ABSTRACT

On Various Notions of the Shadowing Property
in Non-Compact Spaces

Ian Grigsby, Ph.D.

Mentor: Jonathan Meddaugh, Ph.D.

We discuss various notions of the shadowing property on non-compact spaces. In the first part, we discuss the shadowing property acting on general sequence spaces. We develop criteria upon the weights of the space in which shadowing of particular types of orbits occurs. Then we move on to operators acting on Fréchet spaces, and show that an system exhibits the shadowing property if its Waelbroeck spectrum misses the unit circle. Additionally, we discuss the non-uniform pseudo-orbit tracing property, a variant of the shadowing property which allows for different error tolerances depending on where a point lies in the space.

On Various Notions of the Shadowing Property
in Non-Compact Spaces

by

Ian Grigsby, B.S., M.S.

A Dissertation

Approved by the Department of Mathematics

Dorina Mitrea, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of

Doctor of Philosophy

Approved by the Dissertation Committee

Jonathan Meddaugh, Ph.D., Chairperson

Brian Raines, Ph.D.

David Ryden, Ph.D.

David Kahle, Ph.D.

Accepted by the Graduate School

August 2021

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I would like to thank Dr. Jonathan Meddaugh, whose countless hours helped me become something that challenged me to the core- a research mathematician. Without his guidance, I would not have made it this far.

I would also like to thank the faculty and staff here at Baylor University. Moving across the country is no small ordeal, and your loving support has helped me grow into who I am today, both as a mathematician and as a person.

Lastly, I would like to thank my wife, Hannah. There have been some times in my research journey in which I did not know if I had the strength and resolve to persevere through challenging problems, but you have never stopped believing in me. You are the reason that I woke up every morning and hit the books, even when it was not fun to do so. You are the reason that I made it this far. Thank you for being the best thing that could have ever happened to me.

I would like to dedicate this work to my loving wife, Hannah Grigsby.

CHAPTER ONE

Preliminaries

1.1 An Overview

The shadowing property has been of recent interest in the study of dynamical systems. The motivation behind this property is simple. Imagine that we have a system, (X, f), and wish to study it using computers. Now computers must sometimes use approximations (think using 3.14 in place of π), all of which come with a small error term. As our system evolves over time, we are left with a question: Do these error terms grow out of control? Are we left with a situation in which the true mathematical solutions are incompatible with these approximate solutions?

If the system exhibits the shadowing property, then we can ensure that these errors do not grow out of control. In fact, there have been many results within the realm of compact metric spaces about what implies the shadowing property and what the shadowing property implies. In this dissertation, we examine the lesser viewed setting of noncompact spaces. We focus in particular on Fréchet spaces, a generalization of Banach spaces, each of which we define later.

To begin, we examine the concrete case of shifts on sequence spaces. Shift spaces encompass a well-studied set of spaces, and are often the topic of study in discrete dynamical systems as well as in symbolic dynamics. Shifts of finite type are a particularly widely studied in symbolic dynamics, as they describe the dynamics of finite-state machines, which in turn describe systems that can be in one of only a finite number of states at any given time.

As we seek to be as general as possible, we discuss backwards weighted shift maps acting on sequence spaces. These maps take in a sequence of scalars, either real or complex numbers, and output the sequence with the first term removed and with each subsequent term multiplied by some constant, called a weight, which varies based upon the location of the term in the sequence.

In Chapter Two, we put conditions upon the weights of the map that guarantee the shadowing property or some variant of it. It is important to note that we will be studying pseudo-orbits whose elements are sequences in their own right; that is, sequences of sequences. We first discuss uniformly bounded pseudo-orbits, that is pseudo-orbits in which the terms of all sequences in the pseudo-orbit are bounded by a single constant M. From there, we move on to consider bounded pseudo-orbits, that is pseudo-orbits comprised of sequences all bounded by the same constant M, yet whose terms may not be all bounded by the same constant.

Once we have accomplished the above, in Chapter Three we move onto linear operators acting on general Fréchet spaces. In light of Bernardes, et al. [4], we seek to connect the shadowing property to the spectrum of the operator. Hence, we begin by discussing what makes a suitable definition of the spectrum of a Fréchet space operator, as applying the usual definition of the spectrum to a Fréchet space operator fails to maintain some of the nice properties exhibited on Banach space operators, such as the spectrum being a closed set.

Once we have a suitable definition of the spectrum in this setting, we place criteria upon the spectrum that allow us to know if a system exhibits the shadowing property. In similar fashion to Bernardes et al., these conditions revolve around the hyperbolicity of the operator, that is, whether or not the spectrum intersects the unit circle.

Lastly, in Chapter Four, we examine other forms of shadowing relevant to the non-compact setting. In particular, we develop a notion of shadowing called the non-uniform pseudo-orbit tracing property. This property considers pseudo-orbits with error tolerance based upon location in the space. As an example, a point's distance from the boundary of a manifold may be in direct correlation with how precise we require the pseudo-orbit to be; the closer to the boundary, the more exact, for instance. We list some immediate results of the definition of this property, as well as explore its relationship to a slight variant of the same property; one in which there are certain points where no error tolerance is permitted.

From there, we explore the relationship of the non-uniform pseudo-orbit tracing property property of a space to the system's compactification, if it has one. In particular, we explore what assumptions we can put on the compactification of the system in order to show that if the compactified system exhibits the shadowing property, then the original system exhibits the non-uniform pseudo-orbit tracing property. Ultimately, we end with a result regarding the average shadowing property of a backwards weighted shift map on a sequence space.

1.2 Dynamical Systems

Now we wish to discuss how functions acting on the space transform the space. More specifically, we seek to be able to discuss how a function acting on the space transitions a point through the space. This is the study of dynamical systems.

Definition 1.2.1. A *dynamical system* is a pair (X, f), where X is a space and $f : X \to X$ is a continuous function acting on the space.

In order to discuss the dynamics of the space, one must be able to talk about how a point moves through space. This brings up the notion of the orbit of a point.

Definition 1.2.2. Let (X, T) be a dynamical system, and let $x \in X$. Then the *orbit of x under T* is the set $\mathcal{O}(x, T) \coloneqq \{T^n x \mid n \in \omega\}$.

Orbits are particularly important in the study of the long term behavior of the system. Orbits may contain only one point as in the case of fixed points, contain finitely many distinct points as in the case of periodic or preperiodic points, or contain infinitely many distinct points. In the last case, a point's orbit could intersect every open set of the system, in which case we would say that the point has a dense orbit.

Now in the study of dynamical systems, there are many desired properties that guarantee certain properties in the space. We give the following as examples of certain properties that may be desired:

Definition 1.2.3. A dynamical system (X, T) is *topologically transitive* if for every pair of non-empty open sets $U, V \subset X$, there exists $k > 0$ such that $T^k(U) \cap V \neq \emptyset$.

Topological transitivity tells us that points from one arbitrary open set eventually move under iteration to any other arbitrary open set. This notion was first studied by G. D. Birkhoff in the 1920s for flows, as mentioned by Kolyada and Snoha [15].

Interestingly, the notion of topological transitivity does not necessarily imply nor is implied by the existence of a dense orbit. Kolyada and Snoha provide examples of a space which has a dense orbit yet is not topologically transitive and a space which is topologically transitive yet does not have a dense orbit in their work [15]. Under the additional assumption of no isolated points, we may show that the existence of a dense orbit implies topological transitivity, while the additional assumptions of the space being separable and second category give us that topological transitivity implies the existence of a dense orbit.

If every iterate of a map is topologically transitive, we have a stronger condition than topological transitivity.

Definition 1.2.4. A dynamical system (X, T) is *totally transitive* if T^k is transitive for all $k \geq 1$.

Now knowing the idea of topological transitivity, it is natural to wonder if given a topologically transitive system (X, T), whether or not $(X \times X, T \times T)$ is topologically transitive.

Definition 1.2.5. A dynamical system (X, T) is *topologically weakly mixing* if $(X \times X, T \times T)$ is transitive.

If a system is topologically weakly mixing, then it is topologically transitive. It was an open question for some time as to whether a topologically transitive system was weakly mixing. An answer was given in the negative, as shown by Block and Coppel [5].

While topological transitivity guarantees that any arbitrary open set eventually met any other arbitrary open set under iteration, there was no guarantee that the two sets remained intertwined.

Definition 1.2.6. A dynamical system (x, T) is *topologically mixing* if for every pair of non-empty open sets $U, V \subset X$, there is $K \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$ for all $k \geq K$.

All of the above properties describe how points move throughout the space X under the action of T, and how much a given map "mixes" the points in the space. In fact, all of these properties are preserved if the space undergoes a continuous deformation, known as a topological conjugacy.

Definition 1.2.7. Let (X, f) and (Y, g) be two dynamical systems. The systems are said to be *topologically conjugate* if there exists a homeomorphism $h\colon Y \to X$ such that $f \circ h = h \circ g$, and we say that h is a *topological conjugation*.

If a property is preserved under conjugacy, that is, if a system (X, f) exhibits the property and if (X, f) is topologically conjugate to (Y, g), then (Y, g) also exhibits the property, we say that it is a topological property.

1.3 Linear Dynamical Systems

In order to begin our discussion on the dynamics of linear operators acting on infinite dimensional Fréchet spaces, we must first discuss why we choose such a setting to begin with. We begin this discussion by defining the types of spaces that we will be working with for this dissertation.

Definition 1.3.1. A *topological vector space (TVS)* is a vector space X over a field $\mathbb{K}$ endowed with a topology such that vector addition and scalar multiplication, that is the maps $+\colon X \times X \to X$ and $\cdot\colon \mathbb{K} \times X \to X$, are continuous.

The first examples encountered of topological vector spaces are the spaces of $M \times N$ matrices with real entries acting as a vector space over the reals. Arguably the most often encountered topological vector spaces are Banach spaces. First, we define what it means for a space to be complete.

Definition 1.3.2. A sequence $\langle x_n \rangle_{n\in\omega}$ is said to be a *Cauchy sequence* if for all $\epsilon > 0$, there exists some number $N \in \mathbb{N}$ such that for all $m, n > N$, $d(x_m, x_n) < \epsilon$. A metric space X is said to be *complete* if every Cauchy sequence in X has a limit point in X.

We now define what it means for a space to be a Banach space.

Definition 1.3.3. A *Banach space* X is a complete normed vector space.

Banach spaces are a foundational element in the study of functional analysis. Many powerful theorems exist in this setting, such as the Hahn-Banach theorem, the Open Mapping Theorem, and the Closed Graph Theorem. These spaces and the maps acting on them have a large body of work associated to them, as they encompass a large class of spaces. For the purpose of this dissertation, all of the spaces which we consider will be metric spaces whose metric may not be given by a norm. Due to this, the spaces we consider will not necessarily be Banach spaces. This leads us to the definition of a *Fréchet space*. First, we define two notions of convexity.

Definition 1.3.4. A set U in a topological vector space X is said to be *convex* if for any two points $x, y \in U$, the point $(1-t)x + ty$ is an element of U for all scalars $t \in [0, 1]$. A topological vector space X is *locally convex* if the origin has a basis of convex neighborhoods.

Definition 1.3.5. A topological vector space X is a *Fréchet space* if it is locally convex and complete with respect to some translation-invariant metric.

The requirement for the metric to be translation-invariant tells us that given any three points $x, y, z \in X$, then $d(x+z, y+z) = d(x, y)$.

Fréchet spaces are generalizations of Banach spaces. Therefore, all Hilbert and Banach spaces are Fréchet spaces. Importantly, not all authors require that a Fréchet space be locally convex. For the purposes of this dissertation, we will require local convexity of the spaces.

As noted by Grosse-Erdmann and Peris [12], the topology of a Fréchet space is generated by an increasing ($p_n(x) \leq p_m(x)$ for all $n \leq m$ and all $x \in X$) sequence of seminorms $\langle p_n \rangle$, with metric given by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, p_i(x-y)\}.$$

This sequence is also supposed to be separating, meaning that $p_n(x) = 0$ for all $n \in \mathbb{N}$ implies that $x = 0$ [12].

With respect to this metric, we are able to define a functional $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$ given by

$$\|x\| = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, p_i(x)\}.$$

Importantly, this is not necessarily a norm, as it is generated through an increasing sequence of seminorms. Therefore, we call this an *F-norm*. An F-norm has the following characteristics for all $x, y \in X$ and $\lambda \in \mathbb{R}$, as discussed by Grosse-Erdmann and Manguillot [12]:

1) $\|\lambda x\| \leq (|\lambda| + 1)\|x\|$.

2) $\|x + y\| \leq \|x\| + \|y\|$.

3) $\|x\| = 0$ implies that $x = 0$.

4) $\lim_{\lambda \to 0} \|\lambda x\| = 0$.

Many theorems that hold on Banach spaces hold on Fréchet spaces. For instance, we still have the Open Mapping Theorem, the Closed Graph Theorem, and the Hahn-Banach Theorem. The first two of these theorems involves a continuous function, called an operator, acting on the space. We formalize this notion below.

Definition 1.3.6. Let X and Y be Fréchet spaces. A continuous linear map $T : X \to Y$ is called an *operator*. $L(X, Y)$ denotes the space of all such operators. If $X = Y$, then we simplify this notation to $L(X)$ and state that T is an *operator on X*.

Unlike their Banach space counterparts, we are unable to associate a norm to operators acting on a Fréchet space. We now mention a criterion that shows whether or not a map between two Fréchet spaces is indeed an operator:

Proposition 1.3.7. Let X and Y be Fréchet spaces with defining increasing sequences of seminorms $\langle p_n \rangle_{n\in\omega}$ and $\langle q_n \rangle_{n\in\omega}$, respectively. Then a linear map $T : X \to Y$ is an operator if and only if, for any $m \geq 1$, there are $n \geq 1$ and $M > 0$ such that

$$q_m(Tx) \leq Mp_n(x), x \in X.$$

Recall that space of $M \times N$ matrices acting as a topological vector space over the reals. Each $M \times N$ matrix A may also be viewed as a linear operator $A \colon \mathbb{R}^N \to \mathbb{R}^M$. We now define what it means for an operator to be linear.

Definition 1.3.8. Given a dynamical system (X, T), we say that T is a *linear* operator if for any vectors $x, y \in X$ and scalars $a, b \in \mathbb{K}$, we have that

$$T(ax + by) = aT(x) + bT(y).$$

Example 1.3.9. The differential operator,

$$D : f \to f',$$

is an operator when acting on $H(\mathbb{C})$, the space of entire functions. We define our increasing sequence of seminorms by

$$p_n(f) = \sup_{|z|\leq n} |f(z)|.$$

Example 1.3.10. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then

$$\ell^p := \left\{ x = \langle x_n \rangle_{n\in\omega} \in \mathbb{K}^\omega ; \sum_{n=0}^{\infty} |x_n|^p < \infty \right\}, 1 \leq p < \infty$$

and

$$c_0 := \left\{ x = \langle x_n \rangle_{n\in\omega} \in \mathbb{K}^\omega ; \lim_{n\to\infty} x_n = 0 \right\}$$

are Banach spaces. The *backward shift* $B : X \to X$, with $X = \ell^p$ or c_0, defined by

$$B(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

is an operator on X with $\|B\| = 1$.

The space of all sequences, $\mathbb{K}^\omega$, is a Fréchet space when endowed with the increasing sequence of seminorms defined by

$$p_n(x) = \sup_{0\leq k\leq n} |x_k|, \;\; x = \langle x_k \rangle_{k\in\omega} .$$

$B : \mathbb{K}^\omega \to \mathbb{K}^\omega$ is an operator as well.

Lastly, we define the following.

Definition 1.3.11. Given a dynamical system (X, T), we say that (X, T) is *chaotic* if (X, T) has a dense orbit and there exists a dense set of period points under T.

In essence, the notion of chaos tells us that minor changes to initial inputs results in wildly different phenomena in the long term. A common pop culture reference to this is the so-called "butterfly effect", where Lorenz posits that the small change in the atmosphere caused by the flap of a butterfly's wings in Brazil could cause a tornado in Texas [17].

Importantly, given a finite space $\mathbb{K}^N$ for some $N \in \omega$, a linear operator T acting on $\mathbb{K}^N$ may be regarded as a matrix. Therefore, as Grosse-Erdmann states, "the dynamics of linear operators on a finite dimensional space $X = \mathbb{K}^N$ are easy to describe, thanks to the Jordan decomposition theorem." [12]. In fact, Grosse-Erdmann shows that there are no hypercyclic linear operators (linear operators in which some point of X has a dense orbit) on $\mathbb{K}^N$, and hence on any finite-dimensional Fréchet space, thereby making the study of such systems rather restricted.

One may then be led to believe that requiring an operator to be linear may prove to be restricting as well. Fortunately, this is not the case. In fact, according to Grosse-Erdmann,

> "...every continuous map on a compact metric space is conjugate to the restriction of a linear operator on some invariant set. Even more strikingly, the same opearator can be taken for all nonlinear systems, and the operator is even chaotic. In other words: the dynamics of any (compact) non-linear dynamical system can be described by the dynamics of a single chaotic operator." [12]

Therefore, linear chaos exists in the infinite dimensional setting, and is in fact a very powerful tool to describe the dynamics of even more complicated systems.

