ABSTRACT

Boundary Data Smoothness for Solutions of Nonlocal Boundary Value Problems

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In this dissertation, we investigate boundary data smoothness for solutions of nonlocal boundary value problems over discrete and continuous domains. Essentially, we show that under certain conditions partial derivatives and differences exist, with respect to boundary conditions, for solutions of nonlocal boundary value problems and solve the variational equation in the derivative case and a specific linear difference equation in the difference case.

With respect to the continuous domain, we begin by defining a difference quotient for the solution and showing that for h not equal to zero, the boundary conditions for the variational equation are indeed satisfied. Afterwards, we look at the solution in terms of an initial value problem, and with the help of a telescoping sum, the Mean Value Theorem, a theorem attributed to Peano, and a system of equations, we find that the limit of the difference quotient exists and does solve the variational equation. With respect to the discrete domain, we take a very similar approach once again invoking the Mean Value Theorem and also some difference calculus.

Lastly, we provide a corollary and a few examples as well as some ideas for future work.

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A Dissertation

Approved by the Department of Mathematics

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Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

Approved by the Dissertation Committee

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Accepted by the Graduate School
May 2011

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Page bearing signatures is kept on file in the Graduate School.

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ACKNOWLEDGMENTS

This dissertation would not be possible without the encouragement and motivation from the many influences in my life.

I first want to thank Baylor University and the Department of Mathematics for the opportunity to pursue this Ph.D. I thank the faculty of the Department of Mathematics for not only dedicating their precious time to help with my understanding of mathematics but also for sharing their love of the subject. In particular, I would like to thank Dr. Raymond Cannon, Dr. Edwin Oxford, and Dr. Brian Raines for their constant support throughout my undergraduate years and recommending I continue with my studies in graduate school.

I also thank my committee members for their time and counseling on this work; especially my advisor, Dr. Johnny Henderson whose unwavering support for my studies and research, even in the bleakest times, never went unnoticed and was essential to this achievement. The privilege of working with a man whose brilliance is only surpassed by his great personality and devotion is an honor that I have truly cherished.

I would also like to thank my fellow graduate students who have been instrumental in my success; most notably, Britney Hopkins, Eugenie Kim, and Casey Sherman who endured the intense first year of graduate school with me.

Finally, I thank my family whose constant faith in me has been the foundation upon which I have built my success, and I know that their steadfast support will continue to help push me to my fullest potential.

CHAPTER ONE

Introduction

In this work, we will be concerned with characterizing partial derivatives and partial differences of certain nonlocal boundary value problems with respect to boundary data. A nonlocal boundary value problem incorporates a linear combination of function values into a single boundary condition. We adapt the techniques found within two previous works, one related to continuous domains [23] and the other related to discrete domains [8], to show that these partial derivatives and partial differences do indeed solve the variational equation and a certain linear difference equation, respectively, with specific boundary conditions. We achieve this goal by proving that the boundary conditions asserted in each case are satisfied. Then we use a telescoping sum and the Mean Value Theorem (and in addition, in the continuous case, a theorem attributed to Peano) to prove that the partials do indeed exist and solve their respective equations. Lastly, we assert a corollary relating several of the partial derivates.

In Chapter 2, we consider the origins and history of the work done in this dissertation.

In Chapter 3, we deal with the continuous domain where we have partial derivatives with respect to the boundary data for solutions of the general nth order nonlocal boundary value problem

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ x \in (a, b),\ n \geq 2, \tag{1.1}$$

satisfying

$$\begin{aligned} y^{(i)}(x_j) &= y_{ij},\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1, \\ y^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y(\eta_{ip}) &= y_{ik},\ 0 \leq i \leq m_k - 1, \end{aligned} \tag{1.2}$$

where $2 \le k \le n,\ m \in \mathbb{N},\ m_1, \dots, m_k$ are positive integers such that $\sum_{i=1}^k m_i = n$, and $a < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$ and $y_{01}, \dots, y_{m_k-1,k}, r_{01}, \dots, r_{m_k-1,m} \in \mathbb{R}$. We begin by outlining the second order nonlocal continuous problem found in [16]. Afterwards, we move to what we call the stacked and the spread nth order nonlocal problems. At this point, we provide an alternate BVP proof to the spread nth order nonlocal problem. Each of these problems gives insight to the main theorem of the chapter that stems from (1.1), (1.2). While the arguments are often similar for existence of each partial derivative, each case involves a unique characteristic which then motivates a new consideration in computation. We then conclude with a corollary and an example.

In Chapter 4, we deal with the discrete domain where we have partial derivatives and partial differences with respect to the boundary data for solutions of the general nth order nonlocal boundary value problem

$$w(t+n) = f(t, w(t), w(t+1), \dots, w(t+n-1)),\ t \in \mathbb{Z},\ n \ge 2, \tag{1.3}$$

satisfying

$$\begin{aligned} &w(t_j + i) = w_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \\ &w(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}) = w_{ik},\ 0 \le i \le m_k - 1, \end{aligned} \tag{1.4}$$

where $2 \le k \le n,\ m \in \mathbb{N},\ m_1, \dots, m_k$ are positive integers such that $\sum_{i=1}^k m_i = n,\ t_1 < t_1 + m_1 - 1 < t_2 < t_2 + m_2 - 1 < \cdots < t_{k-1} + m_{k-1} - 1 < \eta_{01} < \eta_{01} + 1 < \cdots < \eta_{0m} < \eta_{0m} + 1 < \cdots < \eta_{m_k-1,1} < \eta_{m_k-1,1} + 1 < \cdots < \eta_{m_k-1,m} < \eta_{m_k-1,m} + 1 < t_k$ in $\mathbb{Z}$, and $\alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k} \in \mathbb{R}$. Again, we start off with a proof of the second order nonlocal discrete problem. We then progress to the general nth order problem using (1.3), (1.4) and polish the chapter off with an example.

In Chapter 5, we summarize the results found within this dissertation as well as discuss future work that stems from it.

CHAPTER TWO

History

Interest in multipoint and nonlocal problems has seen a rise in enthusiasm in recent years as can be seen in [1], [7], [12], [19], [35], [36], [44], [45], [46], [47], and [48]. We also see that the study of smoothness of boundary conditions for solutions of boundary value problems is an evolving field of mathematics as can be seen in [2], [3], [4], [5], [6], [8], [9], [10], [11], [13], [14], [15], [16], [17], [18], [20], [21], [22], [23], [24], [25], [30], [26], [27], [28], [29], [31], [32], [34], [37], [38], [39], [40], [41], [42], and [43]. Note that these works encompass much more than simply a continuous domain; specifically, a few relate to discrete domains which are of much interest to this work. The major motivators of the work found within this dissertation are [8], [16], and [23].

In [23], Henderson investigated the general nth order boundary value problem and found that if certain restrictions are placed on a boundary value problem that one is then able to differentiate the solution with respect to the boundary conditions. The resulting function not only exists but solves the variational equation and satisfies certain boundary conditions depending upon which parameter was used. Extension of the methods are adapted, in many cases, for use within this dissertation. For example, the idea of taking the solution of the nth order boundary value problem and viewing it as an initial value problem leading to a system of equations via a theorem of Peano.

The work of Ehrke, Henderson, Kunkel, and Sheng, [16], built upon [23] by introducing the nonlocal condition. Although this work was only done for the second order problem the idea of using the nonlocal condition proved very influential for the main result found in Chapter 3.

Subsequently, Henderson, Hopkins, Kim and Lyons, in [24], and Henderson and Lyons, in [28], extended the work of [16] to a more general classes of nonlocal problems [see, Chapter 3].

Lastly, in [8], Benchohra, Hamani, Henderson, Ntouyas, and Ouahab viewed boundary data smoothness as it relates to discrete domains, and quite recently, Hopkins, Kim, Lyons, and Speer, in [30], dealt with similar boundary data smoothness questions for certain discrete boundary value problems. These works combined with [16] and [23] led directly to the results presented within Chapter 4.

CHAPTER THREE

Differential Equations with Nonlocal Boundary Conditions

In Chapter 3, we investigate the influence of the smoothness of the boundary conditions on solutions as we impose a few restrictions on nonlocal boundary value problems. In Section 3.1, we outline the work done by Ehrke, Kunkel, Henderson, and Sheng found in [16] which, as was stated previously, is a basis of the theorems in this chapter. Next, we look into two different types of nth order nonlocal boundary conditions, found within [24] and [28]. This will lead to the main result of this chapter that deal with the general nth order nonlocal boundary value problem. In conclusion, we provide a corollary and an example.

3.1 Preliminary Definitions, Conditions, and Theorems

Consider the nth order differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ x \in (a, b),\ n \geq 2, \tag{3.1}$$

satisfying the nonlocal boundary conditions

$$\begin{aligned} &y^{(i)}(x_j) = y_{ij},\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1, \\ &y^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y(\eta_{ip}) = y_{ik},\ 0 \leq i \leq m_k - 1, \end{aligned} \tag{3.2}$$

where $2 \leq k \leq n,\ m \in \mathbb{N},\ m_1, \ldots, m_k \in \mathbb{N}$ with $\sum_{i=1}^{k} m_i = n, a < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$, and $r_{01}, \ldots, r_{m_k-1,m}, y_{01}, \ldots, y_{m_k-1,k} \in \mathbb{R}$.

Based upon (3.1), we can state the conditions that we place upon it thoughout the chapter:

(i) $f(x, y_1, \ldots, y_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

(ii) $\dfrac{\partial f}{\partial y_i}(x, y_1, \ldots, y_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ are continuous, $i = 1, 2, \ldots, n$, and

(iii) Solutions of initial value problems for (3.1) extend to (a, b).

Remark 3.1. Condition (iii) is not a necessary condition, but lets us avoid continually making statements about maximal intervals of existence inside (a, b).

Definition 3.1. Given a solution $y(x)$ of (3.1), we define the *variational equation along* $y(x)$ by

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(x, y(x), y'(x), \ldots, y^{(n-1)}(x))z^{(i-1)}. \tag{3.3}$$

We now introduce the definition of $(m_1, \ldots, m_k)$-disconjugacy which will be of use in the subsequent conditions. Notice the definition, when imposed, implies uniqueness of solutions.

Definition 3.2. Let $2 \le k \le n$ be given and let $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$. Given $a < x_1 < x_2 < \ldots < x_n < b$, if

$$y^{(i)}(x_j) = z^{(i)}(x_j),\ 0 \le i \le m_j - 1,\ 1 \le j \le k,$$

implies $y(x) \equiv z(x)$ on (a, b), where $y(x)$ and $z(x)$ are solutions of (3.1), then we say that (3.1) is $(m_1, \ldots, m_k)$-*disconjugate on* (a, b).

