPTER SIX

Conclusion

In Chapter Four, we applied the GKN-EM Theorem in $H \oplus W$ to find various self-adjoint operators in extended Hilbert spaces. This demonstrates the power of this extended GKN Theorem. In order to find these self-adjoint operators, it was necessary to appropriately define the functions Ω and Ψ. We also needed to choose adequate GKN sets for both T_0 in the base space and for $\widehat{T_0}$ in the extension space. Note that the GKN-EM Theorem in $H \oplus W$ results in a one-parameter family of self-adjoint operators, where the parameter is the self-adjoint operator $\mathcal{B} : W \to W$. In this dissertation, we always chose a specific self-adjoint operator $\mathcal{B}$ in order to obtain a self-adjoint operator with certain desired properties.

In Chapter Two and Chapter Five, we studied the Krall differential expression in a weighted L^2 space where the weight had jumps at both endpoints. This differential expression has the Krall orthogonal polynomials as eigenfunctions in the weighted L^2 space. We found that, in an extended Hilbert space, the self-adjoint operator generated by the sixth-order Krall differential expression in the extended Hilbert space $L^2[-1,1] \oplus \mathbb{C}^2$ is given by

$$\begin{aligned} \widehat{T}\big(f, (f(-1), f(1))\big) = &\Big(T_1 f, \Big(24Af''(-1) - 24(AB + A)f'(-1), \\ &\qquad 24Bf''(1) + 24(AB + B)f'(1)\Big)\Big) \\ \mathcal{D}(\widehat{T}) = &\left\{\big(f, (f(-1), f(1))\big) \mid f \in \delta\right\}, \end{aligned}$$

where T_1 is the maximal operator generated by the Krall differential expression in $L^2[-1,1]$ and δ is the domain of the self-adjoint operator in $L^2_\mu[-1,1]$ generated by this expression. We also found a second self-adjoint operator in $L^2[-1,1] \oplus \mathbb{C}^2$ by changing the GKN set for T_0. For both of these self-adjoint operators, we chose

$\mathcal{B} = 0$ for the self-adjoint operator in the extension space $\mathbb{C}^2$. Since the GKN-EM Theorem in $H \oplus W$ yields a one-family parameter of self-adjoint operators, choosing any other self-adjoint operator $\mathcal{B}$ would have yielded another self-adjoint operator in $L^2[-1,1] \oplus \mathbb{C}^2$.

This Krall differential expression is a special case of differential operators with the following inner product:

$$(f,g) = M_1 f(0)\overline{g}(0) + M_2 f'(0)\overline{g}'(0) + \int_0^\infty f(x)\overline{g}(x)w(x)\ dx.$$

By placing restrictions on the coefficients M_1 and M_2, we obtain four different differential expressions.

When $M_1 = M_2 = 0$, then the resulting differential expression is a second-order expression. In fact, we have the classical second-order Laguerre differential expression in the space $L^2((0,\infty); x^\alpha e^{-x})$ defined by

$$\ell[y](x) := \frac{1}{x^\alpha e^{-x}} \left(-(x^{\alpha+1}e^{-x}y'(x))' + kx^\alpha e^{-x}y(x)\right)$$

for $x \in (0,\infty)$. This self-adjoint operator generated by the Laguerre differential expression in $L^2((0,\infty); x^\alpha e^{-x})$ has the Laguerre orthogonal polynomials as eigenfunctions.

When $M_1 > 0$ and $M_2 = 0$, and the interval is $(-1,1)$ instead of $(0,\infty)$, the resulting differential expression is the fourth-order Laguerre-type differential expression given by

$$\ell[y](x) := \left(\left(1-x^2\right)^2 y''(x)\right)'' - \left(8 + 4A(1-x^2)y'(x)\right)' + ky(x),$$

where $k \geq 0$ is a fixed parameter and $x \in (-1,1)$. The associated inner product is given by

$$(f,g) = \frac{f(1)\overline{g}(1)}{2} + \frac{A}{2}\int_{-1}^1 f(x)\overline{g}(x)\ dx + \frac{f(-1)\overline{g}(-1)}{2}.$$

In fact, the self-adjoint operator generated by the fourth-order Laguerre-type differential expression has the Laguerre-type orthogonal polynomials as eigenfunctions.

As noted above, the Krall sixth-order differential expression is a special case of this situation. With the Laguerre-type differential expression, the jumps at both endpoints are $\frac{1}{2}$. However, by choosing jumps at the endpoints that are not equal, we obtain a sixth-order differential expression: the Krall differential expression.

The case where $M_1 = 0$ and $M_2 > 0$ results in an eighth-order differential expression. Note that this is the first case where we have a jump in the derivative at the endpoints, rather than a jump in the function.

When we have $M_1, M_2 > 0$, we obtain a tenth-order Laguerre-Sobolev differential expression. It is this expression that we will work with in the future. We hope to find a self-adjoint operator generated by this differential operator in an extended Hilbert space. To do so, we must use the GKN-EM Theorem in $H \oplus W$. This entails choosing suitable GKN sets for the minimal operators generated by this differential expression in the the base space and in the extended space. Note that the self-adjoint operator found in Chapter Five relied on analysis that had been done by Loveland on the self-adjoint operator in $L^2_\mu[-1,1]$. Similar analysis in the base space must be done for this expression before we can analyze it in an extended space.

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