1.4 The Shadowing Property

As mentioned before, the shadowing property gives us a way of knowing whether the approximate orbits of a system, which often appear when studying dynamics using computers, are reasonably well followed by a true orbit of the system. Given an operator T acting on some metric space X, we define the following:

Definition 1.4.1. For a system (X, T), a δ-*pseudo-orbit* is a sequence $\langle x_n \rangle_{n\in\omega}$ that satisfies $d(Tx_n, x_{n+1}) < \delta$ for all $n \in \omega$. A point $z \in X$ is said to ϵ-*shadow* a δ-pseudo-orbit $\langle x_n \rangle_{n\in\omega}$ if $d(T^n z, x_n) < \epsilon$ for all $n \in \omega$.

Definition 1.4.2. A system (X, T) is said to have the *shadowing property* if for all $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit $\langle x_n \rangle_{n\in\omega}$ in X is ϵ-shadowed by some point $z \in X$.

Much of this theory originated in the late 1960s and 1970s. Anosov and Bowen were the first to research the subject, where Anosov used it to study geodesic flows on closed Reimannian manifolds [1] and Bowen used it to study Axiom A diffeomorphisms [8]. A widely known result, arrived at independently by both Anosov and Bowen, in this area of research is the Shadowing Lemma, which we state here:

Lemma 1.4.3. The Shadowing Lemma [Anosov and Bowen] Let Λ be a hyperbolic invariant set of a diffeomorphism f acting on a metric space X. There exists a neighborhood U of Λ with the following property: for any $\epsilon > 0$, there exists $\delta > 0$, such that any (finite or infinite) δ-pseudo-orbit that stays in U also stays in a ϵ-neighborhood of some true orbit.

As we shall see later, hyperbolicity and the shadowing property are deeply related ideas. Historically, Bowen utilized his knowledge of the shadowing property in order to study Markov partitions [7]. Conley used the shadowing property to prove that if the chain recurrent set of a diffeomorphisim is hyperbolic, then the periodic points are dense in the chain recurrent set [10]. Later, topological conjugacy results for perturbations of diffeomorphisms with hyperbolic sets were proved by Walters [24] using the shadowing property. While much more has been done in the study of the shadowing property, we begin with some examples of two systems; one exhibiting the shadowing property and the other not.

Example 1.4.4. Consider $X = \mathbb{Z}$ with the map $f(x) = x + 1$ acting on it. Given any $\epsilon > 0$ and taking δ to be any number less than or equal to 1, we have that any δ-pseudo-orbit is in fact a true orbit. Therefore this system exhibits the shadowing property.

Example 1.4.5. Consider $Y = \{2^n : n \in \mathbb{Z}\}$ and $g(y) = 2y$. Notice that any true orbit $\langle g^n(y) \rangle_{n\in\omega}$ grows infinitely large as n grows towards infinity. However, given any $\delta > 0$, there exists an $M \in \mathbb{Z}$ such that if $m < M$, then $d(g(2^m), 2^m) < \delta$. Therefore, for any $m < M$, the fixed sequence $\langle 2^m \rangle_{n\in\omega}$ constitutes a δ-pseudo-orbit that cannot be shadowed by a true orbit.

The above two examples provide an illustration of two spaces that are topologically conjugate (take $\phi : Y \to X$ given by $\phi(y) = \log_2(y)$). Therefore, we have shown that the shadowing property is indeed a metric property, not a topological property. This means that the shadowing property depends heavily upon the metric of the space. One is not in general guaranteed to retain the shadowing property if the underlying space undergoes a continuous deformation.

Requiring that the conjugacy is uniformly continuous is an example of how one could preserve the shadowing property. This scenario would arise if the spaces in question were compact, as the a continuous function on a compact space is uniformly continuous. If the conjugacy is not uniformly continuous, then the shadowing property may or may not be preserved.

As mentioned, the presence of the shadowing property in a given dynamical system on a compact metric space implies many things. For instance, we have the following due to Kulczycki et al. [16]:

Theorem 1.4.6. Let X be a compact metric space. If $T\colon X \to X$ is a continuous map with the shadowing property, then the following conditions are equivalent:

1) (X, T) is totally transitive,

2) (X, T) is topologically weakly mixing,

3) (X, T) is topologically mixing,

4) (X, T) is surjective and has the specification property,

5) (X, T) is surjective and has the almost specification property,

6) (X, T) is surjective and has the asymptotic average shadowing property,

7) (X, T) is surjective and has the average shadowing property,

Moreover, if T is c-expansive, that is if for every distinct pair $x, y \in X$, there exists an $N \in \mathbb{N}$ such that $d(T^N x, T^N y) \geq c$, then any of the above conditions is equivalent to the periodic specification property of (X, T).

The specification property, according to Karl Sigmund, roughly states that "one can approximate distinct pieces of orbits by single periodic orbits with a certain uniformity" [21]. In general we have that (3) $\implies$ (2) $\implies$ (1) [12], but the converse implications will not necessarily hold without the presence of the shadowing property.

As encountered above, there are many various notions of the shadowing property. Each of these variants has their own notion for what constitutes a pseudo-orbit and what it means for an orbit to shadow a given pseudo-orbit. In the meantime, we mention the limit shadowing property and thick shadowing property as examples here.

Definition 1.4.7. Let (X, T) be a dynamical system with an operator acting on it. A sequence $\langle x_n \rangle_{n\in\omega}$ is a *limit pseudo-orbit* if $\lim_{n\to\infty} d(Tx_n, x_{n+1}) = 0$. We say that (X, T) exhibits the *limit shadowing property* if for any limit pseudo-orbit $\langle x_n \rangle_{n\in\omega}$, there exists some point $z \in X$ such that $\lim_{n\to\infty} d(T^n z, x_n) = 0$.

Limit shadowing is distinct from the shadowing property in the sense that it studies pseudo-orbits whose error terms converge to zero and requires that a shadowing point converges to the pseudo-orbit asymptotically. This property in conjunction with other assumptions can be used to show that a system exhibits the shadowing property. For instance, Kawaguchi showed that if a continuous self-map f of a compact metric space X has the limit shadowing property, then the restriction of f to the non-wandering set exhibits the shadowing property [13].

Definition 1.4.8. Let (X, T) be a dynamical system. A set A is said to have lower density 1 if

$$\liminf_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} = 1.$$

A sequence $\langle x_n \rangle_{n\in\omega}$ is a *thick δ-pseudo-orbit* if there is a set A with lower density 1 such that $d(Tx_n, x_{n+1}) < \delta$ for all $n \in A$.

We say that (X, T) exhibits the *thick shadowing property* if for all $\epsilon > 0$, there exists some $\delta > 0$ such that any thick δ-pseudo-orbit $\langle x_n \rangle_{n\in\omega}$ is thickly shadowed by a point $z \in X$, that is there is a set $B \subseteq \mathbb{N}$ that contains arbitrarily large intervals such that $d(T^n z, x_n) < \epsilon$ for all $n \in B$.

In general, the presence of one variant of the shadowing property does not guarantee the presence of another variant. This is due to the fact that oftentimes the definitions of what constitutes a pseudo-orbit are quite different or incompatible with each other. While not much may be said in general about the equivalence of any of these various notions on their face, we do know that under certain conditions conclusions may be drawn. William Brian, Jonathan Meddaugh, and Brian Raines [9] provide a proof that it is possible under certain assumptions to demonstrate that different versions of the shadowing property are equivalent, showing that under the assumption of chain transitivity that the shadowing property is equivalent to the thick shadowing property.

Theorem 1.4.9. [Brian, Meddaugh & Raines] Let (X, f) be a compact dynamical system. If (X, f) is chain transitive, then the following properties are equivalent:

1) shadowing (i.e., $(\mathbb{N}, \mathbb{N})$-shadowing).

2) thick shadowing (i.e., $(\mathcal{D}, \mathcal{T})$-shadowing).

3) $(\mathcal{T}, \mathcal{T})$-shadowing.

4) $(\mathbb{N}, \mathcal{T})$-shadowing.

More theorems in this vain provided by Lee and Sakai may be found in [20] and [14], where they show that if a system is expansive, then many notions of shadowing are equivalent, such as the shadowing property and the limit shadowing property. Importantly, they assume that space is compact.

In summary, the shadowing property and its variants are properties that ensure that the dynamics of approximate orbits are modeled by true orbits. We now begin our study of the shadowing property and its variants in the setting of Fréchet spaces.

CHAPTER TWO

Shadowing in Weighted Shifts

2.1 Preliminaries

Let $\mathbb{K}$ be a field (typically $\mathbb{R}$ or $\mathbb{C}$). A space X is called a *sequence space* if it is a subspace of $\mathbb{K}^\omega = \left\{\overline{x} = \langle x_n \rangle_{n\in\omega} : x_n \in \mathbb{K}\right\}$. An operator T acting on a sequence space X is a *weighted backwards shift* if there is a sequence of weights $\langle \beta_j \rangle_{j\in\omega}$ in $\mathbb{K}$ such that $T\colon X \to X$ is given by $T(\langle x_n \rangle_{n\in\omega}) = \langle \beta_n x_{n+1} \rangle_{n\in\omega}$. A notable class sequence spaces are the ℓ^p spaces, defined in the previous chapter. Weighted shifts are particularly nice operators on these spaces, as many properties rely solely upon the weights of the map. For instance, a well-defined operator $T : \ell^p \to \ell^p$ must have that $\sup_{n\in\omega} |\beta_n| < \infty$.

A larger class of sequence spaces are the weighted ℓ^p spaces. We define this in the following manner.

Definition 2.1.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then for $w_i \in \mathbb{K}$ for all $i \in \omega$, the *weighted ℓ^p sequence space* is given by

$$X = \ell^p(w_i) := \left\{ x = \langle x_i \rangle_{i\in\omega} \in \mathbb{K}^\omega ; \sum_{i=0}^{\infty} w_i |x_i|^p < \infty \right\}, 1 \leq p < \infty$$

Now, consider the weighted space $\ell^1\left(\frac{1}{2^i}\right)$, which has a norm given by $\|\overline{x}\| = \sum_{i=0}^{\infty} \frac{1}{2^i}|x_i|$. If we expand to a general sequence space X, where the sequences do not have to be summable, we can take the norm from $\ell^1\left(\frac{1}{2^i}\right)$ and use it on X to generate an extended metric on the space, that is a metric in which the distance between two points is allowed to be infinite, where

$$d(\overline{x}, \overline{y}) = \sum_{i=0}^{\infty} \frac{1}{2^i} |x_i - y_i|.$$

For the remainder of the chapter, we will be working on spaces in which the distance of two points may be infinite, requiring the use of the above extended metric.

As we will be working with sequences of sequences in our examination of the shadowing property, we end this section with a note on notation. We will reserve overline notation to denote only sequences, and will be using the notation x_{n_m} to denote the m-th term of the sequence $\overline{x_n}$.

2.2 The Shadowing Property

The first question we explore is whether there are conditions under which a backwards weighted shift operator T has the shadowing property. In fact, one can easily show that a shift of this type exhibits the shadowing property when the weights of the operator have a supremum of less than $\frac{1}{2}$. This is due to the fact that if T is a contraction (meaning there exists a real number $k \in [0, 1)$ such that for any $\overline{x}, \overline{y} \in X$, $d(T\overline{x}, T\overline{y}) \leq kd(\overline{x}, \overline{y})$) then (X, T) exhibits the shadowing property. This is a known result, and we include the proof for completeness.

Proposition 2.2.1. Let (X, T) be a dynamical system, and suppose that there is an $L \in (0, 1)$ such that $d(Tx, Ty) < L \cdot d(x, y)$ for all $x, y \in X$. Then (X, T) exhibits the shadowing property.

Proof. Let $\epsilon > 0$ be given, and define $\delta = (1 - L) \cdot \epsilon$. Let $\langle x_n \rangle_{n \in \omega}$ be a δ-pseudo-orbit. Notice that for all $n \in \mathbb{N}$, we have that

$$\begin{aligned} d(T^n x_0, x_n) &= d(T(T^{n-1} x_0), x_n) \\ &\le d\big(T(T^{n-1} x_0), T(x_{n-1})\big) + d(x_n, T(x_{n-1})) \\ &< L \cdot d\big(T^{n-1} x_0, x_{n-1}\big) + \delta \\ &= L \cdot d\big(T^{n-1} x_0, x_{n-1}\big) + (1 - L) \cdot \epsilon \\ &= L \cdot \Big(d\big(T^{n-1} x_0, x_{n-1}\big) - \epsilon\Big) + \epsilon. \end{aligned}$$

Therefore, we have that

$$d(T^n x_0, x_n) - \epsilon < L \cdot \Big(d\big(T^{n-1} x_0, x_{n-1}\big) - \epsilon\Big).$$

But then, by induction, we have that

$$d(T^n x_0, x_n) - \epsilon < L^n \cdot \Big(d(x_0, x_0) - \epsilon\Big).$$

Therefore, we have that

$$\begin{aligned} d(T^n x_0, x_n) &< L^n \cdot \Big(d(x_0, x_0) - \epsilon\Big) + \epsilon \\ &= \epsilon \cdot (1 - L^n) < \epsilon \end{aligned}$$

□

Theorem 2.2.2. If T is a weighted backwards shift on X such that $\sup_{j \in \omega} |\beta_j| < \frac{1}{2}$, then (X, T) exhibits the shadowing property.

Proof. Let T be as in the statement. Let $\beta = \sup_{j \in \omega} |\beta_j| < \frac{1}{2}$. Notice that for any two points $\overline{x}$ and $\overline{y}$ in X, we have the following:

$$\|T\overline{x} - T\overline{y}\| = \sum_{i=0}^{\infty} \frac{1}{2^i} |\beta_i| \cdot |x_{i+1} - y_{i+1}| \leq 2\beta \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i - y_i|$$
$$\leq 2\beta \sum_{i=0}^{\infty} \frac{1}{2^i} |x_i - y_i| = 2\beta d(\overline{x}, \overline{y}).$$

Since $2\beta < 1$, T is a contraction. Therefore, (X, T) exhibits the shadowing property.

□

Another scenario which provides the shadowing property for an operator T is if the operator uniformly, under enough iterations, maps every sequence to the zero sequence. Recall that T is *nilpotent* if $T^n = 0$ for some $n \in \omega$.

Theorem 2.2.3. If a weighted backwards shift operator T on X is nilpotent, then (X, T) exhibits the shadowing property.

Proof. Let $n \in \omega$ be such that $T^n = 0$. Let $\epsilon > 0$ be given. By continuity of T and its iterates, choose $\delta > 0$ such that $\delta < \frac{\epsilon}{n+1}$ and if $d(\overline{x}, \overline{y}) < \delta$, then $d(T^k\overline{x}, T^k\overline{y}) < \frac{\epsilon}{n+1}$ for all $k \leq n$. Let $\langle \overline{x_n} \rangle_{n=0}^{\infty}$ be a δ-pseudo-orbit. Defining $z = \overline{x_0}$, we have that

$$d(T^m\overline{z}, \overline{x_m}) \leq d(T^m\overline{z}, T^m\overline{x_0}) + \sum_{j=0}^{m-1} d(T^{m-j}\overline{x_j}, T^{m-(j+1)}\overline{x_{j+1}}).$$

Now, if $m \leq n$, then we have

$$d(T^m\overline{z}, \overline{x_m}) < 0 + \frac{m\epsilon}{n+1} < \epsilon$$

If $m > n$, notice that

$$d(T^m\overline{z}, \overline{x_m}) \leq d(T^m\overline{z}, T^m\overline{x_0}) + \sum_{j=0}^{m-1} d(T^{m-j}\overline{x_j}, T^{m-(j+1)}\overline{x_{j+1}})$$
$$= \sum_{j=m-n}^{m-1} d(T^{m-j}\overline{x_j}, T^{m-(j+1)}\overline{x_{j+1}}),$$

as T is nilpotent. But then

$$d(T^m\overline{z}, \overline{x_m}) < \frac{n\epsilon}{n+1} < \epsilon.$$

Thus $\overline{z}$ ϵ-shadows $\langle \overline{x_n} \rangle_{n=0}^{\infty}$, proving that (X, T) has the shadowing property. □

The above two theorems have shown that weighted shifts with sufficiently small weights have the shadowing property. We now show that weighted shifts with weights that grow sufficiently fast have the shadowing property.

Theorem 2.2.4. Let T be a weighted backwards shift operator on X. If

$$\sup_{n\in\mathbb{N}} \sum_{i=0}^{n} \frac{2^i}{\prod_{j=i}^{n} |\beta_j|} < \infty,$$

then (X, T) exhibits the shadowing property.

Proof. Suppose T is as above, and let $\epsilon > 0$ be given. Suppose also that $\sup_{n\in\mathbb{N}} \sum_{i=0}^{n} \frac{2^i}{\prod_{j=i}^{n} |\beta_j|} = S$ and let $\delta = \frac{\epsilon}{2S}$.