The next condition guarantees uniqueness of solutions of (3.1), (3.2) and is a nonlocal analogue of $(m_1, \ldots, m_k)$-disconjugacy:

(iv) Let $2 \le k \le n$, $m \in \mathbb{N}$, and $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$. Given $a < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$ and $r_{01}, \ldots, r_{m_k-1,m} \in \mathbb{R}$, if, for $0 \le i \le m_j - 1,\ 1 \le j \le k-1$,

$$y^{(i)}(x_j) = z^{(i)}(x_j),$$

and, for $0 \le i \le m_k - 1$,

$$y^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y(\eta_{ip}) = z^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z(\eta_{ip}),$$

where $y(x)$ and $z(x)$ are solutions of (3.1), then, on (a, b),

$$y(x) \equiv z(x).$$

Notice if $r_{01} = \cdots = r_{m_k-1,m} = 0$, then (3.1) is $(m_1, \ldots, m_k)$-disconjugate on (a, b).

The last condition provides uniqueness of solutions of (3.3) along all solutions of (3.1) and again is a nonlocal analogue of $(m_1, \ldots, m_k)$-disconjugacy:

(v) Let $2 \leq k \leq n$, $m \in \mathbb{N}$,, and $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$. Given $a < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$ and $r_{01}, \ldots, r_{m_k-1,m} \in \mathbb{R}$, and a solution $y(x)$ of (3.1), if, for $0 \leq i \leq m_j - 1$, $1 \leq j \leq k-1$,

$$u^{(i)}(x_j) = 0,$$

and, for $0 \leq i \leq m_k - 1$,

$$u^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} u(\eta_{ip}) = 0,$$

where $u(x)$ is a solution of (3.3) along $y(x)$, then, on (a, b),

$$u(x) \equiv 0.$$

Again, notice if $r_{01} = \cdots = r_{m_k-1,m} = 0$, then (3.3) is $(m_1, \ldots, m_k)$-disconjugate on (a, b) along solutions $y(x)$ of (3.1).

Now we introduce a theorem of Peano that we seek an analogue to throughout the chapter.

Theorem 3.1. *[A Peano Theorem] Assume that, with respect to* (3.1), *conditions* (i)-(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) := y(x, x_0, c_1, c_2, \ldots, c_n)$ *denote the solution of* (3.1) *satisfying the initial conditions* $y^{(i-1)}(x_0) = c_i$, $1 \leq i \leq n$. *Then,*

(*a*) *for each* $1 \leq j \leq n$, $\dfrac{\partial y}{\partial c_j}(x)$ *exists on* (a, b), *and* $\alpha_j(x) := \dfrac{\partial y}{\partial c_j}(x)$ *is the solution of the variational equation* (3.3) *along* $y(x)$ *satisfying the initial* *conditions*

$$\alpha_j^{(i-1)}(x_0) = \delta_{ij},\ 1 \le i \le n.$$

(*b*) $\dfrac{\partial y}{\partial x_0}(x)$ *exists on* (a, b), *and* $\beta(x) := \dfrac{\partial y}{\partial x_0}(x)$ *is the solution of the variational equation* (3.3) *along* $y(x)$ *satisfying the initial conditions*

$$\beta^{(i-1)}(x_0) = -y^{(i)}(x_0),\ 1 \le i \le n.$$

(*c*) $\dfrac{\partial y}{\partial x_0}(x) = -\displaystyle\sum_{i=1}^{n} y^{(i)}(x_0)\frac{\partial y}{\partial c_i}(x).$

We also make much use of a well known continuous dependence result which is an application of the Brouwer Invariance of Domain Theorem.

Theorem 3.2. *Assume* (i)-(iv) *are satisfied with respect to* (3.1). *Let* $2 \le k \le n$, $m \in \mathbb{N}$,, *and* $m_1, \ldots, m_k$ *be positive integers such that* $\sum_{i=1}^{k} m_i = n$. *Let* $u(x)$ *be a solution of* (3.1) *on* (a, b), *and let* $a < c < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < d < b$ *and* $r_{01}, \ldots, r_{m_k-1,m} \in \mathbb{R}$ *be given. Then, there exists a* $\delta > 0$ *such that, for*

$$|x_j - t_j| < \delta,\ 1 \le j \le k,$$

$$|\eta_{ip} - \tau_{ip}| < \delta \textit{ and } |r_{ip} - \rho_{ip}| < \delta,\ 0 \le i \le m_k - 1,\ 1 \le p \le m,$$

$$|u^{(i)}(x_j) - y_{ij}| < \delta,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$|u^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip}u(\eta_{ip}) - y_{ik}| < \delta,\ 0 \le i \le m_k - 1,$$

there exists a unique solution $u_\delta(x)$ *of* (3.1) *such that*

$$u_\delta^{(i)}(t_j) = y_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

$$u_\delta^{(i)}(t_k) - \sum_{p=1}^{m} \rho_{ip}u_\delta(\tau_{ip}) = y_{ik},\ 0 \le i \le m_k - 1,$$

and, for $0 \le i \le n-1$, $\{u_\delta^{(i)}(x)\}$ *converges uniformly to* $u^{(i)}(x)$ *as* $\delta \to 0$ *on* $[c, d]$.

3.2 Second Order Problem

In [16], Ehrke, Kunkel, Henderson, and Sheng considered solutions of the second order nonlocal boundary value problem

$$y'' = f(x, y, y'),\ x \in (a, b), \tag{3.4}$$

satisfying

$$y(x_1) = u_1, \qquad y(x_2) - \sum_{p=1}^{m} r_p y(\eta_p) = u_2, \tag{3.5}$$

where $m \in \mathbb{N},\ a < x_1 < \eta_1 < \cdots < \eta_m < x_2$, and $r_1, \ldots, r_m, u_1, u_2 \in \mathbb{R}$.

Now, given a solution $y(x)$ of (3.4), the related *variational equation along* $y(x)$ is defined by

$$z'' = \frac{\partial f}{\partial y_1}(x, y(x), y'(x))z + \frac{\partial f}{\partial y_2}(x, y(x), y'(x))z'. \tag{3.6}$$

Then, under conditions (i)-(v), they derived the following analogue of Theorem 3.1:

Theorem 3.3. *Assume conditions* (i)-(v) *are satisfied. Let* $m \in \mathbb{N}$ *and* $u(x)$ *be a solution of* (3.4) *on* (a, b)*. Let* $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$ *and* $u_1, u_2, r_1, \ldots, r_m \in \mathbb{R}$ *be given so that*

$$u(x) = u(x, x_1, x_2, u_1, u_2, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

where

$$u(x_1) = u_1, \qquad u(x_2) - \sum_{p=1}^{m} r_p u(\eta_p) = u_2.$$

Then,

(a) $\frac{\partial u}{\partial u_1}(x)$ *and* $\frac{\partial u}{\partial u_2}(x)$ *exist on* (a, b)*, and* $y_i(x) := \frac{\partial u}{\partial u_i}(x)$*,* $i = 1, 2$*, are solutions of the variational equation* (3.6) *along* $u(x)$ *satisfying the respective boundary conditions*

$$y_1(x_1) = 1, \qquad y_1(x_2) - \sum_{p=1}^{m} r_p y_1(\eta_p) = 0,$$

$$y_2(x_1) = 0, \qquad y_2(x_2) - \sum_{p=1}^{m} r_p y_2(\eta_p) = 1.$$

(b) $\dfrac{\partial u}{\partial x_1}(x)$ *and* $\dfrac{\partial u}{\partial x_2}(x)$ *exist on* (a, b)*, and* $z_i(x) := \dfrac{\partial u}{\partial x_i}(x)$*,* $i = 1, 2$*, are solutions of the variational equation* (3.6) *along* $u(x)$ *satisfying the respective boundary conditions*

$$\begin{aligned} z_1(x_1) &= -u'(x_1), \qquad z_1(x_2) - \sum_{p=1}^{m} r_p z_1(\eta_p) = 0, \\ z_2(x_1) &= 0, \qquad z_2(x_2) - \sum_{p=1}^{m} r_p z_2(\eta_p) = -u'(x_2). \end{aligned}$$

(c) for $1 \leq s \leq m$*,* $\dfrac{\partial u}{\partial \eta_s}(x)$ *exists on* (a, b)*, and* $w_s(x) := \dfrac{\partial u}{\partial \eta_s}(x)$ *is the solution of* (3.6) *along* $u(x)$ *satisfying the boundary conditions*

$$w_s(x_1) = 0, \qquad w_s(x_2) - \sum_{p=1}^{m} r_p w_s(\eta_s) = r_s u'(\eta_s).$$

(d) for $1 \leq s \leq m$*,* $\dfrac{\partial u}{\partial r_s}(x)$ *exists on* (a, b) *and* $v_s(x) := \dfrac{\partial u}{\partial r_s}(x)$ *is a solution of* (3.6) *along* $u(x)$ *satisfying the boundary conditions*

$$v_s(x_1) = 0, \qquad v_s(x_2) - \sum_{p=1}^{m} r_p v_s(\eta_p) = u(\eta_s).$$

The preceeding result presented within [16] and a similar one presented in [23] are the basis for what is seen in the rest of this chapter.

3.3 Stacked nth Order Problem

Now we begin looking at the nth order differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ x \in (a, b),\ n \geq 2, \tag{3.7}$$

satisfying what we call the stacked boundary conditions

$$\begin{aligned} &y^{(i-1)}(x_1) = u_i,\ 1 \leq i \leq n-1, \\ &y(x_2) - \sum_{p=1}^{m} r_p y(\eta_p) = u_n, \end{aligned} \tag{3.8}$$

where $m \in \mathbb{N}$, $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, and $u_1, \ldots, u_n, r_1, \ldots, r_m \in \mathbb{R}$. This is an extension of the work found within [16]. The work was recently published in *Involve: A Journal of Mathematics* in 2008 by Henderson, Hopkins, Kim, and Lyons, [24].

Given a solution $y(x)$ of (3.7), the *variational equation along* $y(x)$ associated to (3.7) is

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(x, y(x), y'(x), \ldots, y^{(n-1)}(x))z^{(i-1)}. \tag{3.9}$$

We are now able to present the result of the section which is found within [24]. For completeness, we include the proof.