Given a δ-pseudo-orbit $\langle \overline{x_n} \rangle_{n=0}^{\infty}$, define a point $\overline{z}$ by $z_0 = x_{0_0}$ and $z_n = \frac{x_{n_0}}{\prod_{i=0}^{n-1} |\beta_i|}$. Notice the following:

$$d(T^n\overline{z}, \overline{x_n}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right|$$

$$\leq \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \frac{x_{i+n_0}}{\prod_{j=0}^{i-1} \beta_j} - x_{n_i} \right|$$
$$\leq \sum_{i=0}^{\infty} \frac{1}{2^i} \left(\sum_{k=0}^{i-1} \frac{1}{\prod_{j=k}^{i-1} |\beta_j|} \left| x_{i+n-k_k} - \beta_k x_{i+n-(k+1)_{k+1}} \right| \right)$$
$$< \sum_{i=0}^{\infty} \frac{1}{2^i} \left(\sum_{k=0}^{i-1} \frac{1}{\prod_{j=k}^{i-1} |\beta_j|} 2^k \delta \right) < \sum_{i=0}^{\infty} \frac{1}{2^i} \left(\sum_{k=0}^{i-1} \frac{2^k}{\prod_{j=k}^{i-1} |\beta_j|} \frac{\epsilon}{2S} \right)$$
$$= \frac{\epsilon}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} \left(\frac{1}{S} \sum_{k=0}^{i-1} \frac{2^k}{\prod_{j=k}^{i-1} |\beta_j|} \right) < \frac{\epsilon}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} = \epsilon$$

As ϵ and $\langle \overline{x_n} \rangle$ were arbitrary, (X, T) exhibits the shadowing property. $\square$

2.3 Shadowing in a General Sequence Space in which $\sup_{j\in\omega} |\beta_j| < \infty$

The preceding theorems present us with criteria upon the weights in which the shadowing property is guaranteed. While the first two theorems guarantee the property if the weights are sufficiently small, the third criterion requires the weights β_j to grow at a rate on the order of 2^j. In particular, this criterion will not hold if there exists an $M \in \mathbb{N}$ such that $|\beta_j| < M$ for all $j \in \omega$. This provides a stark contrast with the case for the Banach space ℓ^p, as an operator in which this criterion holds is not well-defined. It is worth noting that in a general sequence space X, if the weights tend to infinity, the operator exhibiting this will fail to be continuous, but it will still be well-defined.

In order to discuss the shadowing property on a general sequence space in which the operator has bounded weights, i.e. $\sup_{n\in\omega} |\beta_n| < \infty$, the following definition will prove useful. For $\overline{x} \in \mathbb{K}^\omega$, define the *galaxy* of the sequence, $G_{\overline{x}}$, as all sequences $\overline{y}$ that are only a finite distance from $\overline{x}$, i.e.

$$G_{\overline{x}} = \{\overline{y} : d(\overline{x}, \overline{y}) < \infty\}.$$

Notice that $G_{\overline{0}}$ is just $\ell^1\left(\frac{1}{2^i}\right)$, because for $\overline{x} \in G_{\overline{0}}$, we have that

$$d(\overline{x}, 0) = \sum_{i=0}^{\infty} \frac{1}{2^i}|x_i| < \infty,$$

which is exactly the criterion to be a member of $\ell^1\left(\frac{1}{2^i}\right)$. Also, if $\overline{z} \in G_{\overline{x}} \cap G_{\overline{y}}$, then $G_{\overline{x}} = G_{\overline{y}}$, as $d(\overline{x}, \overline{y}) \leq d(\overline{x}, \overline{z}) + d(\overline{z}, \overline{y}) < \infty$.

Now an important fact that will help us talk about pseudo-orbits $\langle \overline{x_n} \rangle_{n=0}^{\infty}$ not contained within $G_{\overline{0}}$ is as follows.

Lemma 2.3.1. $\langle \overline{x_n} \rangle_{n=0}^{\infty}$ is a δ-pseudo-orbit if and only if $\langle \overline{x_n} - T^n\overline{y} \rangle_{n=0}^{\infty}$ is a δ-pseudo-orbit for every point $\overline{y} \in X$.

Proof. Notice the following:

$$\begin{aligned} d\big(T(\overline{x_{n-1}} - T^{n-1}\overline{y}), \overline{x_n} - T^n\overline{y}\big) &= d\big(T\overline{x_{n-1}} - T^n\overline{y}, \overline{x_n} - T^n\overline{y}\big) \\ &= d\big(T\overline{x_{n-1}}, \overline{x_n}\big) \end{aligned}$$

Thus $\langle \overline{x_n} - T^n\overline{y} \rangle_{n=0}^{\infty}$ is a δ-pseudo-orbit if and only if $\langle \overline{x_n} \rangle_{n=0}^{\infty}$ is. $\square$

This result allows us to think about pseudo-orbits as being centered about the orbit of a point. Combined with the fact that if $\overline{y} \in G_{\overline{x}}$, then $T\overline{y} \in G_{T\overline{x}}$, due to

$$\begin{aligned} d\left(T\overline{x}, T\overline{y}\right) &= \sum_{i=0}^{\infty} \frac{|\beta_i|}{2^i}\left|x_{i+1} - y_{i+1}\right| \\ &\leq \sum_{i=0}^{\infty} \frac{\sup_{i\in\omega}|\beta_i|}{2^i}\left|x_{i+1} - y_{i+1}\right| \\ &= 2\left(\sup_{i\in\omega}|\beta_i|\right)\sum_{i=1}^{\infty}\frac{1}{2^i}\left|x_i - y_i\right| \\ &\leq 2\left(\sup_{i\in\omega}|\beta_i|\right)\sum_{i=0}^{\infty}\frac{1}{2^i}\left|x_i - y_i\right| \end{aligned}$$

$$= 2 \left(\sup_{i \in \omega} |\beta_i| \right) d\left(\overline{x}, \overline{y}\right) < \infty$$

we can now think of pseudo-orbits as being contained within the orbit of the galaxy of some point.

2.4 Shadowing of Uniformly Bounded Pseudo-Orbits

As we now generalize to a weighted shift where the supremum of the weights is allowed to be infinite, we consider weaker versions of the shadowing property. This shift in our focus is due to the fact that sequences may now "jump" between orbits of galaxies, as there exist sequences of weights that make the distance of the image of two points infinitely far apart, whereas their original distance was finite.

Example 2.4.1. Let $Y = \ell^1\left(\frac{1}{2^i}\right)$ and let $T\colon Y \to Y$ a weighted shift with $\beta_i = 2^i$. Then defining $\overline{0} = \langle 0 \rangle_{i \in \omega}$ and $\overline{1} = \langle 1 \rangle_{i \in \omega}$, we have that $d\left(\overline{0}, \overline{1}\right) = 2$, yet $d\left(T\left(\overline{0}\right), T\left(\overline{1}\right)\right) = \infty$.

We begin by defining the following:

Definition 2.4.2. We call a pseudo-orbit $\langle \overline{x_n} \rangle$ *M-uniformly bounded* if $M \in \mathbb{N}$, $|x_{n_m}| < M$ for all $n, m \in \omega$, that is that every entry of every sequence is bounded by M.

We shall say that (X, T) exhibits the *uniformly bounded pseudo-orbit tracing property (UBPOTP)* if for any $\epsilon > 0$ and $M \in \mathbb{N}$, there exists a $\delta > 0$ such that for any M-uniformly bounded δ-pseudo-orbit $\langle \overline{x_n} \rangle$, there exists a point $\overline{z} \in X$ such that $\overline{z}$ ϵ-shadows the pseudo-orbit.

For the remainder of this chapter, we assume that X is a general sequence space with extended metric given by

$$d(\overline{x}, \overline{y}) = \sum_{i=0}^{\infty} \frac{1}{2^i} |x_i - y_i|.$$

The following lemma will prove useful in demonstrating that certain weighted shifts exhibit the UBPOTP.

Lemma 2.4.3. Let T be a weighted backwards shift operator on X with $|\beta_j| \geq 1$ for all $j \in \omega$, and let $\epsilon > 0$, $M \in \mathbb{N}$ be given. Then there exists an L such that if $\langle \overline{x_n} \rangle_{n\in\omega}$ is an M-uniformly bounded pseudo-orbit, then the point $\overline{z}$ defined by $z_0 = x_{0_0}$ and $z_n = \frac{x_{n_0}}{\prod_{j=0}^{n-1} |\beta_j|}$ has the property that

$$d(T^n \overline{z}, \overline{x_n}) \leq \sum_{i=0}^{L} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| + \frac{\epsilon}{2}.$$

Proof. Let $\epsilon > 0$ and $M \in \mathbb{N}$ be given. Notice that there exists an $L \in \mathbb{N}$ such that $2M \sum_{i=0}^{\infty} \frac{1}{2^i} < 2M \sum_{i=0}^{L} \frac{1}{2^i} + \frac{\epsilon}{2}$, as the series converges. Let $\langle \overline{x_n} \rangle_{n\in\omega}$ be an M-uniformly bounded pseudo-orbit. Then, with $\overline{z}$ defined as above, we have that

$$\left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| \leq \left| \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} x_{i+n_0} - x_{n_i} \right| < \frac{M}{\prod_{j=0}^{i-1} |\beta_j|} + M \leq 2M.$$

Thus, for all $n \in \omega$, we have that

$$\begin{aligned} d(T^n \overline{z}, \overline{x_n}) &= \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| \\ &= \sum_{i=0}^{L} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| + \sum_{i=L+1}^{\infty} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| \\ &< \sum_{i=0}^{L} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| + 2M \sum_{i=L+1}^{\infty} \frac{1}{2^i} \end{aligned}$$

$$< \sum_{i=0}^{L} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| + \frac{\epsilon}{2}.$$

$\square$

Theorem 2.4.4. Suppose that T is a weighted backwards shift operator on X with $|\beta_j| \geq 1$ for all $j \in \omega$. Then (X, T) exhibits the UBPOTP.

Proof. Let $M, \epsilon > 0$ be given. Take $L \in \mathbb{N}$ that accompanies M and ϵ from Lemma 2.4.3. Now, define $\beta = \min \left\{ \prod_{j=0}^{i-1} |\beta_j| \mid 0 \leq i \leq L \right\}$ and let $\delta = \frac{\epsilon}{4} \frac{\beta}{2^L(L+1)}$. Let $\langle \overline{x_n} \rangle$ be an M-uniformly bounded δ-pseudo-orbit, and define a point $\overline{z}$ by taking $z_0 = x_{0_0}$ and $z_n = \frac{x_{n_0}}{\prod_{j=0}^{n-1} |\beta_j|}$.

By the above lemma, we will then have that

$$d(T^n \overline{z}, \overline{x_n}) \leq \sum_{i=0}^{L} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| + \frac{\epsilon}{2}.$$

Notice the following for all $i \leq L$:

$$\begin{aligned}
\left| \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} x_{i+n_0} - x_{n_i} \right| &\leq \left| \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} x_{i+n_0} - \frac{1}{\prod_{j=1}^{i-1} |\beta_j|} x_{i+n-1_1} \right| \\
&\quad + \left| \frac{1}{\prod_{j=1}^{i-1} |\beta_j|} x_{i+n-1_1} - \frac{1}{\prod_{j=2}^{i-1} |\beta_j|} x_{i+n-2_2} \right| \\
&\quad + \cdots + \left| \frac{1}{|\beta_{i-1}|} x_{n+1_{i-1}} - x_{n_i} \right| \\
&\leq \delta \left(\frac{1}{\prod_{j=0}^{i-1} |\beta_j|} + \frac{2}{\prod_{j=1}^{i-1} |\beta_j|} + \cdots + \frac{2^{i-1}}{|\beta_{i-1}|} \right) \\
&\leq \delta \sum_{i=0}^{L} \frac{2^i}{\beta} < \frac{\epsilon}{4}
\end{aligned}$$

Thus, we have that

$$\sum_{i=0}^{L} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| < \frac{\epsilon}{4} \sum_{i=0}^{L} \frac{1}{2^i} < \frac{\epsilon}{2}$$

This shows us that $d(T^n \overline{z}, \overline{x_n}) < \epsilon$. Thus (X, T) exhibits the UBPOTP. $\square$

This result extends nicely to the following situation.

Theorem 2.4.5. Suppose that T is a weighted backwards shift operator on X with $\inf_{i \in \mathbb{N}} \prod_{j=0}^{i-1} |\beta_j| > 0$. Then (X, T) exhibits the UBPOTP.

Proof. Let $M, \epsilon > 0$ be given, and assume that $\beta = \inf_{i \in \mathbb{N}} \prod_{j=0}^{i-1} |\beta_j|$. Take $L \in \mathbb{N}$ associated to M and ϵ such that

$$\sum_{i=0}^{\infty} \frac{1}{2^i} \left(\frac{M(\beta+1)}{\beta} \right) \leq \sum_{i=0}^{L} \frac{1}{2^i} \left(\frac{M(\beta+1)}{\beta} \right) + \frac{\epsilon}{2}.$$

. Define $\delta = \frac{\epsilon}{4} \frac{\beta}{2^L (L+1)}$. Let $\langle \overline{x_n} \rangle$ be an M-uniformly bounded δ-pseudo-orbit, define a point $\overline{z}$ by taking $z_0 = x_{0_0}$ and $z_n = \frac{x_{n_0}}{\prod_{j=0}^{n-1} |\beta_j|}$. Notice the following:

$$\begin{aligned}
d(T^n \overline{z}, \overline{x_n}) &= \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| \\
&\leq \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \frac{x_{i+n_0}}{\prod_{j=0}^{i-1} |\beta_j|} - x_{n_i} \right| \\
&< \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \frac{x_{i+n_0}}{\beta} - x_{n_i} \right| \\
&< \sum_{i=0}^{\infty} \frac{1}{2^i} \left(\frac{M}{\beta} + M \right) \\
&= \sum_{i=0}^{\infty} \frac{1}{2^i} \left(\frac{M(\beta+1)}{\beta} \right) \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} \left(\frac{M(\beta+1)}{\beta} \right) + \frac{\epsilon}{2}.
\end{aligned}$$

In a similar argument to the above theorem, it is easily shown that our defined point $\overline{z}$ will indeed ϵ-shadow the given pseudo-orbit. $\square$

Now we show that even if the infinum of magnitude of the product of the weights is equal to zero, a weighted shift (X, T) exhibits the UBPOTP.

Lemma 2.4.6. Suppose that T is a weighted backwards shift operator on X with $|\beta_j| > 0$ for all $j \in \omega$ and let $M, \epsilon > 0$ be given. Moreover, assume that $\sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|}$ converges and that $\inf_{i \in \mathbb{N}} \prod_{j=0}^{i-1} |\beta_j| = 0$. Then there exists an $L \in \mathbb{N}$ dependent on M, ϵ, and the weights β_j such that if $\langle \overline{x_n} \rangle$ is an M-uniformly bounded pseudo-orbit, then the point $\overline{z}$ defined by $z_0 = x_{0_0}$ and $z_n = \frac{x_{n_0}}{\prod_{j=0}^{n-1} |\beta_j|}$ has the property that

$$d(T^n \overline{z}, \overline{x_n}) < \sum_{i=0}^{L} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1} \beta_j \right) x_{n_i} \right| + \frac{\epsilon}{2}.$$

Proof. Let $M, \epsilon > 0$ be given. Since our assumption of the sum converging would not hold if $\prod_{j=0}^{i-1} |\beta_j| = 0$ for some finite i, we do not concern ourselves with this situation. Therefore, the proof can be split into two cases:

Case 1: $\prod_{j=0}^{i-1} |\beta_j|$ Converges to 0. In this case, define L_1 and L_2 as follows.

$$L_1\colon \prod_{j=0}^{i-1} |\beta_j| < 1 \text{ for all } i \geq L_1$$

$$L_2\colon 2M \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} < 2M \sum_{i=0}^{L_2} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} + \frac{\epsilon}{2}$$

Let $L = L_1 + L_2$, and let $\langle \overline{x_n} \rangle_{n\in\omega}$ be a M-uniformly bounded pseudo-orbit. Let $\overline{z}$ be defined as above. We will use the fact that for $i \geq L_1$, we have the following:

$$\begin{aligned}
\frac{1}{2^i}\left|\left(\prod_{j=i}^{i+n-1}\beta_j\right)z_{i+n} - x_{n_i}\right| &= \frac{1}{2^i}\left|\frac{\left(\prod_{j=i}^{i+n-1}\beta_j\right)}{\prod_{j=0}^{i+n-1}|\beta_j|}x_{i+n_0} - x_{n_i}\right| \\
&\leq \frac{1}{2^i}\left|\frac{\left(\prod_{j=i}^{i+n-1}\beta_j\right)}{\prod_{j=0}^{i+n-1}\beta_j}x_{i+n_0} - x_{n_i}\right| \\
&= \frac{1}{2^i}\left|\frac{x_{i+n_0}}{\prod_{j=0}^{i-1}\beta_j} - x_{n_i}\right| \\
&= \frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\left|x_{i+n_0} - \left(\prod_{j=0}^{i-1}\beta_j\right)x_{n_i}\right| \\
&< \frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}(2M)
\end{aligned}$$

Then, for all $n \in \omega$, we have that

$$\begin{aligned}
&d(T^n\overline{z}, \overline{x_n}) = \sum_{i=0}^{\infty}\frac{1}{2^i}\left|\left(\prod_{j=i}^{i+n-1}\beta_j\right)z_{i+n} - x_{n_i}\right| \\
&= \left(\sum_{i=0}^{L}\frac{1}{2^i}\left|\left(\prod_{j=i}^{i+n-1}\beta_j\right)z_{i+n} - x_{n_i}\right|\right) + \left(\sum_{i=L+1}^{\infty}\frac{1}{2^i}\left|\left(\prod_{j=i}^{i+n-1}\beta_j\right)z_{i+n} - x_{n_i}\right|\right) \\
&< \left(\sum_{i=0}^{L}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\left|x_{i+n_0} - \left(\prod_{j=0}^{i-1}\beta_j\right)x_{n_i}\right|\right) + \left(\sum_{i=L+1}^{\infty}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}(2M)\right) \\
&< \sum_{i=0}^{L}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\left|x_{i+n_0} - \left(\prod_{j=0}^{i-1}\beta_j\right)x_{n_i}\right| + \frac{\epsilon}{2}
\end{aligned}$$

Case 2: $\prod_{j=0}^{i-1}|\beta_j|$ Does Not Converge to 0. Let $A = \left\{i \in \mathbb{N} : \prod_{j=0}^{i-1}|\beta_j| < 1\right\}$ and $B = \left\{i \in \mathbb{N} : \prod_{j=0}^{i-1}|\beta_j| \geq 1\right\}$. By the fact that A and B are subsets of ω and both $\sum_{i=0}^{\infty}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}$ and $\sum_{i=0}^{\infty}\frac{1}{2^i}$ converge, define $L \in \mathbb{N}$ such that

$$\sum_{i\in A}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|} \leq \sum_{i\in A\cap[0,L]}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|} + \frac{\epsilon}{8M}$$

and

$$\sum_{i\in B}\frac{1}{2^i}\leq\sum_{i\in B\cap[0,L]}\frac{1}{2^i}+\frac{\epsilon}{8M}.$$

Let $\langle\overline{x_n}\rangle_{n\in\omega}$ be an M-uniformly bounded pseudo-orbit, and define $\overline{z}$ as above. Notice the following:

$$\begin{aligned}
d(T^n\overline{z},\overline{x_n}) &= \sum_{i=0}^{\infty}\frac{1}{2^i}\left|\left(\prod_{j=i}^{i+n-1}\beta_j\right)z_{i+n}-x_{n_i}\right| \\
&= \sum_{i=0}^{\infty}\frac{1}{2^i}\left|\frac{\left(\prod_{j=i}^{i+n-1}\beta_j\right)}{\prod_{j=0}^{i+n-1}|\beta_j|}x_{i+n_0}-x_{n_i}\right| \\
&\leq \sum_{i=0}^{\infty}\frac{1}{2^i}\left|\frac{\left(\prod_{j=i}^{i+n-1}\beta_j\right)}{\prod_{j=0}^{i+n-1}\beta_j}x_{i+n_0}-x_{n_i}\right| \\
&= \sum_{i=0}^{\infty}\frac{1}{2^i}\left|\frac{x_{i+n_0}}{\prod_{j=0}^{i-1}\beta_j}-x_{n_i}\right| \\
&\leq \sum_{i\in A}\frac{1}{2^i}\left|\frac{x_{i+n_0}}{\prod_{j=0}^{i-1}\beta_j}-x_{n_i}\right|+\sum_{i\in B}\frac{1}{2^i}\left|\frac{x_{i+n_0}}{\prod_{j=0}^{i-1}\beta_j}-x_{n_i}\right| \\
&= \sum_{i\in A}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\left|x_{i+n_0}-\left(\prod_{j=0}^{i-1}\beta_j\right)x_{n_i}\right|+\sum_{i\in B}\frac{1}{2^i}\left|\frac{x_{i+n_0}}{\prod_{j=0}^{i-1}\beta_j}-x_{n_i}\right| \\
&< \sum_{i\in A}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}(2M)+\sum_{i\in B}\frac{1}{2^i}(2M) \\
&< 2M\left(\sum_{i\in A}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}+\sum_{i\in B}\frac{1}{2^i}\right)
\end{aligned}$$

But then we have that

$$\begin{aligned}
d(T^n\overline{z},\overline{x_n}) &< 2M\left(\sum_{i\in A\cap[0,L]}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}+\frac{\epsilon}{8M}+\sum_{i\in B\cap[0,L]}\frac{1}{2^i}+\frac{\epsilon}{8M}\right) \\
&= 2M\left(\sum_{i\in A\cap[0,L]}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|}+\sum_{i\in B\cap[0,L]}\frac{1}{2^i}\right)+\frac{\epsilon}{2}
\end{aligned}$$

Then we must have that

$$d(T^n\overline{z}, \overline{x_n}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right|$$
$$< \sum_{i=0}^{L} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j \right) z_{i+n} - x_{n_i} \right| + \frac{\epsilon}{2}$$

□

Theorem 2.4.7. Suppose that T is a weighted backwards shift operator on X with $|\beta_j| > 0$ for all $j \in \omega$. Moreover, assume that $\sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|}$ converges. Then (X, T) exhibits the UBPOTP.

Proof. Let $M, \epsilon > 0$ be given. Now as the result is shown if $\inf_{i \in \mathbb{N}} \prod_{j=0}^{i-1} |\beta_j| > 0$, assume that $\inf_{i \in \mathbb{N}} \prod_{j=0}^{i-1} |\beta_j| = 0$. Take L from the above lemma, and define $\beta = \min \left\{ \prod_{j=0}^{i-1} |\beta_j| \mid 0 \leq i \leq L \right\}$ and let $\delta = \frac{\epsilon}{4} \frac{\beta}{2^L (L+1)}$. Let $\langle \overline{x_n} \rangle$ be an M-uniformly bounded δ-pseudo-orbit and let $\overline{z}$ be defined as in Lemma 2.4.6. Then, by the lemma, we must have that

$$d(T^n\overline{z}, \overline{x_n}) < \sum_{i=0}^{L} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1} \beta_j \right) x_{n_i} \right| + \frac{\epsilon}{2}$$

.

From there, notice that for all $i \leq L$:

$$\frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1} \beta_j \right) x_{n_i} \right| \leq \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| x_{i+n_0} - \beta_0 x_{i+n-1_1} \right|$$
$$+ \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| \beta_0 x_{i+n-1_1} - \beta_0 \beta_1 x_{i+n-2_2} \right| + \cdots$$
$$+ \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| \left(\prod_{j=0}^{i-2} \beta_j \right) x_{n+1_{i-1}} - \left(\prod_{j=0}^{i-1} \beta_j \right) x_{n_i} \right|$$

$$\leq \delta \left(\frac{1}{\prod_{j=0}^{i-1} |\beta_j|} + \frac{2}{\prod_{j=1}^{i-1} |\beta_j|} + \cdots + \frac{2^{i-1}}{|\beta_{i-1}|} \right)$$
$$\leq \delta \sum_{i=0}^{L} \frac{2^i}{\beta} < \frac{\epsilon}{4}$$

Therefore we must have that

$$d(T^n \overline{z}, \overline{x_n}) < \frac{\epsilon}{4} \sum_{i=0}^{L} \frac{1}{2^i} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus (X, T) exhibits the UBPOTP. $\square$

As a special case, the same result holds for any space $\ell^1 \left(\frac{1}{2^i} \right)$. This is due to the fact that $\ell^1 \left(\frac{1}{2^i} \right)$ is a subset of our general sequence space X.

Corollary 2.4.8. Suppose that T is a weighted backwards shift operator on $\ell^1 \left(\frac{1}{2^i} \right)$ with $|\beta_j| > 0$ for all $j \in \omega$. Moreover, assume that $\sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} < \infty$. Then $\left(\ell^1 \left(\frac{1}{2^i} \right), T \right)$ exhibits the UBPOTP.

Proof. Let T be a weighted backwards shift on $\ell^1 \left(\frac{1}{2^i} \right)$. Let $\epsilon > 0$, $M \in \mathbb{N}$ be given. Let $\delta > 0$ be defined as Theorem 2.4.7, and let $\langle \overline{x_n} \rangle_{n \in \omega}$ be an M-uniformly bounded δ-pseudo-orbit. Notice that $\langle \overline{x_n} \rangle_{n \in \omega}$ is also an M-uniformly bounded δ-pseudo-orbit in the general sequence space X from Theorem 2.4.7, and so the point z given by the theorem will ϵ-shadow $\langle \overline{x_n} \rangle_{n \in \omega}$ in X.

In order to prove the corollary, we must show that $z \in \ell^1 \left(\frac{1}{2^i} \right)$. To this end, notice that

$$\sum_{i=0}^{\infty} \frac{1}{2^i} |z_i| = x_{0_0} + \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \frac{x_{i_0}}{\prod_{j=0}^{i-1} |\beta_j|} \right|$$
$$= x_{0_0} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} |x_{i_0}|$$

$$< x_{0_0} + M \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|}$$

$$< \infty$$

Therefore $z \in \ell^1\left(\frac{1}{2^i}\right)$. □

It is important to note that if the above condition of $\sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} < \infty$ does not hold, the UBPOTP can fail.

Theorem 2.4.9. Suppose that T is the weighted backwards shift on X with weights $\beta_j = \frac{1}{2}$ for all $j \in \omega$. Then (X, T) does not exhibit the UBPOTP.

Proof. Let $\epsilon = 1$, and $\delta > 0$ be given. Define a δ-pseudo-orbit $\langle \overline{x_n} \rangle$ as follows: for all $n \in \omega$, let

$$\overline{x_n} = \left\langle \frac{\delta}{2}, \delta, 4\delta, \cdots, 2^{k-1}\delta, 0, \cdots \right\rangle,$$

where $k \in \mathbb{N}$ is an integer such that $\frac{1}{k} < \frac{\delta}{4}$.

Now, suppose that a point $\overline{z} \in X$ shadows $\langle \overline{x_n} \rangle$. Then, since $d(\overline{z}, \overline{x_0}) < 1$, we must have that

$$\left|\frac{\delta}{2} - z_0\right| + \left|\frac{\delta}{2} - \frac{1}{2} z_1\right| + \left|\frac{\delta}{2} - \frac{1}{4} z_2\right| + \cdots + \left|\frac{\delta}{2} - \frac{1}{2^k} z_k\right| < 1.$$

Therefore, there must exist an $i \leq k$ such that $\left|\frac{\delta}{2} - \frac{1}{2^i} z_i\right| < \frac{1}{k} < \frac{\delta}{4}$. But then $\frac{1}{2^i} z_i > \frac{\delta}{4}$ or $z_i > \delta 2^{i-2}$. Thus

$$\|\overline{z}\| = \sum_{i=0}^{\infty} \frac{1}{2^i} |z_i| > \frac{\delta}{4}.$$

Now, as $d(T^k\overline{z}, \overline{x_k}) < 1$, we must have that

$$\left|\frac{\delta}{2} - \frac{1}{2^k}z_k\right| + \left|\frac{\delta}{2} - \frac{1}{2^{k+1}}z_{k+1}\right| + \left|\frac{\delta}{2} - \frac{1}{2^{k+2}}z_{k+2}\right| + \cdots + \left|\frac{\delta}{2} - \frac{1}{2^{2k}}z_{2k}\right| < 1.$$

Therefore, there must exist an $i \leq k$ such that $\left|\frac{\delta}{2} - \frac{1}{2^{k+i}}z_{k+i}\right| < \frac{1}{k} < \frac{\delta}{4}$. But then $\frac{1}{2^{k+i}}z_{k+i} > \frac{\delta}{4}$ or $z_{k+i} > \delta 2^{k+i-2}$. Thus

$$\|\overline{z}\| = \sum_{i=0}^{\infty} \frac{1}{2^i}|z_i| \geq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}.$$

But then, for every interval $[k, 2k], k \in \omega$, there exists an entry $z_{k+i}, i \leq k$, such that $|z_{k+i}| \geq \delta 2^{k+i-2}$. Thus we must have that

$$\|\overline{z}\| = \sum_{i=0}^{\infty} \frac{1}{2^i}|z_i| \geq \sum_{i=0}^{\infty} \frac{\delta}{4} = \infty.$$

But then, since $\|\overline{x_n}\| = \frac{k\delta}{2}$ for all $n \in \omega$, we must have that $d(\overline{z}, \overline{x_0}) = \infty$. Thus there is no point which shadows $\langle \overline{x_n} \rangle$.

□

2.5 Shadowing of Bounded Pseudo-Orbits

We now wish to move away from uniformly bounded pseudo-orbits to a more general setting.

Definition 2.5.1. We call a sequence $\langle \overline{x_n} \rangle$ in $\mathbb{K}^\omega$ *M-bounded* if, for $M \in \mathbb{N}$, $\|\overline{x_n}\| < M$ for all $n \in \omega$.

Notice that if given $\epsilon > 0$ and $\overline{x_n} \in \mathbb{K}^\omega$ with $\|\overline{x_n}\| < M$ for some $M \in \mathbb{N}$, then there must exist an $L_{(n,\epsilon)} \in \mathbb{N}$ such that

$$\sum_{i=0}^{\infty} \frac{1}{2^i}|x_{n_i}| \leq \sum_{i=0}^{L_{(n,\epsilon)}} \frac{1}{2^i}|x_{n_i}| + \frac{\epsilon}{4}.$$

Definition 2.5.2. We shall call a pseudo-orbit $\langle\overline{x_n}\rangle_{n\in\omega}$ *(M, W, ϵ)-bounded* if $\|\overline{x_n}\| < M$ for all $n \in \omega$ and if W is an upper bound of $\left\{L_{(n,\epsilon)}\right\}$. We call a pseudo-orbit *strongly bounded* if for all $\epsilon > 0$, there is a finite M and W such that it is (M, W, ϵ)-bounded.

We say that (X, T) exhibits the *strongly bounded pseudo-orbit tracing property (SBPOTP)*, or *shadowing of (M, W, ϵ)-bounded pseudo-orbits*, if for all $\epsilon > 0$, M and $W \in \mathbb{N}$, there exists a $\delta > 0$ such that for all (M, W, ϵ)-bounded δ-pseudo-orbits $\langle\overline{x_n}\rangle_{n\in\omega}$, there exists a $\overline{z} \in X$ such that $\overline{z}$ ϵ-shadows $\langle\overline{x_n}\rangle_{n\in\omega}$.

It is of worth to note that the existence of such a supremum does not imply that a bounded pseudo-orbit is uniformly bounded. This is due to the fact that the jth coordinate of any sequence $\overline{x}$ is multiplied by $\frac{1}{2^j}$ when calculating the norm of $\overline{x}$. Therefore, for any $M \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that for all $j > N$, $\frac{M}{2^j} < \frac{\epsilon}{4}$. Thus it is rather easy to formulate a pseudo-orbit such that $\sup_{n\in\mathbb{N}} L_{(n,M)}$ exists yet which is not uniformly bounded.

Lemma 2.5.3. Suppose that T is a weighted backwards shift operator on X with $|\beta_j| > 0$ for all $j \in \omega$. Moreover, assume that $\sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1}|\beta_j|} < \infty$. Let $\epsilon > 0$, M, and $W \in \mathbb{N}$ be given. Then there exists an $L \in \mathbb{N}$ such that if $\langle\overline{x_n}\rangle$ is an (M, W, ϵ)-bounded pseudo-orbit, then the point $\overline{z}$ defined by $z_0 = x_{0_0}$ and $z_n = \frac{x_{n_0}}{\prod_{j=0}^{n-1}|\beta_j|}$ has the property that

$$d(T^n\overline{z}, \overline{x_n}) < \sum_{i=0}^{L} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1} \beta_j\right) x_{n_i} \right| + \frac{\epsilon}{2}.$$

Proof. Let $\epsilon > 0$ and $M \in \mathbb{N}$ be given. As $\sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} < \infty$, there exists an $\alpha \in \mathbb{N}$ such that

$$M \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \leq M \sum_{i=0}^{\alpha} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} + \frac{\epsilon}{4}$$

Let $L = \max\{W, \alpha\}$. Let $\langle \overline{x_n} \rangle_{n\in\omega}$ be an (M, W, ϵ)-bounded pseudo-orbit, and define a point $\overline{z}$ as above. Then we must have the following:

$$\begin{aligned}
d(T^n\overline{z}, \overline{x_n}) &= \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \left(\prod_{j=i}^{i+n-1} \beta_j\right) z_{i+n} - x_{n_i} \right| \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1} \beta_j\right) x_{n_i} \right| \\
&\quad + \sum_{i=L+1}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1} \beta_j\right) x_{n_i} \right| \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1} \beta_j\right) x_{n_i} \right| \\
&\quad + \sum_{i=L+1}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} |x_{i+n_0}| + \sum_{i=L+1}^{\infty} \frac{1}{2^i} |x_{n_i}|
\end{aligned}$$

But then we have that

$$\begin{aligned}
\sum_{i=L+1}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} |x_{i+n_0}| &\leq M \sum_{i=\alpha+1}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1} |\beta_j|} \\
&< \frac{\epsilon}{4}
\end{aligned}$$

and that

$$\begin{aligned}\sum_{i=L+1}^{\infty} \frac{1}{2^i}|x_{n_i}| &\leq \sum_{i=W+1}^{\infty} \frac{1}{2^i}|x_{n_i}| \\ &< \frac{\epsilon}{4}\end{aligned}$$

Therefore, we ultimately see that

$$\begin{aligned}d(T^n\overline{z}, \overline{x_n}) &< \sum_{i=0}^{L} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1}|\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1}\beta_j\right) x_{n_i} \right| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \sum_{i=0}^{L} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1}|\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1}\beta_j\right) x_{n_i} \right| + \frac{\epsilon}{2}\end{aligned}$$

□

Theorem 2.5.4. Suppose that T is a weighted backwards shift operator on X with $|\beta_j| > 0$ for all $j \in \omega$. Moreover, assume that $\sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1}|\beta_j|} < \infty$. Then (X, T) exhibits the SBPOTP.

Proof. Let $\epsilon > 0$ and M and $W \in \mathbb{N}$ be given. Define $\beta = \min\left\{\prod_{j=0}^{i-1}|\beta j| \mid 0 \leq i \leq L\right\}$, where L is taken from the above lemma, and let $\delta = \frac{\epsilon}{4}\frac{\beta}{2^L(L+1)}$.