Theorem 3.4. *Assume conditions* (i)-(v) *are satisfied. Let* $n \geq 2$, $m \in \mathbb{N}$, *and* $u(x)$ *be a solution* (3.7) *on* (a, b). *Let* $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$ *and* $r_1, \ldots, r_m, u_1, \ldots, u_n \in \mathbb{R}$ *be given, so that*

$$u(x) = u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

where

$$u^{(i-1)}(x_1) = u_i,\ 1 \leq i \leq n-1, \qquad u(x_2) - \sum_{k=1}^{m} r_k u(\eta_k) = u_n.$$

Then,

(*a*) *for each* $1 \leq i \leq n$, $\dfrac{\partial u}{\partial u_i}(x)$ *exists on* (a, b). *Moreover, for each* $1 \leq j \leq n-1$, $y_j(x) := \dfrac{\partial u}{\partial u_j}(x)$ *solves* (3.9) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned} y_j^{(i-1)}(x_1) &= \delta_{ij},\ 1 \leq i \leq n-1, \\ y_j(x_2) &- \sum_{k=1}^{m} r_k y_j(\eta_k) = 0, \end{aligned}$$

and $y_n(x) := \dfrac{\partial u}{\partial u_n}(x)$ *solves* (3.9) *along* $u(x)$ *satisfying the boundary conditions*

$$y_n^{(i-1)}(x_1) = 0,\ 1 \leq i \leq n-1,$$

$$y_n(x_2) \quad - \quad \sum_{k=1}^{m} r_k y_n(\eta_k) = 1.$$

(*b*) $\dfrac{\partial u}{\partial x_1}(x)$ *and* $\dfrac{\partial u}{\partial x_2}(x)$ *exist on* (a,b), *and* $z_i(x) := \dfrac{\partial u}{\partial x_i}(x)$, $i = 1,2$, *are the solutions of* (3.9) *along* $u(x)$ *satisfying the respective boundary conditions*

$$\begin{aligned} z_1^{(i-1)}(x_1) &= -u^{(i)}(x_1),\ 1 \le i \le n-1, \\ z_1(x_2) &- \sum_{k=1}^{m} r_k z_1(\eta_k) = 0, \end{aligned}$$

and

$$\begin{aligned} z_2^{(i-1)}(x_1) &= 0,\ 1 \le i \le n-1, \\ z_2(x_2) &- \sum_{k=1}^{m} r_k z_2(\eta_k) = -u'(x_2). \end{aligned}$$

(*c*) *for* $1 \le j \le m$, $\dfrac{\partial u}{\partial \eta_j}(x)$ *exists on* (a,b), *and* $w_j(x) := \dfrac{\partial u}{\partial \eta_j}(x)$ *is the solution of* (3.9) *along* $u(x)$ *satisfying*

$$\begin{aligned} w_j^{(i-1)}(x_1) &= 0 \quad,\ 1 \le i \le n-1, \\ w_j(x_2) &- \sum_{k=1}^{m} r_k w_j(\eta_k) = r_j u'(\eta_j). \end{aligned}$$

(*d*) *for* $1 \le j \le m$, $\dfrac{\partial u}{\partial r_j}(x)$ *exists on* (a,b), *and* $v_j(x) := \dfrac{\partial u}{\partial r_j}(x)$ *is the solution of* (3.9) *along* $u(x)$ *satisfying*

$$\begin{aligned} v_j^{(i-1)}(x_1) &= 0,\ 1 \le i \le n-1, \\ v_j(x_2) &- \sum_{k=1}^{m} r_k v_j(\eta_k) = u(\eta_j). \end{aligned}$$

Proof. For part (a), let $1 \le j \le n-1$, and consider $\frac{\partial u}{\partial u_j}$, since the argument for $\frac{\partial u}{\partial u_n}$ is similar we withhold the proof. In this case, we designate for brevity, $u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, u_j)$.

Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define

$$y_{jh}(x) = \frac{1}{h}[u(x, u_j + h) - u(x, u_j)].$$

Note that $u^{(j-1)}(x_1, u_j + h) = u_j + h$ and $u^{(j-1)}(x_1, u_j) = u_j$, so that, for every $h \neq 0$,

$$\begin{aligned} y_{jh}^{(j-1)}(x_1) &= \frac{1}{h}[u_j + h - u_j] \\ &= 1. \end{aligned}$$

Also, for every $h \neq 0,\ 1 \leq i \leq n-1$, and $i \neq j$,

$$\begin{aligned} y_{jh}^{(i-1)}(x_1) &= \frac{1}{h}[u^{(i-1)}(x_1, u_j + h) - u^{(i-1)}(x_1, u_j)] \\ &= \frac{1}{h}[u_i - u_i] \\ &= 0, \end{aligned}$$

and for $h \neq 0$,

$$\begin{aligned} y_{jh}(x_2) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k) =& \frac{1}{h}[u(x_2, u_j + h) - u(x_2, u_j)] \\ & - \sum_{k=1}^{m} \frac{r_k}{h}\left[u(\eta_k, u_j + h) - u(\eta_k, u_j)\right] \\ =& \frac{1}{h}[u_n - u_n] \\ =& 0. \end{aligned}$$

Let

$$\beta = u^{(n-1)}(x_1, u_j),$$

and

$$\epsilon = \epsilon(h) = u^{(n-1)}(x_1, u_j + h) - \beta.$$

By Theorem 3.2, $\epsilon = \epsilon(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.1 for solutions of initial value problems for (3.7), viewing the solution $u(x)$ as the solution of an initial value problem, and denoting $u(x) = y(x, x_1, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta)$ by $y(x, x_1, u_j, \beta)$, we have

$$y_{jh}(x) = \frac{1}{h}[y(x, x_1, u_j + h,\ \beta + \epsilon) - y(x, x_1, u_j, \beta)].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} y_{jh}(x) &= \frac{1}{h}\big[\{y(x,x_1,u_j+h,\ \beta+\epsilon)-y(x,x_1,u_j,\beta+\epsilon)\} \\ &\quad +\{y(x,x_1,u_j,\beta+\epsilon)-y(x,x_1,u_j,\beta)\}\big]. \end{aligned}$$

Applying Theorem 3.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} y_{jh}(x) &= \frac{1}{h}\alpha_j(x,y(x,x_1,u_j+\bar{h},\beta+\epsilon))(u_j+h-u_j) \\ &+ \frac{1}{h}\alpha_n(x,y(x,x_1,u_j,\beta+\bar{\epsilon}))(\beta+\epsilon-\beta) \\ &= \alpha_j(x,y(x,x_1,u_j+\bar{h},\beta+\epsilon))+\frac{\epsilon}{h}\alpha_n(x,y(x,x_1,u_j,\beta+\bar{\epsilon})), \end{aligned}$$

where $\alpha_k(x,y(\cdot)),\ k\in\{j,n\}$, is the solution of the variational equation (3.9) along $y(\cdot)$ satisfying, in each case,

$$\alpha_j^{(i-1)}(x_1)=\delta_{ij},\ 1\le i\le n$$

and

$$\alpha_n^{(i-1)}(x_1)=\delta_{in},\ 1\le i\le n.$$

Furthermore, $u_j+\bar{h}$ is between u_j and u_j+h, and $\beta+\bar{\epsilon}$ is between β and $\beta+\epsilon$. Thus, to show $\lim_{h\to 0} y_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0}\frac{\epsilon}{h}$ exists.

Now $\alpha_n(x,y(\cdot))$ is a nontrivial solution of (3.9) along $y(\cdot)$,
and $\alpha_n^{(i-1)}(x_1,y(\cdot))=0,\ 1\le i\le n-1$. So, by assumption (v),

$$\alpha_n(x_2,y(\cdot))-\sum_{k=1}^{m} r_k\alpha_n(\eta_k,y(\cdot))\neq 0.$$

However, we observed that $y_{jh}(x_2)-\sum_{k=1}^m r_k y_{jh}(\eta_k)=0$, from which we obtain

$$\frac{\epsilon}{h}=\frac{\sum_{k=1}^m r_k\alpha_j(\eta_k,y(x,x_1,u_j+\bar{h},\ \beta+\epsilon))-\alpha_j(x_2,y(x,x_1,u_j+\bar{h},\ \beta+\epsilon))}{[\alpha_n(x_2,y(x,x_1,u_j,\beta+\bar{\epsilon}))-\sum_{k=1}^m r_k\alpha_n(\eta_k,y(x,x_1,u_j,\ \beta+\bar{\epsilon}))]}.$$

As a consequence of continuous dependence, we can let $h\to 0$, so that

$$\lim_{h\to 0}\frac{\epsilon}{h} = \frac{-[\alpha_j(x_2,y(x,x_1,u_j,\beta_2))-\sum_{k=1}^m r_k\alpha_j(\eta_k,y(x,x_1,u_j,\beta))]}{[\alpha_n(x_2,y(x,x_1,u_j,\beta))-\sum_{k=1}^m r_k\alpha_n(\eta_k,y(x,x_1,u_j,\beta))]}$$

$$
\begin{aligned}
&= \frac{-[\alpha_j(x_2, u(x)) - \sum_{k=1}^{m} r_k \alpha_j(\eta_k, u(x))]}{[\alpha_n(x_2, u(x)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))]} \\
&:= A.
\end{aligned}
$$

Let

$$y_j(x) = \lim_{h \to 0} y_{jh}(x),$$

and note by construction of $y_{jh}(x)$,

$$y_j(x) = \frac{\partial u}{\partial u_j}(x).$$

Furthermore,

$$y_j(x) = \lim_{h \to 0} y_{jh}(x) = \alpha_j(x, y(x, x_1, u_j, \beta)) + A\alpha_n(x, (u(x)),$$

which is a solution of the variational equation (3.9) along $u(x)$. In addition because of the boundary conditions satisfied by $y_{jh}(x)$, we also have

$$y_j^{(i-1)}(x_1) = \lim_{h \to 0} y_{jh}^{(i-1)}(x_1) = \delta_{ij},\ 1 \le i \le n-1,$$

$$y_j(x_2) - \sum_{k=1}^{m} r_k y_j(\eta_k) = \lim_{h \to 0} \left[y_j(x_2) - \sum_{k=1}^{m} r_k y_j(\eta_k) \right] = 0.$$

This completes the argument for $\frac{\partial u}{\partial u_j}$.

In part (b) of the theorem, we will produce the details for $\frac{\partial u}{\partial x_1}$, with the arguments for $\frac{\partial u}{\partial x_2}$ being similar. This time, we designate $u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, x_1)$.

Let $\delta > 0$ be as in Theorem 3.2, let $0 < |h| < \delta$ be given, and define

$$z_{1h}(x) = \frac{1}{h}[u(x, x_1 + h) - u(x, x_1)].$$

Note that for $h \neq 0$ and $1 \le i \le n-1$,

$$
\begin{aligned}
z_{1h}^{(i-1)}(x_1) &= \frac{1}{h}[u^{(i-1)}(x_1, x_1 + h) - u^{(i-1)}(x_1, x_1)] \\
&= \frac{1}{h}[u^{(i-1)}(x_1, x_1 + h) - u^{(i-1)}(x_1 + h, x_1 + h)]
\end{aligned}
$$

$$= -\frac{1}{h}[u^{(i)}(c_{x_1,h}, x_1 + h) \cdot h]$$
$$= -u^{(i)}(c_{x_1,h}, x_1 + h),$$

where $c_{x_1,h}$ lies between x_1 and $x_1 + h$.

In addition, we note that

$$\begin{aligned} z_{1h}(x_2) - \sum_{k=1}^{m} r_k z_{1h}(\eta_k) &= \frac{1}{h}[u(x_2, x_1 + h) - \sum_{k=1}^{m} r_k u(\eta_k, x_1 + h) \\ &\quad -\{u(x_2, x_1) - \sum_{k=1}^{m} r_k u(\eta_k, x_1)\}] \\ &= \frac{1}{h}[u_n - u_n] \\ &= 0, \end{aligned}$$

for every $h \neq 0$.

Next, let

$$\beta_n = u^{(n-1)}(x_1, x_1),$$

$$\epsilon_1 = \epsilon_1(h) = u(x_1, x_1 + h) - u_1,$$

$$\epsilon_2 = \epsilon_2(h) = u'(x_1, x_1 + h) - u_2,$$

$$\vdots$$

$$\epsilon_{n-1} = \epsilon_{n-1}(h) = u^{(n-2)}(x_1, x_1 + h) - u_{n-1},$$

and

$$\epsilon_n = \epsilon_n(h) = u^{(n-1)}(x_1, x_1 + h) - \beta_n.$$

By Theorem 3.2, for $1 \leq i \leq n$, $\epsilon_i \to 0$ as $h \to 0$. As in part (a), we employ the notation of Theorem 3.1 for solutions of initial value problems for (3.7), and viewing the solution $u(x)$ as the solution of an initial value problem, we have

$$\begin{aligned} z_{1h}(x) &= \frac{1}{h}[y(x, x_1, u_1 + \epsilon_1, u_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) - y(x, x_1, u_1, u_2, \ldots, \beta_n)] \\ &= \frac{1}{h}[y(x, x_1, u_1 + \epsilon_1, u_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \end{aligned}$$

$$-y(x, x_1, u_1, u_2+\epsilon_2, \ldots, \beta_n+\epsilon_n) + y(x, x_1, u_1, u_2+\epsilon_2, \ldots, \beta_n+\epsilon_n)$$

$$-y(x, x_1, u_1, u_2, \ldots, \beta_n+\epsilon_n) + y(x, x_1, u_1, u_2, \ldots, \beta_n+\epsilon_n)$$

$$-\cdots$$

$$-y(x, x_1, u_1, u_2, \ldots, u_{n-1}, \beta_n+\epsilon_n)$$

$$+y(x, x_1, u_1, u_2, \ldots, u_{n-1}, \beta_n+\epsilon_n)$$

$$-y(x, x_1, u_1, u_2, \ldots, \beta_n)].$$

By Theorem 3.1 and the Mean Value Theorem,

$$\begin{aligned} z_{1h}(x) &= \frac{1}{h}[\epsilon_1\alpha_1(x, y(x, x_1, u_1+\bar{\epsilon}_1, u_2+\epsilon_2, \ldots, \beta_n+\epsilon_n)) \\ &\quad +\epsilon_2\alpha_2(x, y(x, x_1, u_1, u_2+\bar{\epsilon}_2, \ldots, \beta_n+\epsilon_n)) \\ &\quad +\cdots \\ &\quad +\epsilon_n\alpha_n(x, y(x, x_1, u_1, u_2, \ldots, \beta_n+\bar{\epsilon}_n))], \end{aligned}$$

where for $1 \le i \le n-1$, $u_i + \bar{\epsilon}_i$ lies between u_i and $u_i + \epsilon_i$ and $\beta_n + \bar{\epsilon}_n$ lies between β_n and $\beta_n + \epsilon_n$, and for $1 \le j \le n$, $\alpha_j(x, y(\cdot))$ are the solutions of (3.9) along $y(\cdot)$ satisfying,

$$\alpha_j^{(i-1)}(x_1) = \delta_{ij},\ 1 \le i \le n,$$

and

$$\alpha_n^{(i-1)}(x_1) = \delta_{in},\ 1 \le i \le n.$$

As before, to show $\lim_{h\to 0} z_{1h}(x)$ exists, it suffices to show, for $1 \le i \le n$, $\lim_{h\to 0} \epsilon_i/h$ exists.

Now, from above, for each $1 \le i \le n-1$,

$$\begin{aligned} \lim_{h\to 0} \frac{\epsilon_i}{h} &= \lim_{h\to 0} z_{1h}^{(i-1)}(x_1) \\ &= \lim_{h\to 0} u^{(i)}(c_{x_1,h}, x_1+h) \\ &= -u^{(i)}(x_1). \end{aligned}$$

Since $\alpha_n(x, y(\cdot))$ is a nontrivial solution of (3.9) along $y(\cdot)$ and $\alpha_n^{(i-1)}(x_1, y(\cdot)) = 0,\ 1 \leq i \leq n-1$, it follows from assumption (v) that

$$\alpha_n(x_2, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot)) \neq 0.$$

Since $z_{1h}(x_2) - \sum_{k=1}^{m} r_k z_{1h}(\eta_k) = 0$, we have

$$\frac{\epsilon_n}{h} = -\sum_{i=1}^{n-1} \frac{\epsilon_i}{h} \cdot \frac{A_i}{\alpha_n(x_2, y(x, x_1, u_j, \beta_n + \bar{\epsilon}_n)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(x, x_1, u_j, \beta_n + \bar{\epsilon}_n))},$$

where

$$\begin{aligned} A_i &= \alpha_i(x_2, y(x, x_1, \ldots, u_i + \bar{\epsilon}_i, \ldots, \beta_n + \epsilon_n)) \\ &\quad - \sum_{k=1}^{m} r_k \alpha_i(\eta_k, y(x, x_1, \ldots, u_i + \bar{\epsilon}_i, \ldots, \beta_n + \epsilon_n)). \end{aligned}$$

And so,

$$\begin{aligned} \lim_{h \to 0} \frac{\epsilon_n}{h} &= \sum_{i=1}^{n-1} u^{(i)}(x_1) \Big[\alpha_i(x_2, y(x, x_1, u_1, u_2, \ldots, \beta_n)) \\ &\quad - \sum_{k=1}^{m} r_k \alpha_i(\eta_k, y(x, x_1, u_1, u_2, \ldots, \beta_n)) \Big] \\ &\quad \div \Big[\alpha_n(x_2, y(x, x_1, u_1, u_2, \ldots, \beta_n)) \\ &\quad - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(x, x_1, u_1, u_2, \ldots, \beta_n)) \Big] \\ &= \sum_{i=1}^{n-1} \frac{u^{(i)}(x_1)[\alpha_i(x_2, u(x)) - \sum_{k=1}^{m} r_k \alpha_i(\eta_k, u(x))]}{\alpha_n(x_2, u(x)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))} \\ &:= B. \end{aligned}$$

From the above expression,

$$\begin{aligned} z_{1h}(x) &= \sum_{i=1}^{n-1} \frac{\epsilon_i}{h} \alpha_i(x, y(x, x_1, \ldots, u_i + \bar{\epsilon}_i, \ldots, \beta_n + \epsilon_n)) \\ &\quad + \frac{\epsilon_n}{h} \alpha_n(x, y(x, x_1, u_1, u_2, \ldots, \beta_n + \bar{\epsilon}_n)), \end{aligned}$$

and we can evaluate the limit as $h \to 0$ by Theorem 3.2. If we let $z_1(x) = \lim_{h \to 0} z_{1h}(x)$, then

$$z_1(x) = \frac{\partial u}{\partial x_1}(x),$$

and

$$
\begin{aligned}
z_1(x) &= \lim_{h\to 0} z_{1h}(x) \\
&= -\sum_{i=1}^{n-1} u^{(i)}(x_1)\alpha_i(x, y(x, x_1, u_1, u_2, \ldots, \beta_n)) \\
&\quad + B\alpha_n(x, y(x, x_1, u_1, u_2, \ldots, \beta_n)) \\
&= -\sum_{i=1}^{n-1} u^{(i)}(x_1)\alpha_i(x, u(x)) + B\alpha_n(x, u(x)),
\end{aligned}
$$

which is a solution of (3.9) along $u(x)$. In addition, from above observations, $z_1(x)$ satisfies the boundary conditions,

$$z_1^{(i-1)}(x_1) = \lim_{h\to 0} z_{1h}^{(i-1)}(x_1) = -u^{(i)}(x_1),\ 1 \le i \le n-1,$$

and

$$z_1(x_2) - \sum_{k=1}^{m} r_k z_1(\eta_k)) = \lim_{h\to 0}\left[z_{1h}(x_2) - \sum_{k=1}^{m} r_k z_{1h}(\eta_k)\right] = 0.$$

This completes the proof for $\frac{\partial u}{\partial x_1}$.

The proofs of (c) and (d) are in very much the same spirit.