Let $\langle\overline{x_n}\rangle$ be an (M, W, ϵ)-bounded δ-pseudo-orbit, and define $\overline{z}$ as in the lemma above. Then, by the lemma, we will then have that

$$d(T^n\overline{z}, \overline{x_n}) < \sum_{i=0}^{L} \frac{1}{2^i} \frac{1}{\prod_{j=0}^{i-1}|\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1}\beta_j\right) x_{n_i} \right| + \frac{\epsilon}{2}.$$

But then, we must have that for all $i \leq L$:

$$\frac{1}{\prod_{j=0}^{i-1}|\beta_j|} \left| x_{i+n_0} - \left(\prod_{j=0}^{i-1}\beta_j\right) x_{n_i} \right|$$

$$
\begin{aligned}
&\leq \frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\left|x_{i+n_0}-\beta_0 x_{i+n-1_1}\right| \\
&\qquad + \frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\left|\beta_0 x_{i+n-1_1}-\beta_0\beta_1 x_{i+n-2_2}\right| + \dots \\
&\qquad \dots + \frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\left|\left(\prod_{j=0}^{i-2}\beta_j\right)x_{n+1_{i-1}} - \left(\prod_{j=0}^{i-1}\beta_j\right)x_{n_i}\right| \\
&= \frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\left|x_{i+n_0}-\beta_0 x_{i+n-1_1}\right| + \frac{1}{\prod_{j=1}^{i-1}|\beta_j|}\left|x_{i+n-1_1}-\beta_1 x_{i+n-2_2}\right| + \dots \\
&\qquad \dots + \frac{1}{|\beta_{i-1}|}\left|x_{n+1_{i-1}}-\beta_{i-1}x_{n_i}\right| \\
&< \frac{1}{\prod_{j=0}^{i-1}|\beta_j|}\delta + \frac{1}{\prod_{j=1}^{i-1}|\beta_j|}2\delta + \dots + \frac{1}{|\beta_{i-1}|}2^{i-1}\delta \\
&= \delta\left(\frac{1}{\prod_{j=0}^{i-1}|\beta_j|} + \frac{2}{\prod_{j=1}^{i-1}|\beta_j|} + \dots + \frac{2^{i-1}}{|\beta_{i-1}|}\right) \\
&\leq \delta\sum_{i=0}^{L}\frac{2^i}{\beta} < \frac{\epsilon}{4}
\end{aligned}
$$

But then we have that

$$d(T^n\overline{z},\overline{x_n}) < \frac{\epsilon}{4}\sum_{i=0}^{L}\frac{1}{2^i} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus (X,T) exhibits the SBPOTP. $\square$

A special case of these theorems, yet again, is that ℓ^p spaces. Yet again this is due to ℓ^1 being a subspace of our general sequence space X.

Theorem 2.5.5. Suppose that T is a weighted backwards shift operator on $\ell^1\left(\frac{1}{2^i}\right)$ with $|\beta_j| \geq 0$ for all $j \in \omega$. Moreover, assume that $\sum_{i=0}^{\infty}\frac{1}{2^i}\frac{1}{\prod_{j=0}^{i-1}|\beta_j|} < \infty$. Then $\left(\ell^1\left(\frac{1}{2^i}\right), T\right)$ exhibits the SBPOTP.

Based on the definitions, we have the following somewhat trivial relationship.

Theorem 2.5.6. Given a weighted backwards shift operator T acting on the space X, the shadowing property implies the SBPOTP, and the SBPOTP implies the UBPOTP.

CHAPTER THREE

Shadowing on Fréchet Spaces

3.1 Introduction

As menctioned earlier, much has been done in the realm of Banach spaces in the exploration of the shadowing property, such as [4] and [3]. For instance, in "Expansivity and Shadowing in Linear Dynamics", the authors provided work which linked the hyperbolicity of an operator to the presence of various forms of shadowing. A theorem due to the pair is presented below:

Theorem 3.1.1. [Bernardes and Messaoudi] Every invertible hyperbolic operator T on a Banach space X has the shadowing property, the limit shadowing property and the ℓ_p shadowing property for all $1 \leq p < \infty$.

In particular, with regards to the last chapter, the pair arrives at the following:

Theorem 3.1.2. [Bernardes and Messaoudi] Let $X = \ell_p(\mathbb{N})\,(1 \leq p < \infty)$ or $X = c_0(\mathbb{N})$, let $w = (w_n)_{n\in\mathbb{N}}$ be a bounded sequence of nonzero scalars and consider the unilateral weighted backward shift

$$B_w : (x_n)_{n\in\mathbb{N}} \in X \mapsto (w_{n+1}x_{n+1})_{n\in\mathbb{N}} \in X.$$

Then B_w has the positive shadowing property if and only if one of the following conditions holds:

1) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{\frac{1}{n}} < 1$, or

2) $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{\frac{1}{n}} > 1$.

It is important to notice that weighted backwards shifts are not invertible, so that the above theorem is not a corollary of the first theorem. From this second result, we have the following, which is a natural generalization of the above theorem based upon our previous work.

Theorem 3.1.3. Let X be a general sequence space, T a weighted shift operator on X such that $\sup_{j\in\omega} \beta_j < \infty$. Then T has the shadowing property if and only if one of the following holds:

1) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{\frac{1}{n}} < \frac{1}{2}$.

2) $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{\frac{1}{n}} > \frac{1}{2}$.

Proof. Let X be a sequence space, and let $\epsilon > 0$ be given. Let T a weighted shift acting on X with $\sup_{j\in\omega} \beta_j < \infty$. Write $X = \bigcup_{n\in I} G_{\overline{z_n}}$, the disjoint union of all distinct galaxies of X, where $\overline{z_n}$ is a representative of each galaxy and I is some indexing set. Notice that $G_{\overline{0}}$ has the shadowing property if and only if one the conditions of theorem holds, due to Theorem 3.1.0.2.

Let $\langle \overline{x_n} \rangle_{n=0}^{\infty}$ be a δ-pseudo-orbit, where δ is arbitrary. Now $x_0 \in G_{\overline{z_N}}$ for some N. Consider the δ-pseudo-orbit $\langle \overline{x_n} - T^n \overline{z_N} \rangle_{n=0}^{\infty}$. As $\overline{x_n} \in G_{T^n \overline{z_N}}$ for all $n \in \omega$, we have that $\langle \overline{x_n} - T^n \overline{z_N} \rangle_{n=0}^{\infty} \subseteq G_{\overline{0}}$. Let $\overline{y} \in G_{\overline{0}}$ ϵ-shadow the pseudo-orbit $\langle \overline{x_n} - T^n \overline{z_N} \rangle_{n=0}^{\infty}$. Notice the following:

$$d(\overline{x_n}, T^n (\overline{y} + \overline{z_N})) = d(\overline{x_n}, T^n \overline{y} + T^n \overline{z_N}) = d(\overline{x_n} - T^n \overline{z_N}, T^n \overline{y}) < \epsilon$$

Thus the point $\langle \overline{y} + \overline{z_N} \rangle_{n=0}^{\infty}$ ϵ-shadows the pseudo-orbit $\langle \overline{x_n} \rangle_{n=0}^{\infty}$. □

Notice that the criterion is changed from conditions one and two being less than or greater than 1, respectively, to being less than or greater that $\frac{1}{2}$. This is a consequence of us using the metric associated with the weighted space $\ell^1\left(\frac{1}{2^i}\right)$.

In this paper, extend this result to Fréchet spaces, which, as noted earlier, are generalizations of Banach spaces. To do this, we must use different notions of the spectrum of an operator. We conclude with results similar to the ones presented by Bernardes et al. using different notions of the spectrum of an operator, such as the Waelbroek and continuous spectrum.

3.2 Preliminaries

Bernardes et al. effectively solved the question of whether a linear operator acting on a Banach space exhibits the shadowing property, with the property being completely determined by the spectrum of the operator. As we move away from the setting of Banach spaces, we may lose the "nice" properties associated with the usual definition of the spectrum, such as the fact that the spectrum of a bounded linear operator acting on a Banach space is always a compact, nonempty set. For instance, there are operators acting on Fréchet spaces which have an empty spectrum, a spectrum that is not closed, and most certainly a spectrum which is unbounded. Perhaps as importantly, it isn't immediately clear if we retain nice decomposition theorems on these spaces, most of which relied heavily upon the definition of the spectrum. We define the spectrum of a bounded linear operator on a Banach space here for reference:

Definition 3.2.1. For an operator T on a Banach space X, we define the *resolvent of* T, $\rho(T)$, as the set of all $\lambda \in \mathbb{C}$ for which the operator $(\lambda I - T)^{-1}$ is a bounded linear operator. The *spectrum of* T, $\sigma(T)$, is defined as $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Given a bounded linear operator on a Banach space, its spectrum is always closed, bounded (and therefore compact), and nonempty in $\mathbb{C}$. For bounded linear operators, it is well known through a consequence of the Bounded Inverse Theorem that the spectrum consists precisely of the complex numbers λ for which $\lambda I - T$ is not bijective.

The above concerns lead to a couple of different notions of the spectrum of an operator acting on a Fréchet space. The first idea is to change what we require out of the operator $(\lambda I - T)^{-1}$ for λ to be included in $\rho(T)$. In *Spectral Radii of Bounded Operators on Topological Vector Spaces* [22], Troitsky defines multiple notions of the spectrum:

Definition 3.2.2. Let T be a linear operator on a topological vector space. For $\lambda \in \mathbb{C}$, we say that $\lambda \in \rho^l(T)$ if $\lambda I - T$ is invertible in the algebra of linear operators. We say that $\lambda \in \rho^c(T)$ if the inverse of $\lambda I - T$ is continuous. The spectral sets $\sigma^l(T)$ and $\sigma^c(T)$ are definied to be the complements of the reslovent sets $\rho^l(T)$ and $\rho^c(T)$, respectively.

It is worthy of note that this is different than the usual notion of the "continuous spectrum", which is normally defined as the set of all complex numbers for which the operator $\lambda I - T$ is injective and has dense range, yet is not surjective.

The other notion of the spectrum we will use is the *Waelbroeck spectrum*, denoted $\sigma_\omega(T)$. This spectrum is defined as follows:

Definition 3.2.3. Let $\hat{\mathbb{C}}$ be the one-point compactification of the complex plane. $\lambda \in \hat{\mathbb{C}}$ is an element of the resolvent set $\rho_\omega(T)$ if and only if there is a neighborhood V_λ of λ in $\hat{\mathbb{C}}$ such that there is a function $\mu \to R_\mu$ on $V_\lambda \cap \mathbb{C}$ to $\mathcal{L}(X)$ satisfying, for each $\mu \in V_\lambda \cap \mathbb{C}$, the conditions

1) $R_\mu(\mu I - T) = (\mu I - T)R_\mu = I$, and

2) $\{R_\mu : \mu \in V_\lambda \cap \mathbb{C}\}$ is bounded in $\mathcal{L}(X)$.

The *Waelbroeck spectrum of* T, $\sigma_\omega(T)$ is defined as $\sigma_\omega(T) = \hat{\mathbb{C}} \setminus \rho_\omega(T)$.

On a Fréchet space, there are examples which show that these notions of spectra are not necessarily equal, as noted in [19]. The usual notion of the spectrum and the Waelbroeck spectrum are equal on a certain classes of operators, such as completely bounded operators and compact operators. In fact, Maeda [18] shows that for operators in the algebra of the identity operator with completely bounded operators, a subalgebra of the linear operators on a locally convex space, the usual notion of the spectrum and the Waelbroeck spectrum are equal and are even compact subsets of $\mathbb{C}$.

The definition of the Waelbroeck spectrum has the advantage that it is a closed set which contains the usual spectrum of an operator as noted in [6]. Moreover, if X is a Banach space, then $\sigma(T) = \sigma_\omega(T)$.

As we will be concerned with where the different notions of the spectra are contained within the complex plane, we define $\mathbb{D} \subset \mathbb{C} = \{z \in \mathbb{C} \colon \|z\| < 1\}$ and $\mathbb{T} \subset \mathbb{C} = \{z \in \mathbb{C} \colon \|z\| = 1\}$.

Lastly, as defining a single norm on the operator, such as for bounded linear operators on Banach spaces, no longer makes sense, we define the following, due to [22]:

Definition 3.2.4. Let T be an operator on a seminormed vector space (X, p). As in the case with normed spaces, p generates an *operator seminorm* $p(T)$ defined by

$$p(T) = \sup_{p(x) \neq 0} \frac{p(Tx)}{p(x)}.$$

More generally, let $T \colon X \to Y$ be a linear operator between two seminormed spaces (X, p) and (Y, q). Then we define a mixed operator seminorm associated with p and q via

$$\mathfrak{m}_{p,q}(T) = \sup_{p(x) \neq 0} \frac{q\,(Tx)}{p(x)}.$$

Troitsky notes that this is just "a measure of how far in the sminorm q the points of the p-unit ball can go under T." [22]. This may mean that $\mathfrak{m}_{pq}(T)$ is infinite, as T may not be a bounded operator. Troitsky also mentions how the mixed operator seminorm may be used to give a criterion upon when an operator is continuous.

Proposition 3.2.5. [Troitsky] Let T be an operator from Fréchet spaces X to Y, and let P and Q denote the generating families of seminorms, respectively. Then T is continuous if an only if for every $q \in Q$, there exists $p \in P$ such that $\mathfrak{m}_{p,q}(T)$ is finite.

In his paper, Troitsky defines the spectral radii for each of his different spectra. This is defined to be the distance for which the radius of a neighborhood about zero must be larger than in order to capture all elements of the spectrum. For the continuous spectrum of an operator T on a space X with generating family of seminorms P, Troitsky shows that the spectral radius, $r_c(T)$, is given by

$$r_c(T) = \sup_{q \in P} \inf_{p \in P} \limsup_{n \to \infty} \sqrt[n]{\mathfrak{m}_{p,q}(T^n)}.$$

3.3 Shadowing on Fréchet Spaces

Lemma 3.3.1. If T is a linear operator acting on a Fréchet space X such that $\sigma_\omega(T) \subseteq \mathbb{D}$, then there exists constants $t \in (0,1)$ and $C \geq 1$ such that $\mathfrak{m}_{p_n,p_m}(T^k) < C \cdot t^k$.

Proof. First, assume that $\sigma_\omega(T) \subseteq \mathbb{D}$ and let P be the generating family of seminorms on the space. Then $\overline{\sigma^c(T)} \subseteq \mathbb{D}$ as well. Then by the spectral radius formula for operators on Fréchet spaces, we have that

$$\sup_{q \in P} \inf_{p \in P} \limsup_{n \to \infty} \sqrt[n]{\sup_{p(x)=1} q(T^n x)} < 1.$$

Therefore, there exists a $t < 1$ such that

$$\sup_{q \in P} \inf_{p \in P} \limsup_{n \to \infty} \sqrt[n]{\sup_{p(x)=1} q(T^n x)} < t.$$

Noting that the generating family of seminorms is taken to be increasing, that is $p_n(x) \leq p_m(x)$ for all $x \in X, n < m$, we have that this is equivalent to

$$\lim_{m \to \infty} \lim_{n \to \infty} \limsup_{k \to \infty} \sqrt[k]{\sup_{p_n(x)=1} p_m(T^k x)} < t.$$

Therefore, there exists some $M \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \limsup_{k \to \infty} \sqrt[k]{\sup_{p_n(x)=1} p_m(T^k x)} < t$$

for all $m \geq M$. Now we also must have that there exists an $N \in \mathbb{N}$ such that

$$\limsup_{k\to\infty} \sqrt[k]{\sup_{p_n(x)=1} p_m(T^k x)} < t$$

for all $m \geq M, n \geq N$. Similarly, there must exist a $K \in \mathbb{N}$ such that

$$\sqrt[k]{\sup_{p_n(x)=1} p_m(T^k x)} < t$$

for all $m \geq M, n \geq N, k \geq K$. Therefore, we must have that

$$\sup_{p_n(x)=1} p_m(T^k x) < t^k$$

for all $m \geq M, n \geq N, k \geq K$. Therefore, there exists a $C \in \mathbb{N}$ such that

$$\mathfrak{m}_{p_n,p_m}(T^k) = \sup_{p_n(x)=1} p_m(T^k x) < Ct^k$$

for all $m, n, k \in \omega$. $\square$

Theorem 3.3.2. Let T be an operator acting on a Fréchet space X. Suppose that $\sigma_\omega(T) \subset \mathbb{D}$. Then T exhibits the shadowing property, as well as the limit shadowing property.

Proof. Let $\epsilon > 0$ be given. There exists an $L \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} \frac{1}{2^i} < \sum_{i=0}^{L} \frac{1}{2^i} + \frac{\epsilon}{2}$. Now as T is a contraction, there exists a constant $C \in \mathbb{N}$ and a $t \in (0,1)$ such that $p_i(T)^n < C \cdot t^n$ for all $n \in \omega$. Let $\delta = \frac{(1-t)\epsilon}{4C \cdot 2^L}$, and let $\langle x_n \rangle_{n\in\omega}$ be a δ-pseudo-orbit. Notice then that

$$d(T^n x_0, x_n) = \sum_{i=0}^{\infty} \frac{1}{2^i} \min\left(1, p_i(T^n x_0 - x_n)\right)$$

$$
\begin{aligned}
&< \sum_{i=0}^{L} \frac{1}{2^i} p_i(T^n x_0 - x_n) + \frac{\epsilon}{2} \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} \big(p_i(T^n x_0 - T^{n-1}x_1) + p_i(T^{n-1}x_1 - T^{n-2}x_2) + \dots \\
&\qquad\qquad\qquad \dots + p_i(Tx_{n-1} - x_n)\big) + \frac{\epsilon}{2} \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} \big(p_i(T)^{n-1} p_i(Tx_0 - x_1) + p_i(T)^{n-2} p_i(Tx_1 - x_2) + \dots \\
&\qquad\qquad\qquad \dots + p_i(Tx_{n-1} - x_n)\big) + \frac{\epsilon}{2} \\
&= \sum_{i=0}^{L} \frac{1}{2^i} \left(\sum_{k=0}^{n-1} p_i(T)^{n-1-k} p_i(Tx_k - x_{k+1}) \right) + \frac{\epsilon}{2} \\
&< \sum_{i=0}^{L} \frac{1}{2^i} \left(\sum_{k=0}^{n-1} C \cdot t^{n-1-k} p_i(Tx_k - x_{k+1}) \right) + \frac{\epsilon}{2}
\end{aligned}
$$

Now since $d(Tx_k, x_{k+1}) < \delta = \frac{(1-t)\epsilon}{4C \cdot 2^L}$, we must have that

$$
p_i(Tx_k - x_{k+1}) < \frac{(1-t)\epsilon}{4C}.
$$

Therefore, we have that

$$
\begin{aligned}
d(T^n x_0, x_n) &< \sum_{i=0}^{L} \frac{1}{2^i} \left(\sum_{k=0}^{n-1} C \cdot t^{n-1-k} \frac{(1-t)\epsilon}{4C} \right) + \frac{\epsilon}{2} \\
&= \sum_{i=0}^{L} \frac{1}{2^i} \left(\frac{(1-t)\epsilon}{4} \sum_{k=0}^{n-1} t^{n-1-k} \right) + \frac{\epsilon}{2} \\
&< \sum_{i=0}^{L} \frac{1}{2^i} \left(\frac{(1-t)\epsilon}{4} \sum_{k=0}^{\infty} t^{k} \right) + \frac{\epsilon}{2} \\
&= \sum_{i=0}^{L} \frac{1}{2^i} \left(\frac{(1-t)\epsilon}{4} \cdot \frac{1}{1-t} \right) + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{4} \sum_{i=0}^{L} \frac{1}{2^i} + \frac{\epsilon}{2}
\end{aligned}
$$

$$< \epsilon$$

Therefore, we have shown that (X, T) exhibits the shadowing property.

To see that (X, T) exhibits the limit shadowing property, let $\langle x_n \rangle_{n\in\omega} \subseteq X$ be such that $d(Tx_n, x_{n+1}) \to 0$ as $n \to \infty$. Let $\epsilon > 0$ be given. We show that for this ϵ, there exists some $N \in \mathbb{N}$ such that for all $n > N$, $d(T^n x_0, x_n) < \epsilon$. To this end, notice the following for all $j < n$:

$$\begin{aligned}
d(T^n x_0, x_n) &= \sum_{i=0}^{\infty} \frac{1}{2^i} \min\Big(1, p_i(T^n x_0 - x_n)\Big) \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} \min\Big(1, p_i(T^n x_0 - x_n)\Big) + \frac{\epsilon}{2} \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} p_i(T^n x_0 - x_n) + \frac{\epsilon}{2} \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} \Big(p_i(T^n x_0 - T^{n-1} x_1) + p_i(T^{n-1} x_1 - T^{n-2} x_2) + \ldots \\
&\qquad\qquad \ldots + p_i(Tx_{n-1} - x_n)\Big) + \frac{\epsilon}{2} \\
&\leq \sum_{i=0}^{L} \frac{1}{2^i} \Big(p_i(T)^{n-1} p_i(Tx_0 - x_1) + p_i(T)^{n-2} p_i(Tx_1 - x_2) + \ldots \\
&\qquad\qquad \ldots + p_i(Tx_{n-1} - x_n)\Big) + \frac{\epsilon}{2} \\
&= \sum_{i=0}^{L} \frac{1}{2^i} \left(\sum_{k=0}^{n-1} p_i(T)^k p_i(Tx_{n-k-1} - x_{n-k}) \right) + \frac{\epsilon}{2} \\
&< \sum_{i=0}^{L} \frac{1}{2^i} \left(\sum_{k=0}^{n-1} C \cdot t^k p_i(Tx_{n-k-1} - x_{n-k}) \right) + \frac{\epsilon}{2} \\
&= \sum_{i=0}^{L} \frac{C}{2^i} \left(\sum_{k=0}^{j} (t^k \cdot p_i(Tx_{n-k-1} - x_{n-k}) \right. \\
&\qquad\qquad \left. + \sum_{k=j+1}^{n} (t^k \cdot p_i(Tx_{n-k-1} - x_{n-k})) \right) + \frac{\epsilon}{2}
\end{aligned}$$

$$\leq \sum_{i=0}^{L} \frac{C}{2^i} \left(\sup_{0 \leq k \leq j} p_i(Tx_{n-k-1} - x_{n-k}) \cdot \sum_{k=0}^{j} t^k \right.$$
$$\left. + \sup_{k \in \omega} p_i(Tx_{n-k-1} - x_{n-k}) \cdot \sum_{k=j+1}^{n} t^k \right) + \frac{\epsilon}{2}$$

Let $M_i = \max\left(\frac{1}{1-t}, \sup_{k\in\omega} p_i(Tx_k - x_{k+1})\right)$. Fix $j_i \in \omega$ such that $\sum_{k=j_i+1}^{\infty} t^k < \frac{\epsilon}{8CM_i}$. Clearly as i increases, so does M_i and hence j_i. Now since $p_i(Tx_l - x_{l+1}) \to 0$ as $l \to \infty$, there exists some N_i such that if $n \geq N_i$, then $p_i(Tx_n - x_{n+1}) < \frac{\epsilon}{8CM_i}$. Let $N = \max(N_0, \ldots, N_L)$. But then, if $n > N + j_i + 1$,

$$\begin{aligned}
&\sup_{0 \leq k \leq j_i} p_i(Tx_{n-k-1} - x_{n-k}) \cdot \sum_{k=0}^{j_i} t^k + \sup_{k\in\omega} p_i(Tx_k - x_{k+1}) \cdot \sum_{k=j_i+1}^{n} t^k \\
&\quad < \frac{\epsilon}{8CM_i} \cdot \sum_{k=0}^{j_i} t^k + \sup_{k\in\omega} p_i(Tx_k - x_{k+1}) \cdot \frac{\epsilon}{8CM_i} \\
&\quad \leq \frac{\epsilon}{8CM_i} \cdot 2M_i \\
&\quad < \frac{\epsilon}{4C}
\end{aligned}$$

If we let $j = \max(j_0, \ldots, j_L)$, by the above note, we must have that if $n > N+j+1$, then $d(T^n x_0, x_n) < \epsilon$. As ϵ was arbitrary, this shows that

$$\lim_{n\to\infty} d(T^n x_0, x_n) = 0.$$

This shows that (X, T) exhibits the limit shadowing property.

$\square$

Now when working on a compact space, the shadowing property is equivalent to the ability to shadow arbitrarily long pseudo-orbits. We define this property here, and then provide a proof of the aforementioned result.

Definition 3.3.3. A system (X, T) is said to exhibit *shadowing of arbitrarily long pseudo-orbits* if for all $\epsilon > 0$, there exists $\delta > 0$ such that every finite δ-pseudo-orbit $\langle x_n \rangle$ in X is ϵ-shadowed by some point $z \in X$.

Theorem 3.3.4. Let X be a compact space and $T \colon X \to X$ be an operator. Then (X, T) exhibits the shadowing property if and only if (X, T) exhibits shadowing of arbitrarily long pseudo-orbits.

Proof. Let (X, T) be as above, and let $\epsilon > 0$ be given. If (X, T) exhibits the shadowing property, then there exists a δ such that any δ-pseudo-orbit $\langle x_i \rangle_{i \in \omega}$ is ϵ-shadowed by some point $z \in X$. Notice that if given an arbitrarily long δ-pseudo-orbit $\langle x_i \rangle_{i=0}^{n}$, one can extend this to an infinite δ-pseudo-orbit $\langle y_i \rangle_{i \in \omega}$ defined by:

$$y_i = \begin{cases} x_i & i \leq n \\ T^{i-n}(x_n) & i > n \end{cases}$$

Thus $\langle y_i \rangle_{i \in \omega}$ is a δ-pseudo-orbit, and hence there exists a point $z \in X$ that ϵ-shadows it. But then z ϵ-shadows $\langle x_i \rangle_{i=0}^{n}$. Therefore, as this is true for any $n \in \omega$, (X, T) exhibits shadowing of arbitrarily long pseudo-orbits.

Now suppose that (X, T) exhibits shadowing of arbitrarily long pseudo-orbits, and let $\epsilon > 0$ be given. Take $\delta > 0$ from $\frac{\epsilon}{2}$-arbitrarily-long shadowing.

Let $\langle x_i \rangle_{i\in\omega}$ be a δ-pseudo-orbit. For each $n \geq 1$, let $z_n \in X$ be the point that $\frac{\epsilon}{2}$-shadows $\langle x_i \rangle_{i=0}^{n}$. We have that $z_n \in B_{\frac{\epsilon}{2}}(x_0)$ for all $n \in \omega$. As $\overline{B_{\frac{\epsilon}{2}}(x_0)}$ is compact, there is a subsequence $\left\langle z_{n_j} \right\rangle_{j\in\omega}$ of $\langle z_n \rangle_{n\in\omega}$ that converges to a point $z \in B_{\frac{\epsilon}{2}}(x_0)$.

As $T^i(z_n) \in B_{\frac{\epsilon}{2}}(x_i)$ for all $n \geq 1$ and $i \leq n$, and by continuity of T^i for all $i \in \mathbb{N}$, we must have that $T^i(z) \in \overline{B_{\frac{\epsilon}{2}}(x_i)} \subset B_\epsilon(x_i)$ for all $i \in \omega$. But then z ϵ-shadows $\langle x_i \rangle_{i\in\omega}$. $\square$

As we move to the noncompact setting, we unfortunately lose this equivalence. However this property is still worth exploring, and we do so in the following.

Theorem 3.3.5. Let T be an operator acting on a Fréchet space X. Suppose that $\sigma_\omega(T) \subset \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Then T exhibits shadowing of arbitrarily long pseudo-orbits.

Proof. Let $\epsilon > 0$ be given. Notice that since $0 \notin \sigma_\omega(T)$, we must have that T is invertible, since $(T - 0 \cdot I)^{-1} = T^{-1}$ exists and is continuous.

As $\sigma_\omega(T) \subset \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, we must have that $\sigma_\omega(T^{-1}) \subset \mathbb{D}$. Therefore, by previous theorem, T^{-1} exhibits the shadowing property. Let δ be given from the definition of $\frac{\epsilon}{C\cdot 2^{L+2}}$-shadowing on T^{-1}, where C is the constant from the first lemma and L is a constant such that $\sum_{i=0}^{\infty} \frac{1}{2^i} < \sum_{i=0}^{L} \frac{1}{2^i} + \frac{\epsilon}{2}$. Let $\langle x_n \rangle_{n\in\omega}$ be a T δ-pseudo-orbit. Let $N \in \mathbb{N}$. Define $\left\langle y_n^N \right\rangle_{n\in\omega}$ as follows:

$$
\begin{aligned}
y_0^N &= Tx_N & T^{-1}y_0^N &= x_N \\
y_1^N &= Tx_{N-1} & T^{-1}y_1^N &= x_{N-1} \\
&\vdots \\
y_N^N &= Tx_0 & T^{-1}y_N^N &= x_0
\end{aligned}
$$

$$y^N_{N+1} = x_0 \qquad\qquad T^{-1}y^N_{N+1} = T^{-1}x_0$$

$$y^N_{N+2} = T^{-1}x_0 \qquad\qquad T^{-1}y^N_{N+2} = T^{-2}x_0$$

$$\vdots$$

In general,

$$y^N_i = \begin{cases} Tx_{N-i} & i \leq N \\ T^{-i+N+1}x_0 & i > N \end{cases}$$

Notice that by definition, we have that $d(T^{-1}y^N_n, y^N_{n+1}) < \delta$ for all $n \in \omega$. Therefore, there must exist some $z^N \in X$ such that $d(T^{-n}z^N, y^N_n) < \frac{\epsilon}{C \cdot 2^{L+2}}$ for all $n \in \omega$. Let $x^N = T^{-(N+1)}z^N$. For $m \leq N$, notice the following:

$$\begin{aligned} d(T^m x^N, x_m) &= d(T^m(T^{-(N+1)}z^N), x_m) = d(T^{m-N-1}z^N, x_m) \\ &= \sum_{i=0}^{\infty} \frac{1}{2^i} \min\left(1, p_i(T^{m-N-1}z^N - x_m)\right) \\ &= \sum_{i=0}^{\infty} \frac{1}{2^i} \min\left(1, p_i(T^{-1}(T^{m-N}z^N - Tx_m))\right) \\ &\leq \sum_{i=0}^{L} \frac{1}{2^i} \min\left(1, p_i(T^{-1})p_i(T^{m-N}z^N - Tx_m))\right) + \frac{\epsilon}{2} \\ &< \sum_{i=0}^{L} \frac{1}{2^i} \min\left(1, Ct \cdot p_i(T^{m-N}z^N - Tx_m))\right) + \frac{\epsilon}{2} \end{aligned}$$

Defining $k = N - m$, so that $m = N - k$, we see that

$$d(T^m x^N, x_m) < \sum_{i=0}^{L} \frac{1}{2^i} \min\left(1, Ct \cdot p_i(T^{-k}z^N - Tx_{N-k}))\right) + \frac{\epsilon}{2}$$

$$< \sum_{i=0}^{L} \frac{1}{2^i} \min\left(1, C \cdot p_i(T^{-k}z^N - y_k^N))\right) + \frac{\epsilon}{2}$$

Now since $d(T^{-n}z^N, y_n^N) < \frac{\epsilon}{C \cdot 2^{L+2}}$ for all $n \in \omega$, we must have that $C \cdot p_i(T^{-k}z^N - y_k^N) < \frac{\epsilon}{4}$ for $k \leq L$. But then:

$$\begin{aligned} d(T^m x^N, x_m) &< \sum_{i=0}^{L} \frac{1}{2^i} \min\left(1, \frac{\epsilon}{4}\right) + \frac{\epsilon}{2} \\ &= \frac{\epsilon}{4} \sum_{i=0}^{L} \frac{1}{2^i} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

□

Unfortunately proofs regarding the shadowing property for the case above on Banach spaces relied heavily upon the linearity of the norm on the Banach space. When expanding to a Fréchet space, the F-norm associated with the space is not linear, and hence such a proof strategy was unavailable to the author.

The last theorem is a result of the following proposition, which can be found in [23].

Proposition 3.3.6. [Vasilescu] Let X be a Fréchet space and $T\colon X \to X$ a linear operator. Suppose that $\sigma_\omega(T) = F_1 \cup F_2$, where F_1 and F_2 are closed and disjoint subsets of $\hat{\mathbb{C}}$, the one-point compactification of the complex plane. Then the space X is a direct sum of two closed subspaces X_1 and X_2 with the following properties:

1) $T\left(\mathrm{dom}\left(T\big|_{X_j}\right)\right) \subset X_j$ for $j = 1, 2$.

2) $\sigma_\omega\left(T\big|_{X_j}\right) = F_j$ for $j = 1, 2$.

Proposition 3.3.6 allows us decompose (X, T) into two systems. We will use this theorem to consider the case in which the spectrum of T acting on the first subspace is contained within the unit circle while and the spectrum of T acting on the second subspace lies outside the closure of the unit circle.

Theorem 3.3.7. Let T be an operator acting on a Fréchet space X. Suppose that $\sigma_\omega(T) \cap \mathbb{T} = \emptyset$. Then (X, T) exhibits shadowing of arbitrarily long pseudo-orbits.

Proof. Let $\sigma_1 = \sigma_\omega(T) \cap \mathbb{D}$ and $\sigma_2 = \sigma_\omega(T) \cap \left(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}\right)$. Now these sets form a partition of $\sigma_\omega(T)$ into two nonempty closed sets. Therefore, by the above decomposition, there are T-invariant closed subspaces X_1 and X_2 of X such that

$$X = X_1 \oplus X_2 \qquad \sigma_\omega\left(T\big|_{X_1}\right) = \sigma_1 \qquad \sigma_\omega\left(T\big|_{X_2}\right) = \sigma_2$$

By Theorem 3.3.2, we have that $\left(X_1, T\big|_{X_1}\right)$ exhibits the shadowing property, while by Theorem 3.3.5, $\left(X_2, T\big|_{X_2}\right)$ exhibits shadowing of arbitrarily long pseudo-orbits.

Let $\epsilon > 0$ and $N > 0$ be given, and let $\delta = \min\{\delta_1, \delta_2\}$ where δ_1 comes from $\frac{\epsilon}{2}$-shadowing on X_1 and δ_2 comes from $\frac{\epsilon}{2}$-shadowing of N-long pseudo-orbits on X_2. Let $\langle x_i \rangle_{i=0}^N$ be a N-long δ-pseudo-orbit. As X can be decomposed into $X_1 \oplus X_2$, write $\langle x_i \rangle_{i=0}^N = \langle x_{i_1} + x_{i_2} \rangle_{i=0}^N$.