For (c), we fix $1 \le j \le m$, and this time we designate $u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, \eta_j)$. Let $\delta > 0$ be as in Theorem 3.2 and $0 < |h| < \delta$ be given. Define

$$w_{jh}(x) = \frac{1}{h}[u(x, \eta_j + h) - u(x, \eta_j)].$$

Note that for every $h \ne 0$,

$$w_{jh}^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1.$$

Next, let

$$\beta = u^{(n-1)}(x_1, \eta_j),$$

and

$$\epsilon = \epsilon(h) = u^{(n-1)}(x_1, \eta_j + h) - \beta.$$

By Theorem 3.2, $\epsilon \to 0$ as $h \to 0$. Again, we use the notation of Theorem 3.1 for solutions of initial value problems for (3.7), viewing the solution $u(x)$ as the solution of an initial value problem, and denoting $u(x) = y(x, x_1, u_1, \ldots, u_{n-1}, \beta)$ by $y(x, x_1, \beta)$, we have

$$w_{jh}(x) = \frac{1}{h}[y(x, x_1, \beta + \epsilon) - y(x, x_1, \beta)].$$

By Theorem 3.1 and the Mean Value Theorem,

$$w_{jh}(x) = \frac{\epsilon}{h}\alpha_n(x, y(x, x_1, \beta + \bar{\epsilon})),$$

where $\alpha_n(x, y(\cdot))$ is the solution of (3.1) along $y(\cdot)$ satisfying

$$\alpha_n^{(i-1)}(x_1) = \delta_{in},\ 1 \le i \le n-1,$$

and $\beta + \bar{\epsilon}$ lies between β and $\beta + \epsilon$. Once again, to show $\lim_{h\to 0} w_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0} \epsilon/h$ exists.

Since $\alpha_n(x, y(\cdot))$ is a nontrivial solution of (3.9) along $y(\cdot)$ and $\alpha_n^{(i-1)}(x_1, y(\cdot)) = 0,\ 1 \le i \le n-1$, it follows from assumption (v) that

$$\alpha_n(x_2, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot)) \neq 0.$$

Hence,

$$\frac{\epsilon}{h} = \frac{w_{jh}(x_2) - \sum_{k=1}^{m} r_k w_{jh}(\eta_i)}{\alpha_n(x_2, y(x, x_1, \beta_2 + \bar{\epsilon})) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(x, x_1, \beta + \bar{\epsilon}))}.$$

We look in more detail at the numerator of this quotient. Consider

$$\begin{aligned} w_{jh}(x_2) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k) &= \frac{1}{h}\Big[u(x_2, \eta_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j + h) \\ &\quad - u(x_2, \eta_j) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j)\Big] \\ &= \frac{1}{h}\big[u(x_2, \eta_j + h) - \sum_{k\in\{1,\ldots,m\}\setminus\{j\}} r_k u(\eta_k, \eta_j + h) \\ &\quad - r_j u(\eta_j + h, \eta_j + h) + r_j u(\eta_j + h, \eta_j + h) \end{aligned}$$

$$
\begin{aligned}
& -r_j u(\eta_j, \eta_j + h)] - \frac{u_n}{h} \\
= & \frac{u_n}{h} - \frac{u_n}{h} + \frac{r_j u(\eta_j + h, \eta_j + h) - r_j u(\eta_j, \eta_j + h)}{h} \\
= & \frac{r_j}{h}[u(\eta_j + h, \eta_j + h) - u(\eta_j, \eta_j + h)] \\
= & \frac{r_j}{h} \int_{\eta_j}^{\eta_j + h} u'(s, \eta_j + h) ds \\
= & \frac{r_j}{h} u'(c_{j,h}, \eta_j + h)(\eta_j + h - \eta_j) \\
= & r_j u'(c_{j,h}, \eta_j + h),
\end{aligned}
$$

where $c_{\eta_j,h}$ is between η_j and $\eta_j + h$. So, as $h \to 0$, we obtain

$$
r_j u'(c_h, \eta_j + h) \to r_j u'(\eta_j, \eta_j) = r_j u'(\eta_j).
$$

When we return to the quotient defining ϵ/h, we compute the limit,

$$
\begin{aligned}
\lim_{h \to 0} \frac{\epsilon}{h} & = \frac{r_j u'(\eta_j)}{\alpha_n(x_2, y(x, x_1, u_1, \beta)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(x, x_1, u_1, \beta))} \\
& = \frac{r_j u'(\eta_j)}{\alpha_n(x_2, u(x)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))} \\
& := C.
\end{aligned}
$$

If we let $w_j(x) = \lim_{h \to 0} w_{jh}(x)$, then

$$
w_j(x) = \frac{\partial u}{\partial \eta_j}(x),
$$

and

$$
\begin{aligned}
w_j(x) & = \lim_{h \to 0} w_{jh}(x) \\
& = C\alpha_n(x, y(x, x_1, u_1, \beta)) \\
& = C\alpha_n(x, u(x)),
\end{aligned}
$$

which is a solution of (3.9) along $u(x)$.

In addition, from above observations, $w_j(x)$ satisfies the boundary conditions,

$$
w_j^{(i-1)}(x_1) = \lim_{h \to 0} w_{jh}^{(i-1)}(x_1) = 0, \ 1 \leq i \leq n-1,
$$

and

$$w_j(x_2) - \sum_{k=1}^{m} r_k w_j(\eta_k) = \lim_{h\to 0}\left[w_j(x_2) - \sum_{k=1}^{m} r_k w_j(\eta_k)\right] = r_j u'(\eta_j).$$

This concludes the proof of (c).

It remains to verify part (d). Fix $1 \le j \le m$ as before and consider $\frac{\partial u}{\partial r_j}$. Again, let $\delta > 0$ be as in Theorem 3.2 and $0 < |h| < \delta$. Define

$$v_{jh}(x) = \frac{1}{h}[u(x, r_j + h) - u(x, r_j)],$$

where, for brevity, we designate $u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, r_j)$.

Note that

$$v_{jh}^{(i-1)}(x_1) = \frac{1}{h}[u_i - u_i] = 0,$$

for every $h \neq 0$ and $1 \le i \le n-1$.

Also, we see that

$$\begin{aligned}
v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) &= \frac{1}{h}\Big[u(x_2, r_j + h) - u(x_2, r_j) \\
&\quad - \sum_{k=1}^{m} r_k (u(\eta_k, r_j + h) - u(\eta_k, r_j))\Big] \\
&= \frac{1}{h}\Big[u(x_2, r_j + h) - u(x_2, r_j) \\
&\quad - \sum_{k=1}^{m} r_k u(\eta_k, r_j + h) + \sum_{k=1}^{m} r_k u(\eta_k, r_j)\Big] \\
&= \frac{1}{h}u(x_2, r_j + h) - \frac{1}{h}\sum_{k=1}^{m} r_k u(\eta_k, r_j + h) - \frac{u_n}{h} \\
&= \frac{1}{h}\Big[u(x_2, r_j + h) - \sum_{k\in\{1,\ldots,m\}\setminus\{j\}} r_k u(\eta_k, r_j + h) \\
&\quad - r_j u(\eta_j, r_j + h) - h u(\eta_j, r_j + h) \\
&\quad + h u(\eta_j, r_j + h)\Big] - \frac{u_n}{h} \\
&= \frac{1}{h}\Big[u(x_2, r_j + h) - \sum_{k\in\{1,\ldots,m\}\setminus\{j\}} r_k u(\eta_k, r_j + h) \\
&\quad - (r_j + h) u(\eta_j, r_j + h)\Big] + u(\eta_j, r_j + h) - \frac{u_n}{h}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{u_n}{h} + u(\eta_j, r_j + h) - \frac{u_n}{h} \\
&= u(\eta_j, r_j + h).
\end{aligned}
$$

And so by Theorem 3.2,

$$\lim_{h \to 0} \left[v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) \right] = u(\eta_j, r_j).$$

Now recall that, $u^{(n-2)}(x_1, r_j) = u_{n-1}$, and define

$$\beta = u^{(n-1)}(x_1, r_j),$$

and

$$\epsilon = \epsilon(h) = u^{(n-1)}(x_1, r_j + h) - \beta.$$

As usual, by Theorem 3.2, $\epsilon \to 0$ as $h \to 0$. Once again, using the notation for solutions of initial value problems for (3.7) and denoting $u(x) = y(x, x_1, u_1, \ldots, u_{n-1}, \beta)$ by $y(x, x_1, \beta)$, we have

$$v_{jh}(x) = \frac{1}{h}[y(x, x_1, \beta + \epsilon) - y(x, x_1, \beta)].$$

By Theorem 3.1 and the Mean Value Theorem,

$$
\begin{aligned}
v_{jh}(x) &= \frac{1}{h}\alpha_n(x, y(x, x_1, \beta + \bar{\epsilon}))(\beta + \epsilon - \beta) \\
&= \frac{\epsilon}{h}\alpha_n(x, y(x, x_1, \beta + \bar{\epsilon})),
\end{aligned}
$$

where $\alpha_n(x, y(\cdot))$ is the solution of (3.9) along $y(\cdot)$ satisfying

$$
\begin{aligned}
\alpha_n^{(i-1)}(x_1, y(\cdot)) &= 0, \ 1 \le i \le n-1, \\
\alpha_n^{(n-1)}(x_1, y(\cdot)) &= 1,
\end{aligned}
$$

and $\beta + \bar{\epsilon}$ lies between β and $\beta + \epsilon$. As in previous cases, it follows from assumption (v) that

$$\alpha_n(x_2, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot)) \neq 0.$$

Hence,

$$\frac{\epsilon}{h} = \frac{v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k)}{\alpha_n(x_2, y(x, x_1, \beta + \bar{\epsilon})) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(x, x_1, \beta + \bar{\epsilon}))},$$

and so from above,

$$\begin{aligned} \lim_{h\to 0} \frac{\epsilon}{h} &= \frac{r_j u(\eta_j)}{\alpha_n(x_2, y(x, x_1, \beta)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(x, x_1, \beta))} \\ &= \frac{r_j u(\eta_j)}{\alpha_n(x_2, u(x)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))} \\ &:= D. \end{aligned}$$

If we set $v_j(x) = \lim_{h\to 0} v_{jh}(x)$, we obtain

$$v_j(x) = \frac{\partial u}{\partial r_j}(x).$$

In particular,

$$\begin{aligned} v_j(x) &= \lim_{h\to 0} v_{jh}(x) \\ &= D\alpha_n(x, y(x, x_1, \beta)) \\ &= D\alpha_n(x, u(x)), \end{aligned}$$

which is a solution of (3.9) along $u(x)$.