Now as $T\left(\operatorname{dom}\left(T\big|_{X_j}\right)\right) \subset X_j$ for $j = 1, 2$, we have that $\langle x_{i_j} \rangle_{i=0}^N$ is completely contained in X_1 for $j = 1$ and X_2 for $j = 2$. Therefore there points $z_1 \in X_1$ and $z_2 \in X_2$ that z $\frac{\epsilon}{2}$-shadow $\langle x_{i_1} \rangle_{i=0}^N$ and $\langle x_{i_2} \rangle_{i=0}^N$, respectively. But then the point $z = z_1 + z_2$ ϵ-shadows $\langle x_i \rangle_{i=0}^N$ by the triangle inequality. Therefore, (X, T) exhibits shadowing of arbitrarily long pseudo-orbits. $\square$

Note that it is not immediately obvious as to whether this result holds for $\sigma^c(T)$. The heart of the result comes from the ability to decompose the space into two invariant subspaces, which has been done classically using the Waelbroeck spectrum. While there are larger classes for which a similar decomposition theorems exist (see [2]), hinting at a larger class of operators for which the above result holds, work still may be done to hopefully show that the result is true when $\sigma^c(T) \cap \mathbb{T} = \emptyset$.

CHAPTER FOUR

Other Notions of Shadowing

Now we seek to generalize the shadowing property into other variants of shadowing. As mentioned in the first chapter, there are many variants of the shadowing property. In this chapter, we develop a new notion of shadowing in noncompact spaces. Lastly, we come back to the spaces mentioned in Chapter Two and present a result regarding the average shadowing property and pseudo-orbits of weighted shifts.

4.1 Non-Uniform Pseudo-Orbit Tracing Properties

We begin by defining the non-uniform pseudo-orbit tracing property.

Definition 4.1.1. Let (X, T) be a metric space, and let $\delta_\epsilon : X \to (0, 1]$ be a (not necessarily continuous) function. We call $\langle x_i \rangle_{i\in\omega}$ a *δ_ϵ-non-uniform-pseudo-orbit* if $d\left(T(x_i), x_{i+1}\right) < \delta_\epsilon(x_i)$ for all $i \in \omega$.

A system (X, T) is said to have the *non-uniform pseudo-orbit tracing property (NUPOTP)* if for all $\epsilon > 0$, there exists a function $\delta_\epsilon : X \to (0, 1]$ such that for every δ_ϵ-non-uniform pseudo-orbit $\langle x_i \rangle_{i\in\omega}$, there exists a point $z \in X$ such that $d(T^i(z), x_i) < \epsilon$ for all $i \in \omega$.

This property allows consideration of pseudo-orbits with different error tolerances based upon where each point is located within the space as opposed to a single error tolerance. For instance, perhaps it is necessary that near the boundary of a manifold, pseudo-orbits must be more precise, whereas far from the boundary such precision is no longer needed.

Example 4.1.2. Consider again the space $Y = \{2^n : n \in \mathbb{Z}\}$ and $g(y) = 2y$. As we have discussed earier in Example 1.4.5, (Y, g) does not exhibit the shadowing property. It does, however, exhibit the NUPOTP. For any $\epsilon > 0$, we define $\delta_\epsilon(2^n) = \frac{1}{2^n}$. By this definition, any δ_ϵ-pseudo-orbit $\langle y_i \rangle_{i\in\omega}$ is a true orbit, as

$$\left\{ y \in Y : d\left(g\left(2^n\right), y\right) < \frac{1}{2^n} \right\} = \left\{g\left(2^n\right)\right\}.$$

. Therefore the point y_0 shadows $\langle y_i \rangle_{i\in\omega}$ trivially.

Now, we want to ensure that this definition is relatable to the shadowing property. Note that the shadowing property is achieved if each function δ_ϵ has a positive lower bound.

Proposition 4.1.3. If (X, T) exhibits the NUPOTP, with $\langle \delta_\epsilon \rangle_{\epsilon>0}$ a family of functions witnessing it, and we have that $\inf_{x\in X} \delta_\epsilon(x) > 0$ for all $\epsilon > 0$, then (X, T) exhibits the shadowing property.

Proof. Let $\epsilon > 0$ be given. Take $\delta = \inf_{x\in X} \delta_\epsilon(x) > 0$, and let $\langle x_n \rangle_{n\in\omega}$ be a δ-pseudo-orbit. Notice that $d(Tx_n, x_{n+1}) < \delta \leq \delta_\epsilon(x_n)$, so that $\langle x_n \rangle_{n\in\omega}$ is a δ_ϵ-pseudo-orbit. Therefore, by the NUPOTP, there exists a point $z \in X$ that ϵ-shadows $\langle x_n \rangle_{n\in\omega}$. Thus (X, T) also exhibits the shadowing property. □

As a corollary to the above, we have the following.

Corollary 4.1.4. If X is compact and δ_ϵ can be chosen to be continuous for all $\epsilon > 0$, then (X, T) witnesses the shadowing property.

Proof. Let $\epsilon > 0$ be given. As the continuous image of a compact set is compact, we have that $\delta_\epsilon(X)$ is a compact subset of $(0, 1]$. Therefore there exists some point $a \in (0, 1]$ such that $0 < a < \inf_{x \in X} \delta_\epsilon(x)$. Thus, by 4.1.3, (X, T) witnesses the shadowing property. □

Within the definition of the NUPOTP, there was room to be more general and more specific with our requirements on the δ_ϵ functions. For instance, one may wish to have all of the functions be continuous so that no brash jumps in error tolerance requirements occurs. Alternatively, there may be points in the space where no error may be permitted, requiring that δ_ϵ at those points be equal to zero. We now define variants of the NUPOTP describing these two situations.

Definition 4.1.5. A system (X, T) is said to have the *continuous non-uniform pseudo-orbit tracing property (CNUPOTP)* if for all $\epsilon > 0$, there exists a *continuous* function $\delta_\epsilon : X \to (0, 1]$ such that for every δ_ϵ-non-uniform pseudo-orbit $\langle x_i \rangle_{i \in \omega}$, there exists a point $z \in X$ such that $d(T^i(z), x_i) < \epsilon$ for all $i \in \omega$.

Definition 4.1.6. A system (X, T) is said to have the *non-uniform pseudo-orbit tracing property* (NUPOTP*)* if for all $\epsilon > 0$, there exists a function $\delta_\epsilon : X \to [0, 1]$ such that for every δ_ϵ-non-uniform pseudo-orbit $\langle x_i \rangle_{i \in \omega}$, there exists a point $z \in X$ such that $d(T^i(z), x_i) < \epsilon$ for all $i \in \omega$.

Notice that the CNUPOTP is stronger than the NUPOTP, which in turn is stronger than the NUPOTP*. We will now examine the relationship between the shadowing property, CNUPOTP, the NUPOTP, and the NUPOTP* properties. In this pursuit, we have the following.

Proposition 4.1.7. Let (X, T) be a dynamical system.

1) If (X, T) exhibits the shadowing property, then (X, T) exhibits the CNUPOTP.

2) If (X, T) exhibits the CNUPOTP, then (X, T) exhibits the NUPOTP.

3) If (X, T) exhibits the NUPOTP, then (X, T) exhibits the NUPOTP*.

Proof. Let $\epsilon > 0$ be given.

If (X, T) exhibits the shadowing property, then there exists $\delta > 0$ witnessing ϵ-shadowing. Define $\delta_\epsilon = \delta$. As δ_ϵ is constant (and hence continuous) and any δ_ϵ-pseudo-orbit is ϵ-shadowed, then (X, T) exhibits the CNUPOTP.

If (X, T) exhibits the CNUPOTP, then the same δ_ϵ witnessing the CNUPOTP satisfies the requirements of the NUPOTP. If, lastly, (X, T) exhibits the NUPOTP, then the same δ_ϵ witnessing the NUPOTP satisfies the requirements of the NUPOTP*.

□

We now show some examples of spaces where some of the above properties hold while others do not. Conveniently, all of our examples will revolve around the double tent map $T : \mathbb{R} \to \mathbb{R}$, which is given by

$$T(x) = \begin{cases} -\frac{1+\sqrt{5}}{2}(x+2) & \text{if } x \leq -1 \\ \frac{1+\sqrt{5}}{2}x & \text{if } -1 \leq x \leq 1 \\ -\frac{1+\sqrt{5}}{2}(x-2) & \text{if } x \geq 1 \end{cases}$$

The following counterexamples will arise from restricting the domain of T to particular subsets of the real line.

Example 4.1.8. Let $X = \left[-\frac{1+\sqrt{5}}{2}, 0\right) \cup \left(0, \frac{1+\sqrt{5}}{2}\right]$, $T : X \to X$ given above. Then (X, T) exhibits the CNUPOTP (and hence the NUPOTP and the NUPOTP*), but does not witness the shadowing property.

Proof. Notice that each closed interval $\left[-\frac{1+\sqrt{5}}{2}, 0\right]$ and $\left[0, \frac{1+\sqrt{5}}{2}\right]$ exhibits the shadowing property by [11]. Therefore, if given an $\epsilon > 0$, one may define continuous functions $\delta_\epsilon^\pm$, where $\delta_\epsilon^- : \left[-\frac{1+\sqrt{5}}{2}, 0\right] \to (0,1]$ and $\delta_\epsilon^\pm$, where $\delta_\epsilon^+ : \left[0, \frac{1+\sqrt{5}}{2}\right] \to (0,1]$, that witness the CNUPOTP by Proposition 4.1.7. But then, the continuous function

$$\delta_\epsilon(x) = \begin{cases} \frac{\frac{1+\sqrt{5}}{2}-1}{2} \cdot \delta_\epsilon^-(x) & \text{if } x \leq -1 \\ \frac{\frac{1+\sqrt{5}}{2}-1}{2}|x| \cdot \delta_\epsilon^-(x) & \text{if } -1 \leq x < 0 \\ \frac{\frac{1+\sqrt{5}}{2}-1}{2}|x| \cdot \delta_\epsilon^+(x) & \text{if } 0 < x \leq 1 \\ \frac{\frac{1+\sqrt{5}}{2}-1}{2} \cdot \delta_\epsilon^+(x) & \text{if } x \geq 1 \end{cases},$$

will satisfy the requirements of the CNUPOTP on (X, T).

To show this, let $\epsilon > 0$ be given with $\epsilon < \frac{1}{2}$, and let $\langle x_n \rangle_{n\in\omega}$ be a δ_ϵ-non-uniform pseudo-orbit. We claim that $\langle x_n \rangle_{n\in\omega}$ is entirely contained in either $\left[-\frac{1+\sqrt{5}}{2}, 0\right)$ or $\left(0, \frac{1+\sqrt{5}}{2}\right]$.

To this end, notice that for $n \in \omega$, we have that $d\left(Tx_n, x_{n+1}\right) < \delta_\epsilon(x_n)$. If $x_n \in \left[-\frac{1+\sqrt{5}}{2}, -1\right]$ or $x_n \in \left[1, \frac{1+\sqrt{5}}{2}\right]$, then as $\delta_\epsilon^{\pm}(x_n) < 1$, we must have that $x_{n+1} \in \left[-\frac{1+\sqrt{5}}{2}, 0\right)$ or $x_{n+1} \in \left(0, \frac{1+\sqrt{5}}{2}\right]$, respectively. If $x_n \in (-1, 0)$, then $d\left(Tx_n, x_{n+1}\right) < |x_n|$, so that $x_{n+1} \in \left[-\frac{1+\sqrt{5}}{2}, 0\right)$, as $Tx_n = \frac{1+\sqrt{5}}{2}x_n < x_n$. A similar argument shows that if $x_n \in (0, 1)$, then $x_{n+1} \in \left(0, \frac{1+\sqrt{5}}{2}\right]$. Therefore, for all $n \in \omega$, if $x_n \in \left[-\frac{1+\sqrt{5}}{2}, 0\right)$ or if $x_n \in \left(0, \frac{1+\sqrt{5}}{2}\right]$, then so is x_{n+1}, respectively. Therefore, any δ_ϵ-non-uniform pseudo-orbit is necessarily contained in either $\left[-\frac{1+\sqrt{5}}{2}, 0\right)$ or $\left(0, \frac{1+\sqrt{5}}{2}\right]$.

Assume that $\langle x_n \rangle_{n\in\omega}$ is completely contained in $\left[-\frac{1+\sqrt{5}}{2}, 0\right)$. As $\delta_\epsilon(x_n) \leq \delta_\epsilon^-(x_n)$, $\langle x_n \rangle_{n\in\omega}$ is a δ_ϵ^--non-uniform pseudo-orbit. Therefore, there exists some $z \in \left[-\frac{1+\sqrt{5}}{2}, 0\right]$ that ϵ-shadows it.

We show that $z \neq 0$. Notice that if $x_n \in (-1, 0)$, then $x_{n+1} < x_n$. This is due to the fact that

$$d(Tx_n, x_n) = x_n - \frac{1+\sqrt{5}}{2}x_n = \left(1 - \frac{1+\sqrt{5}}{2}\right)x_n = \left(\frac{1+\sqrt{5}}{2} - 1\right)|x_n|,$$

so that since $d(Tx_n, x_{n+1}) < \delta_\epsilon(x_n) < \frac{\frac{1+\sqrt{5}}{2}-1}{2}|x_n| = \frac{1}{2} \cdot d(Tx_n, x_n)$, we have that $x_{n+1} < x_n$. Due to this, there exists some number $N \in \mathbb{N}$ such that $x_N \leq -1$. But then $d(x_N, 0) \geq 1 > \epsilon$, so that $z \neq 0$. Therefore $z \in \left[-\frac{1+\sqrt{5}}{2}, 0\right)$.

A similar argument can be constructed if $\langle x_n \rangle_{n\in\omega}$ is contained in $\left(0, \frac{1+\sqrt{5}}{2}\right]$. Therefore (X, T) exhibits the CNUPOTP.

Despite this, for given $\epsilon > 0$, no fixed δ can be picked that will satisfy the requirements of the shadowing property. This is due to the fact that any positive δ would permit points close to 0 to switch sides as above.

More explicitly, let $\epsilon = \frac{1}{8}$, and fix $0 < \delta < 1$. Now for sufficiently small δ and any point $x \in X$ such that $d(x, 0) < \frac{\delta}{1+\sqrt{5}}$, we have that $d(Tx, 0) < \frac{\delta}{2}$. Let x_0 be one of these such points, and let $x_1 = -x_0$. Note that this is possible because

$$d(Tx_0, -x_0) \leq d(Tx_0, 0) + d(0, -x_0) < \frac{\delta}{2} + \frac{\delta}{1+\sqrt{5}} < \delta.$$

Define a pseudo-orbit $\langle x_n \rangle_{n \in \omega}$ in this manner, with $x_n = x_0$ if n is even and $x_n = -x_0$ if n is odd.

Now suppose that there is a point $z \in X$ that $\frac{1}{8}$-shadows this pseudo-orbit. Then $z \in \left(0, \frac{1+\sqrt{5}}{2}\right]$ or $z \in \left[-\frac{1+\sqrt{5}}{2}, 0\right)$. Either way, eventually there is an n such that $d(T^n z, 0) > \frac{1}{2}$, so that

$$d(T^n z, x_n) \geq \left| d(T^n z, 0) - d(0, x_n) \right| > \frac{1}{2} - \frac{\delta}{1+\sqrt{5}} > \frac{1}{2} - \frac{1}{1+\sqrt{5}} > \frac{1}{8}.$$

□

Example 4.1.9. Let $X = [-2, 2]$, $T : X \to X$ given above. Then (X, T) exhibits the NUPOTP*, but not the NUPOTP (and hence not the CNUPOTP nor shadowing).

Proof. The points $x = \pm 2$ get mapped to $x = 0$, which allows a pseudo-orbit to "switch sides", after forcing any shadowing point to be on one side of the space.

More explicitly, assume that (X, T) exhibits the NUPOTP. Let $\epsilon = \frac{1}{2}$, and let $\delta_\epsilon \colon X \to (0, 1]$. Define a δ_ϵ-pseudo-orbit in the following manner: Let $x_0 = 2$. Now x_1 must come from a neighborhood of radius $\delta_\epsilon(x_0) > 0$ about the point $Tx_0 = 0$. As this radius is positive, x_1 can therefore be either positive or negative. Take x_1 to be negative and within this neighborhood. Define $x_n = T^{n-1}x_1$ for $n \geq 2$.

No point $z \in X$ will be able to shadow $\langle x_n \rangle_{n \in \omega}$, as since $d(z, x_0) < \frac{1}{2}$, we must have that $z > 0$. But as T maps nonnegative values to nonnegative values, and since $x_n = T_{n-1}x_1$ eventually is less than or equal to $-\frac{1}{2}$, we have that $d(T^n z, x_n) \geq \frac{1}{2}$. $\square$

To summarize, we encode the above information into two helpful charts. The first indicates implications under the criteria of the referenced propositions, while the second indicates the existence of counterexamples.

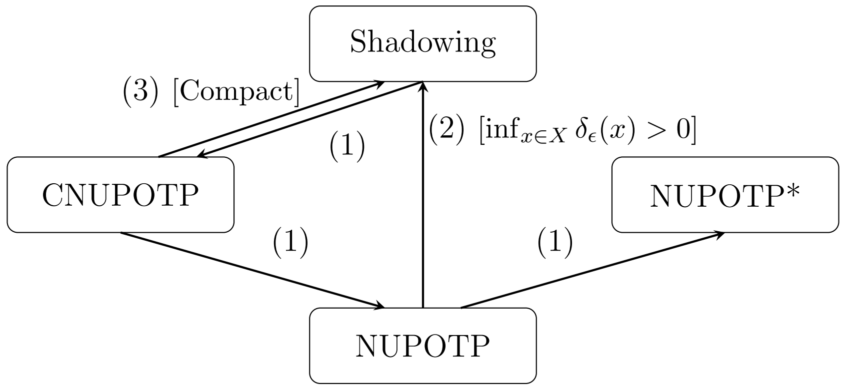

Figure 4.1: (1) Proposition 4.1.7, (2) Proposition 4.1.3, (3) Corollary 4.1.4

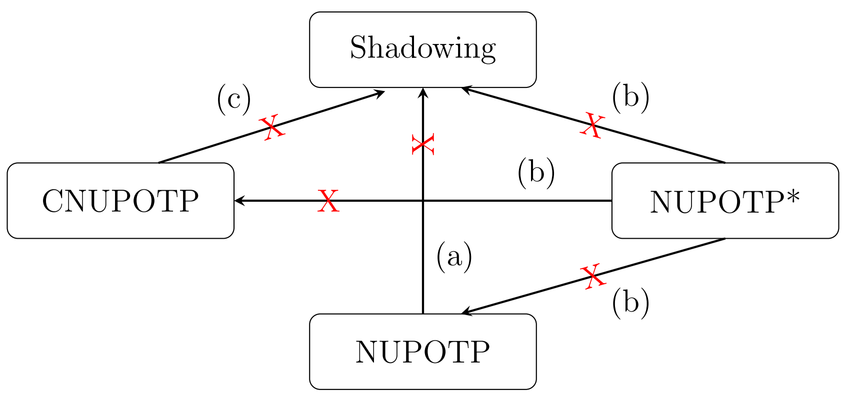

Figure 4.2: (a) Example 4.1.2 , (b) Example 4.1.9, (c) Example 4.1.8

4.2 CNUPOTP and Compactifications

As stated, much work has been done in examining the shadowing property on compact spaces. The notion of a compactification provides a way to bridge the gap between the metric spaces we have studied and this field of work.

Definition 4.2.1. A *compactification* of a topological space X is a compact space $\hat{X}$ containing X.

We now consider systems which have a compactification and ask the following: What relationship does the shadowing property on a compactification of a system have with the the non-uniform shadowing property on the original space? We first consider one-point compactifications and show that for isometric embeddings, existence of the shadowing property on the one-compactification proves the existence of the non-uniform shadowing property on the original space if the compactifying point is attracting or repelling.

Proposition 4.2.2. Suppose that a system $\Big((X,d)\,,T\Big)$ has a one-point compactification $\Big(\Big(\hat{X},\hat{d}\Big)\,,\hat{T}\Big)$ where $\hat{T}\big|_X = T$, $\hat{T}(\infty)=\infty$ and $\hat{d}\big|_{X\times X}=d$. If

1) $\hat{d}\left(\hat{T}(x),\infty\right) > \hat{d}(x,\infty)$ for all $x\in X$

2) $(\hat{X},\hat{T},\hat{d})$ exhibits the shadowing property,

then $\Big((X,d)\,,T\Big)$ exhibits the CNUPOTP.

Proof. Let $\epsilon>0$ be given, and choose $\delta>0$ to witness ϵ-shadowing on $\Big(\Big(\hat{X},\hat{d}\Big)\,,\hat{T}\Big)$. Let $\eta_x = \hat{d}(\hat{T}x,\infty)-\hat{d}(x,\infty)>0$, and define $\delta_\epsilon(x)=\min\left(1,\frac{\eta_x}{2},\frac{d(x,Tx)}{2}\right)\cdot\delta$. Notice that this is continuous, as η_x and $d(x,Tx)$ are continuous.

Let $\langle x_i \rangle_{i\in\omega}$ be a δ_ϵ-pseudo-orbit. As this is also a δ-pseudo-orbit under $\hat{T}$, then there is a point $z \in \hat{X}$ that ϵ-shadows it. If $z \neq \infty$, then we are done, because then there is a point in the original space that ϵ-shadows the pseudo-orbit.

Suppose that $z = \infty$. Then $\hat{d}(\infty, x_i) < \epsilon$ for all $i \in \omega$. Also, we have the following:

$$\begin{aligned}
\hat{d}(\infty, x_{i+1}) &\geq \left|\hat{d}(\hat{T}x_i, x_{i+1}) - \hat{d}(\infty, \hat{T}x_i)\right| \\
&= \left|\hat{d}(\hat{T}x_i, x_{i+1}) - \hat{d}(\infty, x_i) - \eta_{x_i}\right| \\
&> \hat{d}(\infty, x_i) + \frac{\eta_{x_i}}{2},
\end{aligned}$$

as $\delta_\epsilon(x_i) < \frac{\eta_{x_i}}{2}$. Therefore we have that

$$\hat{d}(\infty, x_i) > \hat{d}(\infty, x_0) + \frac{1}{2}\sum_{j=0}^{i-1} \eta_{x_j}.$$

Now $\eta_{x_j} > 0$ for all $j \in \omega$. If $\eta_{x_j} > k > 0$ for some k for all $j \in \omega$, then we reach our contradiction, as the distance would grow beyond epsilon.

Assume then that $\eta_{x_j} \to 0$ as $j \to \infty$. As $\hat{X}$ is a compact space, $\langle x_i \rangle_{i\in\omega}$ contains a convergent subsequence $\left\langle x_{i_j} \right\rangle_{j\in\omega}$ with $\eta_{x_{i_j}} \to 0$ as $j \to \infty$. Say $x_{i_j} \to x \in \hat{X}$. By the continuity of $\hat{d}$, we must have that $\eta_x = 0$. But then $\hat{d}(\hat{T}x, \infty) = \hat{d}(x, \infty)$. Therefore, $x = \infty$. This cannot be the case by the definitions of δ_ϵ and $\langle x_i \rangle_{i\in\omega}$. $\square$

Proposition 4.2.3. Suppose that a system $\Big((X, d)\,,\ T \Big)$ has a one-point compactification $\left(\left(\hat{X}, \hat{d}\right), \hat{T} \right)$ where $\hat{T}\big|_X = T$, $\hat{T}(\infty) = \infty$, and $\hat{d}\big|_{X\times X} = d$. If

1) $\hat{d}\left(\hat{T}(x), \infty\right) < \hat{d}\,(x, \infty)$ for all $x \in X$

2) $(\hat{X}, \hat{T}, \hat{d})$ exhibits the shadowing property,

then $\Big((X, d)\,,\ T \Big)$ exhibits the CNUPOTP.

Proof. Let $\epsilon > 0$ be given, and choose $\delta > 0$ to witness ϵ-shadowing on $\left(\left(\hat{X}, \hat{d}\right), \hat{T}\right)$. Let $\eta_x = \hat{d}\,(x, \infty) - \hat{d}\left(\hat{T}(x), \infty\right)$. Define $\delta_\epsilon(x) = \min\left(1, \frac{\eta_x}{2}\right) \cdot \delta$. Notice that this is continuous, as η_x is continuous.

Let $\langle x_n \rangle_{n\in\omega} \subset X$ be a δ_ϵ-pseudo-orbit. Notice that this is a δ-pseudo-orbit in $\hat{X}$, so that there exists a point $z \in \hat{X}$ such that $\hat{d}\left(\hat{T}^n z, x_n\right) < \epsilon$ for all $n \in \omega$. If $z \in X$, then we are done.

Assume that $z = \infty$. Notice that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\hat{d}(\infty, x_{n+1}) &\leq \hat{d}(\hat{T}x_n, x_{n+1}) + \hat{d}(\infty, \hat{T}x_n) \\
&= \hat{d}(\hat{T}x_n, x_{n+1}) + \hat{d}(\infty, x_n) - \eta_{x_n} \\
&< \hat{d}(\infty, x_n) - \frac{\eta_{x_n}}{2} < \hat{d}(\infty, x_n).
\end{aligned}$$

But then, notice the following:

$$\begin{aligned}
d(T^n x_0, x_n) &= \hat{d}(T^n x_0, x_n) \\
&\leq \hat{d}(T^n x_0, \infty) + \hat{d}(\infty, x_n) \\
&< \hat{d}(x_0, \infty) + \hat{d}(\infty, x_0) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore we find that $\left((X, d),\ T\right)$ exhibits the CNUPOTP. $\square$

We now seek to expand our view beyond systems with a one-point compactification, beginning with the case of both an attracting and repelling point.

Proposition 4.2.4. Suppose that $\left((X,d)\,,\ T\right)$ has a two-point compactification $\left(\left(\hat{X},\hat{d}\right),\hat{T}\right)$ with $\hat{X} = X \cup \{\infty_a\} \cup \{\infty_b\}$. Suppose that there exist disjoint open neighborhoods $B_{r_a}(\infty_a)$ and $B_{r_b}(\infty_b)$ in $\hat{X}$ such that

1) $\hat{d}\left(\hat{T}x,\infty_a\right) < \hat{d}\left(x,\infty_a\right)$ for all $x \in B_{r_a}(\infty_a)$

2) $\hat{d}\left(\hat{T}x,\infty_b\right) > \hat{d}(x,\infty_b)$ for all $x \in B_{r_b}$

Then if $\left(\left(\hat{X},\hat{d}\right),\hat{T}\right)$ exhibits the shadowing property, $\left((X,d)\,,\ T\right)$ exhibits the CNUPOTP.

Proof. Let $\epsilon > 0$ be given. Let $\eta_x^a = \hat{d}\left(x,\infty_a\right) - \hat{d}\left(\hat{T}x,\infty_a\right) > 0$ and $\eta_x^b = \hat{d}(\hat{T}x,\infty) - \hat{d}(x,\infty) > 0$. Define $\delta_\epsilon(x) = \min\left(1, \frac{\eta_x^a}{2}, \frac{\eta_x^b}{2}, \frac{d(x,Tx)}{2}\right)\cdot\delta$, where δ comes from the definition of $\frac{\epsilon}{2}$ shadowing on $\hat{X}$, where $\epsilon < \min\left(r_a, r_b\right)$. Yet again, δ_ϵ is continuous, since η_x^a, η_x^b, and $d(x,Tx)$ are all continuous.

Notice that the requirements of 4.2.3 are satisfied in $B_{r_a}(\infty_a)$ and the requirements of 4.2.2 are satisfied in $B_{r_b}(\infty_b)$. Therefore, the result is attained if $\langle x_n\rangle$ is completely contained in either $B_{r_a}(\infty_a)$ or $B_{r_b}(\infty_b)$.

Therefore, all that is left to consider is a pseudo-orbit that is not completely contained in one of the two neighborhoods. If $\langle x_n\rangle_{n\in\omega}$ is such a pseudo-orbit, let x_N be the first term that is not in either neighborhood. As $\langle x_n\rangle_{n\in\omega}$ is a δ-pseudo-orbit of $\left(\left(\hat{X},\hat{d}\right),\hat{T}\right)$, there exists a point $\hat{z} \in \hat{X}$ such that $\hat{d}(\hat{T}^n\hat{z}, z_n) < \frac{\epsilon}{2}$ for all $n \in \omega$. Notice that $\hat{d}(\infty_j, x_N) > \epsilon$ for $j = a$ or $j = b$ because $\epsilon < \min\left(r_a, r_b\right)$, so that $\hat{z} \in X$. Therefore $\left((X,d)\,,\ T\right)$ exhibits the CNUPOTP by previous results. $\square$

BIBLIOGRAPHY

[1] D.V. Anosov. "Geodesic flows on closed Riemann manifolds with negative curvature". In: *Proceedings of the Steklov Institute of Mathematics* 90 (1967).

[2] H. Arikan, L. Runov, and V. Zahariuta. "Holomorphic Functional Calculus for Operators On A Locally Convex Space". In: *Results in Mathematics* (2003).

[3] Jr. Bernardes Nilson C. and Ali Messaoudi. "Shadowing and structural stability in linear dynamical systems". In: *arXiv e-prints*, arXiv:1902.04386 (Feb. 2019), arXiv:1902.04386. arXiv: `1902.04386 [math.DS]`.

[4] Jr. Bernardes Nilson C. et al. "Expansivity and Shadowing in Linear Dynamics". In: *arXiv e-prints*, arXiv:1612.02921 (Dec. 2016), arXiv:1612.02921. arXiv: `1612.02921 [math.DS]`.

[5] L.S. Block and W.A. Coppel. *Dynamics in One Dimension*. Dynamics in One Dimension no. 1513. Springer-Verlag, 1992. ISBN: 9783540553090. URL: `https://books.google.com/books?id=BFXvAAAAMAAJ`.

[6] José Bonet. "The Structure of Volterra Operators on Korenblum Type Spaces of Analyitic Functions". In: *Integral Equations and Operator Theory* (Oct. 2019). Available at `https://doi.org/10.1007/s00020-019-2547-x`.

[7] R. Bowen. "On axiom A diffeomorphisms". In: 1975.

[8] Rufus Bowen. "ω-Limit sets for Axiom A diffeomorphisms". In: *Journal of Differential Equations* 18.2 (1975), pp. 333–339. DOI: `10.1016/0022-0396(75)90065-0`.

[9] William Brian, Jonathan Meddaugh, and Brian Raines. "Chain transitivity and variations of the shadowing property". In: *Ergodic Theory and Dynamical Systems* 35 (July 2014), pp. 1–9. DOI: `10.1017/etds.2014.21`.

[10] C. Conley. *Isolated Invariant Sets and the Morse Index*. Regional conference series in mathematics. R.I.:American Mathematical Society, 1978. ISBN: 9780821888834. URL: `https://books.google.com/books?id=vU9UC5x8xB0C`.

[11] Ethan M. Coven, Ittai Kan, and James A. Yorke. "Pseudo-Orbit Shadowing in the Family of Tent Maps". In: *Transactions of the American Mathematical Society* 308.1 (1988), pp. 227–241. ISSN: 00029947. URL: `http://www.jstor.org/stable/2000960`.

[12] Karl-G Grosse-Erdmann and Alfred Peris. *Linear Chaos*. Springer, Jan. 2011. ISBN: 978-1-4471-2169-5. DOI: `10.1007/978-1-4471-2170-1`.

[13] Noriaki Kawaguchi. *On the shadowing and limit shadowing properties*. 2019. arXiv: `1710.00313 [math.DS]`.

[14] Kazuhiro Sakai Keonhee Lee. "Various shadowing properties and their equivalence". In: *Discrete and Continuous Dynamical Systems* 13.2 (2005), pp. 533–540.

[15] S. Kolyada and L. Snoha. "Topological transitivity". In: *Scholarpedia* 4 (2009), p. 5802.

[16] Marcin Kulczycki, Dominik Kwietniak, and Piotr Oprocha. "On almost specification and average shadowing properties". In: *Fundamenta Mathematicae* 224.3 (2014), pp. 241–278. ISSN: 1730-6329. DOI: `10.4064/fm224-3-4`. URL: `http://dx.doi.org/10.4064/fm224-3-4`.

[17] E. N. Lorenz. "Predictability: Does the Flap of a Butterfly's Wings in Brazil Set off a Tornado in Texas". In: *American Association for the Advancement of Science* (1972). URL: `http://gymportalen.dk/sites/lru.dk/files/lru/132_kap6_lorenz_artikel_the_butterfly_effect.pdf`.

[18] Fumiyuki Maeda. "Remarks on Spectra of Operators on a Locally Convex Space". In: (May 1961).

[19] Kirti K. Oberai. "Spectrum of a Spectral Operator". In: (July 1966).

[20] Kazuhiro Sakai. "Various shadowing properties for positively expansive maps". In: *Topology and its Applications* 131.1 (2003), pp. 15–31. ISSN: 0166-8641. DOI: `https://doi.org/10.1016/S0166-8641(02)00260-2`. URL: `https://www.sciencedirect.com/science/article/pii/S0166864102002602`.

[21] Karl Sigmund. "On Dynamical Systems With the Specification Property". In: *Transactions of the American Mathematical Society* 190 (1974), pp. 285–299. ISSN: 00029947. URL: `http://www.jstor.org/stable/1996963`.

[22] Vladimir G. Troitsky. "Spectral Radii of Bounded Operators On Topological Vector Spaces". In: *PanAmerican Mathematical Journal* (2001).

[23] Florian-Horia Vasilescu. *Analytic functional calculus and spectral decompositions / Florian-Horia Vasilescu.* eng. 2nd ed. Mathematics and its applications (Reidel) East European series ; v. 1. Bucuresti, Romania: Editura Academiei ; Dordrecht, Holland ; Boston : D. Reidel Pub. Co., 1982. ISBN: 9027713766.

[24] Peter Walters. "On the pseudo orbit tracing property and its relationship to stability". In: *The Structure of Attractors in Dynamical Systems: Proceedings, North Dakota State University, June 20–24, 1977*. Ed. by Nelson G. Markley, John C. Martin, and William Perrizo. Berlin, Heidelberg: Springer Berlin Heidelberg, 1978, pp. 231–244. ISBN: 978-3-540-35751-3. DOI: 10.1007/BFb0101795. URL: https://doi.org/10.1007/BFb0101795.