In addition, $v_j(x)$ satisfies the boundary conditions

$$v_j(x_1) = \lim_{h\to 0} v_{jh}^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1,$$

and

$$v_j(x_2) - \sum_{k=1}^{m} r_k v_j(\eta_k) = \lim_{h\to 0} \left[v_j(x_2) - \sum_{k=1}^{m} r_k v_j(\eta_k) \right] = u(\eta_j).$$

This completes case (d), which in turn completes the proof of the theorem. □

3.4 Spread nth Order Problem

Once again, we will deal with the differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ x \in (a, b),\ n \geq 2, \tag{3.10}$$

but for this section we invoke what we call the spread boundary conditions

$$y(x_i) = u_i,\ 1 \leq i \leq n-1,\ \ y(x_n) - \sum_{p=1}^{m} r_p y(\eta_p) = u_n, \tag{3.11}$$

where $m \in \mathbb{N},\ a < x_1 < x_2 < \cdots < x_{n-1} < \eta_1 < \cdots < \eta_m < x_n < b$, and $u_1, \ldots, u_n, r_1, \ldots, r_m \in \mathbb{R}$. Given a solution $y(x)$ of (3.10), the *variational equation along* $y(x)$ associated to (3.10) is

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(x, y(x), y'(x), \ldots, y^{(n-1)}(x)) z^{(i-1)}. \tag{3.12}$$

We are now in a position to state the main result which was recently published by Henderson and Lyons in the International Journal of Pure and Applied Mathematics, [28]. For completeness, we include the proof.

Theorem 3.5. *Assume conditions* (i)-(v) *are satisfied. Let* $u(x)$ *be a solution* (3.10) *on* (a, b). *Let* $n \geq 2,\ m \in \mathbb{N}$, *and* $a < x_1 < x_2 < \cdots < x_{n-1} < \eta_1 < \cdots < \eta_m < x_n < b$ *and* $r_1, \ldots, r_m, u_1, \ldots, u_n \in \mathbb{R}$ *be given, so that*

$$u(x) = u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

where

$$u(x_i) = u_i,\ 1 \leq i \leq n-1, \qquad u(x_n) - \sum_{k=1}^{m} r_k u(\eta_k) = u_n.$$

Then,

(*a*) *for each* $1 \leq i \leq n$, $\dfrac{\partial u}{\partial u_i}(x)$ *exists on* (a, b). *Moreover, for each* $1 \leq j \leq n-1$, $y_j(x) := \dfrac{\partial u}{\partial u_j}(x)$ *solves* (3.12) *along* $u(x)$ *satisfying the boundary conditions*

$$y_j(x_i) = \delta_{ij},\ 1 \leq i \leq n-1,$$

$$y_j(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = 0,$$

and $y_n(x) := \dfrac{\partial u}{\partial u_n}(x)$ *solves* (3.12) *along* $u(x)$ *satisfying the boundary conditions*

$$y_n(x_i) = 0, \ 1 \le i \le n-1,$$
$$y_n(x_n) - \sum_{k=1}^{m} r_k y_n(\eta_k) = 1.$$

(*b*) *for each* $1 \le i \le n$, $\dfrac{\partial u}{\partial x_i}(x)$ *exists on* (a, b), *Moreover, for each* $1 \le j \le n-1$, $z_j(x) := \dfrac{\partial u}{\partial x_i}(x)$ *solves* (3.12) *along* $u(x)$ *satisfying the boundary conditions*

$$z_j(x_i) = -u'(x_i)\delta_{ij}, \ 1 \le i \le n-1,$$
$$z_j(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = 0,$$

and $z_n(x) := \dfrac{\partial u}{\partial x_n}(x)$ *solves* (3.12) *along* $u(x)$ *satisfying the boundary conditions*

$$z_n(x_i) = 0, \ 1 \le i \le n-1,$$
$$z_n(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = -u'(x_n).$$

(*c*) *for* $1 \le j \le m$, $\dfrac{\partial u}{\partial \eta_j}(x)$ *exists on* (a, b), *and* $w_j(x) := \dfrac{\partial u}{\partial \eta_j}(x)$ *is the solution of* (3.12) *along* $u(x)$ *satisfying*

$$w_j(x_i) = 0, \ 1 \le i \le n-1,$$
$$w_j(x_n) - \sum_{k=1}^{m} r_k w_j(\eta_k) = r_j u'(\eta_j).$$

(*d*) *for* $1 \le j \le m$, $\dfrac{\partial u}{\partial r_j}(x)$ *exists on* (a, b), *and* $v_j(x) := \dfrac{\partial u}{\partial r_j}(x)$ *is the solution of* (3.12) *along* $u(x)$ *satisfying*

$$v_j(x_i) = 0, \ 1 \le i \le n-1,$$

$$v_j(x_n) - \sum_{k=1}^{m} r_k v_j(\eta_k) = u(\eta_j).$$

For this theorem, we will give two different proofs. The first is the standard approach of looking at the solution of a boundary value problem as the solution of an initial value problem. The second proof uses a theorem of Henderson found in [23] to look at the solution of (3.10), (3.11) in terms of the solution of a related boundary value problem.

3.4.1 Initial Value Problem Proof

We now provide a proof of Theorem 3.5 using typical techniques.

Proof. For part (a), let $1 \leq j \leq n-1$, and consider $\frac{\partial u}{\partial u_j}$, since the argument for $\frac{\partial u}{\partial u_n}$ is similar, we withhold its proof. In this case we designate, for brevity, $u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, u_j)$.

Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define

$$y_{jh}(x) = \frac{1}{h}[u(x, u_j + h) - u(x, u_j)].$$

Note that $u(x_j, u_j + h) = u_j + h$, and $u(x_j, u_j) = u_j$, so that, for every $h \neq 0$,

$$y_{jh}(x_j) = \frac{1}{h}[u_j + h - u_j] = 1.$$

Also, for every $h \neq 0,\ 1 \leq i \leq n-1,\ i \neq j$,

$$\begin{aligned} y_{jh}(x_i) &= \frac{1}{h}[u(x_i, u_j + h) - u(x_i, u_j)] \\ &= \frac{1}{h}[u_i - u_i] = 0, \end{aligned}$$

and for $h \neq 0$,

$$\begin{aligned} y_{jh}(x_n) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k) =& \frac{1}{h}[u(x_n, u_j + h) - u(x_n, u_j)] \\ &- \sum_{k=1}^{m} \frac{r_k}{h}[u(\eta_k, u_j + h) - u(\eta_k, u_j)] \end{aligned}$$

$$= \frac{1}{h}[u_n - u_n] = 0.$$

For $2 \le i \le n$, let

$$\beta_i = u^{(i-1)}(x_j, u_j),$$

and

$$\epsilon_i = \epsilon_i(h) = u^{(i-1)}(x_j, u_j + h) - \beta_i.$$

By Theorem 3.2, for $2 \le i \le n$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.1 for solutions of initial value problems for (3.10), viewing the solution $u(x)$ as the solution of an initial value problem, and denoting the solution $u(x) = y(x, x_j, u_j, \beta_2, \beta_3 \ldots, \beta_n)$, we have

$$y_{jh}(x) = \frac{1}{h}[y(x, x_j, u_j + h, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} y_{jh}(x) &= \frac{1}{h}\big[y(x, x_j, u_j + h, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad - y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad + y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad - + \cdots \\ &\quad - y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &\quad + y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &\quad - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)\big]. \end{aligned}$$

By Theorem 3.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} y_{jh}(x) &= \frac{1}{h}\alpha_1(x, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n))(u_j + h - u_j) \\ &\quad + \frac{1}{h}\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n))(\beta_2 + \epsilon_2 - \beta_2) \\ &\quad + \cdots + \frac{1}{h}\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n))(\beta_n + \epsilon_n - \beta_n), \end{aligned}$$

where $\alpha_k(x, y(\cdot)),\ 1 \le k \le n$, is the solution of the variational equation (3.12) along $y(\cdot)$ satisfying,

$$\alpha_k^{(i-1)}(x_j) = \delta_{ik},\ 1 \le i \le n.$$

Furthermore, $u_j + \bar{h}$ is between u_j and $u_j + h$, and for $2 \le i \le n,\ \beta_i + \bar{\epsilon}_i$ is between β_i and $\beta_i + \epsilon_i$. Now simplifying,

$$\begin{aligned} y_{jh}(x) &= \alpha_1(x, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) \\ &\quad + \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n)) \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_n}{h}\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n)). \end{aligned}$$

Thus, to show $\lim_{h\to 0} y_{jh}(x)$ exists, it suffices to show, for $2 \le i \le n,\ \lim_{h\to 0}\frac{\epsilon_i}{h}$ exists. Now for $1 \le i \le n-1,\ i \ne j$,

$$0 = y_{jh}(x_i) = \alpha_1(x_i, y(\cdot)) + \frac{\epsilon_2}{h}\alpha_2(x_i, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_i, y(\cdot)),$$

and

$$\begin{aligned} 0 &= y_{jh}(x_n) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k) \\ &= \alpha_1(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(\cdot)) \\ &\quad + \frac{\epsilon_2}{h}\Big[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\Big] \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_n}{h}\Big[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\Big]. \end{aligned}$$

Hence, we have a system of $n-1$ equations with $n-1$ unknowns (note the x_jth equation is omitted):

$$-\alpha_1(x_1, y(\cdot)) = \frac{\epsilon_2}{h}\alpha_2(x_1, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_1, y(\cdot))$$

$$-\alpha_1(x_2, y(\cdot)) = \frac{\epsilon_2}{h}\alpha_2(x_2, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_2, y(\cdot))$$

$$\vdots$$

$$\begin{aligned}
-\alpha_1(x_n, y(\cdot)) \quad - & \sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(\cdot)) \\
= & \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\
& + \cdots \\
& + \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right].
\end{aligned}$$

Define the following matrices:

$$-\alpha := \begin{pmatrix} -\alpha_1(x_1, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) \\ -\alpha_1(x_2, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) \\ \vdots \\ -\alpha_1(x_n, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) - \\ \sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) \end{pmatrix}, \quad \epsilon := \begin{pmatrix} \frac{\epsilon_2}{h} \\ \frac{\epsilon_3}{h} \\ \vdots \\ \frac{\epsilon_n}{h} \end{pmatrix},$$

and

$$M(h) := \begin{pmatrix} \alpha_2(x_1, y(\cdot)) & \alpha_3(x_1, y(\cdot)) & \cdots & \alpha_n(x_1, y(\cdot)) \\ \alpha_2(x_2, y(\cdot)) & \alpha_3(x_2, y(\cdot)) & \cdots & \alpha_n(x_2, y(\cdot)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n, y(\cdot)) - & \alpha_3(x_n, y(\cdot)) - & & \alpha_n(x_n, y(\cdot)) - \\ \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot)) & \sum_{k=1}^{m} r_k \alpha_3(\eta_k, y(\cdot)) & \cdots & \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot)) \end{pmatrix}.$$

Then the system of equations written in its matrix form is

$$-\alpha = M(h)\epsilon.$$

Note that in the matrix $M(h)$, the solutions $y(\cdot)$ that each α is along are not identical. Thus we consider the matrix

$$M := \begin{pmatrix} \alpha_2(x_1, u(x)) & \alpha_3(x_1, u(x)) & \cdots & \alpha_n(x_1, u(x)) \\ \alpha_2(x_2, u(x)) & \alpha_3(x_2, u(x)) & \cdots & \alpha_n(x_2, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n, u(x)) - & \alpha_3(x_n, u(x)) - & & \alpha_n(x_n, u(x)) - \\ \sum_{k=1}^{m} r_k \alpha_2(\eta_k, u(x)) & \sum_{k=1}^{m} r_k \alpha_3(\eta_k, u(x)) & \cdots & \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_2, p_3, \ldots, p_n \in \mathbb{R}$ not all zero such that

$$p_2 \begin{pmatrix} \alpha_2(x_1, u(x)) \\ \alpha_2(x_2, u(x)) \\ \vdots \\ \alpha_2(x_n, u(x)) - \\ \sum r \alpha_2(\eta, u(x)) \end{pmatrix} + p_3 \begin{pmatrix} \alpha_3(x_1, u(x)) \\ \alpha_3(x_2, u(x)) \\ \vdots \\ \alpha_3(x_n, u(x)) - \\ \sum r \alpha_3(\eta, u(x)) \end{pmatrix} + \cdots + p_n \begin{pmatrix} \alpha_n(x_1, u(x)) \\ \alpha_n(x_2, u(x)) \\ \vdots \\ \alpha_n(x_n, u(x)) - \\ \sum r \alpha_n(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix},$$

where the limits of summation and the subscripts of r and η have been suppressed. Let

$$w(x, u(x)) := p_2 \alpha_2(x, u(x)) + p_3 \alpha_3(x, u(x)) + \cdots + p_n \alpha_n(x, u(x)).$$

Then

$$w(x_i, u(x)) = 0, \ 1 \leq i \leq n - 1,$$

and

$$w(x_n, u(x)) - \sum_{k=1}^{m} r_k w(\eta_k, u(x)) = 0,$$

which when coupled with hypothesis (v) yields $p_2 = p_3 = \cdots = p_n = 0$. This is a contradiction to the choice of p_i's. Hence $\det(M) \neq 0$ which means M has an inverse. Hence, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse, and therefore, we can solve $-\alpha = M(h)\epsilon$ by finding $[M(h)]^{-1}$ using Crammer's rule. Thus, we have $[M(h)]^{-1}\cdot(-\alpha) = \epsilon$ implying, again, as a result consequence of continuous dependence, $\lim_{h\to 0} \frac{\epsilon_i}{h} := A_i$ exists for $2 \le i \le n$.

Now let $y_j(x) = \lim_{h\to 0} y_{jh}(x)$, and note by construction of $y_{jh}(x)$,

$$y_j(x) = \frac{\partial u}{\partial u_j}(x).$$

Furthermore,

$$\begin{aligned} y_j(x) = \lim_{h\to 0} y_{jh}(x) &= \alpha_1(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\ &\quad + A_2\alpha_2(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\ &\quad + \cdots \\ &\quad + A_n\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\ &= \alpha_1(x, u(x)) + \sum_{i=2}^{n} A_i\alpha_i(x, u(x)), \end{aligned}$$

which is a solution of the variational equation (3.12) along $u(x)$. In addition,

$$y_j(x_i) = \lim_{h\to 0} y_{jh}(x_i) = \delta_{ij},\ 1 \le i \le n-1,$$

and

$$y_j(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = \lim_{h\to 0}\left[y_{jh}(x_n) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k)\right] = 0.$$

This completes the argument for $\frac{\partial u}{\partial u_j}$.

For part (b), let $1 \le j \le n-1$, and consider $\frac{\partial u}{\partial x_j}$, since the argument for $\frac{\partial u}{\partial x_n}$ is similar, we omit its proof. This time we designate $u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, x_j)$.

Let $\delta > 0$ be as in Theorem 3.2, let $0 < |h| < \delta$ be given, and define

$$z_{jh}(x) = \frac{1}{h}[u(x, x_j + h) - u(x, x_j)].$$

Note that for $h \neq 0$,

$$\begin{aligned} z_{jh}(x_j) &= \frac{1}{h}[u(x_j, x_j+h) - u(x_j, x_j)] \\ &= \frac{1}{h}[u(x_j, x_j+h) - u(x_j+h, x_j+h) \\ &\quad +u(x_j+h, x_j+h) - u_1] \\ &= -\frac{1}{h}[u(c_{x_j,h}, x_j+h) \cdot h] \\ &= -u'(c_{x_j,h}, x_j+h), \end{aligned}$$

where $c_{x_j,h}$ lies between x_j and $x_j + h$.

Also, for $1 \leq i \leq n-1,\ i \neq j$, and $h \neq 0$,

$$\begin{aligned} z_{jh}(x_i) &= \frac{1}{h}[u(x_i, x_j+h) - u(x_i, x_j)] \\ &= \frac{1}{h}[u_i - u_i] \\ &= 0. \end{aligned}$$

In addition,

$$\begin{aligned} z_{jh}(x_n) - \sum_{k=1}^{m} r_k z_{jh}(\eta_k) &= \frac{1}{h}[u(x_n, x_j+h) - \sum_{k=1}^{m} r_k u(\eta_k, x_j+h) \\ &\quad -\{u(x_n, x_j) - \sum_{k=1}^{m} r_k u(\eta_k, x_j)\}] \\ &= \frac{1}{h}[u_n - u_n] \\ &= 0, \end{aligned}$$

for every $h \neq 0$.

Next, for $2 \leq i \leq n$, let

$$\beta_i = u^{(i-1)}(x_j, x_j),$$

$$\epsilon_i = \epsilon_i(h) = u^{(i-1)}(x_j, x_j+h) - \beta_i,$$

and

$$\epsilon_1 = \epsilon_1(h) = u(x_j, x_j+h) - u_j.$$

By Theorem 3.2, for $1 \le i \le n$, $\epsilon_i \to 0$ as $h \to 0$. As in part (a), we employ the notation of Theorem 3.1 for solutions of initial value problems for (3.10). Viewing the solution $u(x)$ as the solution of an initial value problem, $u(x) = y(x, x_j, u_j, \beta_2, \beta_3, \ldots, \beta_n)$, and using a telescoping sum, we have

$$\begin{aligned} z_{jh}(x) &= \frac{1}{h}[y(x, x_j, u_j + \epsilon_1, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)] \\ &= \frac{1}{h}[y(x, x_j, u_j + \epsilon_1, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad -y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad +y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad - + \cdots \\ &\quad -y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &\quad +y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)]. \end{aligned}$$

Applying the Mean Value Theorem and Theorem 3.1,

$$\begin{aligned} z_{jh}(x) &= \frac{1}{h}[\epsilon_1 \alpha_1(x, y(x, x_j, u_j + \bar{\epsilon}_1, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) \\ &\quad + \cdots \\ &\quad +\epsilon_n \alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n))], \end{aligned}$$

where, for $1 \le i \le n$, $\bar{\epsilon}_i$ lies between β_i and $\beta_i + \epsilon_i$, and for $1 \le k \le n$, $\alpha_k(x, y(\cdot))$ is the solution of (3.12) along $y(\cdot)$ satisfying,

$$\alpha_k^{(i-1)}(x_j) = \delta_{ik},\ 1 \le i \le n.$$

Hence, to show $\lim_{h \to 0} z_{jh}(x)$ exists, it suffices to show for $1 \le i \le n$, $\lim_{h \to 0} \frac{\epsilon_i}{h}$ exists. From above,

$$\begin{aligned} \lim_{h \to 0} \frac{\epsilon_1}{h} &= \lim_{h \to 0} z_{jh}(x_j) \\ &= -\lim_{h \to 0} u'(c_{x_j,h}, x_j + h) \\ &= -u'(x_j). \end{aligned}$$

Now, for $1 \le i \le n-1,\ i \ne j$,

$$0 = z_{jh}(x_i) = \frac{\epsilon_1}{h}\alpha_1(x_i, y(\cdot)) + \frac{\epsilon_2}{h}\alpha_2(x_i, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_i, y(\cdot)),$$

and

$$\begin{aligned} 0 &= z_{jh}(x_n) - \sum_{k=1}^{m} r_k z_{jh}(\eta_k) \\ &= \frac{\epsilon_1}{h}\left[\alpha_1(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(\cdot))\right] + \\ &\quad \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right]. \end{aligned}$$

Hence, we have a system of $n-1$ equations with $n-1$ unknowns (note the x_jth equation is omitted):

$$u'(x_j)\alpha_1((x_1, y(\cdot)) = \frac{\epsilon_2}{h}\alpha_2(x_1, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_1, y(\cdot))$$

$$u'(x_j)\alpha_1((x_2, y(\cdot)) = \frac{\epsilon_2}{h}\alpha_2(x_2, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_2, y(\cdot))$$

$$\vdots$$

$$\begin{aligned} u'(x_j)\Big[\alpha_1(x_n, y(\cdot)) &- \sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(\cdot))\Big] \\ &= \frac{\epsilon_2}{h}\Big[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\Big] \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_n}{h}\Big[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\Big]. \end{aligned}$$

Now, we define the following matrices:

$$\alpha := \begin{pmatrix} u'(x_j)\alpha_1(x_1, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) \\ u'(x_j)\alpha_1(x_2, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) \\ \vdots \\ u'(x_j)\Big[\alpha_1(x_n, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) - \\ \sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)\Big] \end{pmatrix}, \quad \epsilon := \begin{pmatrix} \frac{\epsilon_2}{h} \\ \frac{\epsilon_3}{h} \\ \vdots \\ \frac{\epsilon_n}{h} \end{pmatrix},$$

and

$$M(h) := \begin{pmatrix} \alpha_2(x_1, y(\cdot)) & \alpha_3(x_1, y(\cdot)) & \cdots & \alpha_n(x_1, y(\cdot)) \\ \alpha_2(x_2, y(\cdot)) & \alpha_3(x_2, y(\cdot)) & \cdots & \alpha_n(x_2, y(\cdot)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n, y(\cdot)) - & \alpha_3(x_n, y(\cdot)) - & & \alpha_n(x_n, y(\cdot)) - \\ \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot)) & \sum_{k=1}^{m} r_k \alpha_3(\eta_k, y(\cdot)) & \cdots & \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot)) \end{pmatrix}.$$

Then, the system of equations written in its matrix form is

$$\alpha = M(h)\epsilon.$$

At this time, it is important to note that in the matrix $M(h)$ each α is not always along the same solution $y(\cdot)$. Therefore, we consider the matrix

$$M := \begin{pmatrix} \alpha_2(x_1, u(x)) & \alpha_3(x_1, u(x)) & \cdots & \alpha_n(x_1, u(x)) \\ \alpha_2(x_2, u(x)) & \alpha_3(x_2, u(x)) & \cdots & \alpha_n(x_2, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n, u(x)) - & \alpha_3(x_n, u(x)) - & & \alpha_n(x_n, u(x)) - \\ \sum_{k=1}^{m} r_k \alpha_2(\eta_k, u(x)) & \sum_{k=1}^{m} r_k \alpha_3(\eta_k, u(x)) & \cdots & \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist

$p_2, p_3, \ldots, p_n \in \mathbb{R}$ not all zero such that

$$p_2 \begin{pmatrix} \alpha_2(x_1, u(x)) \\ \alpha_2(x_2, u(x)) \\ \vdots \\ \alpha_2(x_n, u(x)) - \\ \sum r\alpha_2(\eta, u(x)) \end{pmatrix} + p_3 \begin{pmatrix} \alpha_3(x_1, u(x)) \\ \alpha_3(x_2, u(x)) \\ \vdots \\ \alpha_3(x_n, u(x)) - \\ \sum r\alpha_3(\eta, u(x)) \end{pmatrix} + \cdots + p_n \begin{pmatrix} \alpha_n(x_1, u(x)) \\ \alpha_n(x_2, u(x)) \\ \vdots \\ \alpha_n(x_n, u(x)) - \\ \sum r\alpha_n(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Note we have suppressed the subscripts of r and η as well as the limits of summation. Let

$$w(x, u(x)) := p_2\alpha_2(x, u(x)) + p_3\alpha_3(x, u(x)) + \cdots + p_n\alpha_n(x, u(x)).$$

Then

$$w(x_i, u(x)) = 0, \; 1 \leq i \leq n-1,$$

and

$$w(x_n, u(x)) - \sum_{k=1}^{m} r_k\alpha_n(\eta_k, u(x)) = 0,$$

which when coupled with hypothesis (v) gives $p_2 = p_3 = \cdots = p_n = 0$. This is a contradiction to the choice of p_i's. Hence $\det(M) \neq 0$ which means M has an inverse. Hence, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse, and therefore we can solve $\alpha = M(h)\epsilon$ by finding $[M(h)]^{-1}$ using Crammer's rule. Thus, we have $[M(h)]^{-1} \cdot \alpha = \epsilon$ implying as a result consequence of continuous dependence, $\lim_{h\to 0} \frac{\epsilon_i}{h} := B_i$ exists for $2 \leq i \leq n$.

Now let $z_j(x) = \lim_{h\to 0} z_{jh}(x)$, and note by construction of $z_{jh}(x)$,

$$z_j(x) = \frac{\partial u}{\partial x_j}(x).$$

Furthermore,

$$\begin{aligned} z_j(x) = \lim_{h\to 0} z_{jh}(x) &= -u'(x_j)\alpha_1(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\ &\quad + B_2\alpha_2(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \end{aligned}$$

$$
\begin{aligned}
&+\cdots \\
&+B_n\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\
= \quad & -u'(x_j)\alpha_1(x, u(x)) + \sum_{i=2}^{n} B_i\alpha_i(x, u(x)),
\end{aligned}
$$

which is a solution of the variational equation (3.12) along $u(x)$.

In addition, from above observations, $z_j(x)$ satisfies the boundary conditions

$$
z_j(x_i) = \lim_{h\to 0} z_{jh}(x_i) = -\delta_{ij}u'(x_j),\ 1 \le i \le n-1,
$$

and

$$
z_j(x_n) - \sum_{k=1}^{m} r_k z_j(\eta_k) = \lim_{h\to 0}\left[z_{jh}(x_2) - \sum_{k=1}^{m} r_k z_{jh}(\eta_k)\right] = 0.
$$

This completes the proof for $\frac{\partial u}{\partial x_j}$.

For (c), we fix $1 \le j \le m$, and this time we designate

$u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1,\ \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, \eta_j)$.

Let $\delta > 0$ be as in Theorem 3.2, let $0 < |h| < \delta$ be given, and define

$$
w_{jh}(x) = \frac{1}{h}[u(x, \eta_j + h) - u(x, \eta_j)].
$$

Note that for $h \neq 0$,

$$
\begin{aligned}
w_{jh}(x_j) \quad - \quad & \sum_{k=1}^{m} r_k w_{jh}(\eta_k) \\
= \quad & \frac{1}{h}\Big[u(x_j, \eta_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j + h) \\
& -u(x_j, \eta) + \sum_{k=1}^{m} r_k u(\eta_k, \eta_j)\Big] \\
= \quad & \frac{1}{h}\Big[u(x_j, \eta_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j + h) \\
& -r_j u(\eta_j + h, \eta_j + h) + r_j u(\eta_j + h, \eta_j + h) - u_n\Big] \\
= \quad & \frac{r_j}{h}[u(c_{\eta_{j,h}}, \eta_j + h) \cdot h] \\
= \quad & r_j u'(c_{\eta_{j,h}}, \eta_j + h),
\end{aligned}
$$

where $c_{\eta_j,h}$ lies between η_j and $\eta_j + h$. Also, for $1 \le i \le n-1$ and $h \ne 0$

$$\begin{aligned} w_{jh}(x_i) &= \frac{1}{h}[u(x_i, \eta_j + h) - u(x_i, \eta_j)] \\ &= \frac{1}{h}[u_i - u_i] \\ &= 0. \end{aligned}$$

Next, for $2 \le i \le n$, let

$$\beta_i = u^{(i-1)}(x_j, \eta_j),$$

and

$$\epsilon_i = \epsilon_i(h) = u^{(i-1)}(x_j, \eta_j + h) - \beta_i.$$

By Theorem 3.2, for $2 \le i \le n$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. We employ the notation of Theorem 3.1 for solutions of initial value problems for (3.10). Viewing the solution $u(x)$ as the solution of an initial value problem, $u(x) = y(x, x_j, u_j, \beta_2, \beta_3, \ldots, \beta_n)$, and using a telescoping sum, we have

$$\begin{aligned} w_{jh}(x) &= \frac{1}{h}[y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)] \\ &= \frac{1}{h}[y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad - y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &\quad + y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &\quad - + \cdots \\ &\quad - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)]. \end{aligned}$$

Then, by the Mean Value Theorem and Theorem 3.1,

$$\begin{aligned} w_{jh}(x) &= \frac{1}{h}[\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n))(\beta_2 + \epsilon_2 - \beta_2) \\ &\quad + \cdots \\ &\quad + \alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n))(\beta_n + \epsilon_n - \beta_n)] \\ &= \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n)) \end{aligned}$$

$$+\cdots$$

$$+\frac{\epsilon_n}{h}\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n)),$$

where, for $2 \le i \le n,\ \bar{\epsilon}_i$ lies between β_i and $\beta_i + \epsilon_i$, and, for $1 \le k \le n,\ \alpha_k(x, y(\cdot))$ is the solution of (3.12) along $y(\cdot)$ satisfying

$$\alpha_k^{(i-1)}(x_j) = \delta_{ik},\ 1 \le i \le n.$$

Thus, to show $\lim_{h\to 0} w_{jh}(x)$ exists, it suffices to show, for $2 \le i \le n,\ \lim_{h\to 0}\frac{\epsilon_i}{h}$ exists. Now for $1 \le i \le n-1,\ i \ne j$,

$$0 = w_{jh}(x_i) = \frac{\epsilon_2}{h}\alpha_2(x_i, y(\cdot)) + \frac{\epsilon_3}{h}\alpha_3(x_i, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_i, y(\cdot)),$$

and

$$\begin{aligned} r_j u'(c_{\eta_j,h}, \eta_j + h) &= w_{jh}(x_n) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k) \\ &= \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\ &\quad + \frac{\epsilon_3}{h}\left[\alpha_3(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_3(\eta_k, y(\cdot))\right] \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right]. \end{aligned}$$

Hence, we have a system of $n-1$ equations with $n-1$ unknowns (note the x_jth equation is omitted):

$$0 = \frac{\epsilon_2}{h}\alpha_2(x_1, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_1, y(\cdot)),$$

$$0 = \frac{\epsilon_2}{h}\alpha_2(x_2, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_2, y(\cdot)),$$

$$\vdots$$

$$r_j u'(c_{\eta_j,h}, \eta_j + h) = \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right]$$

$$+\cdots$$
$$+\frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right].$$

Now, define the following matrices:

$$\alpha := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ r_j u'(c_{\eta_j,h}, \eta_j + h) \end{pmatrix}, \quad \epsilon := \begin{pmatrix} \frac{\epsilon_2}{h} \\ \frac{\epsilon_3}{h} \\ \vdots \\ \frac{\epsilon_n}{h} \end{pmatrix},$$

and

$$M(h) := \begin{pmatrix} \alpha_2(x_1, y(\cdot)) & \alpha_3(x_1, y(\cdot)) & \cdots & \alpha_n(x_1, y(\cdot)) \\ \alpha_2(x_2, y(\cdot)) & \alpha_3(x_2, y(\cdot)) & \cdots & \alpha_n(x_2, y(\cdot)) \\ \vdots & \vdots & \ddots & \vdots \\ \begin{array}{c}\alpha_2(x_n, y(\cdot)) - \\ \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\end{array} & \begin{array}{c}\alpha_3(x_n, y(\cdot)) - \\ \sum_{k=1}^{m} r_k \alpha_3(\eta_k, y(\cdot))\end{array} & \cdots & \begin{array}{c}\alpha_n(x_n, y(\cdot)) - \\ \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\end{array} \end{pmatrix}.$$

Then our system of equations written in its matrix form is

$$\alpha = M(h)\epsilon.$$

As in the previous parts, the solutions $y(\cdot)$ need not be identical. Hence, we consider the matrix

$$M := \begin{pmatrix} \alpha_2(x_1, u(x)) & \alpha_3(x_1, u(x)) & \cdots & \alpha_n(x_1, u(x)) \\ \alpha_2(x_2, u(x)) & \alpha_3(x_2, u(x)) & \cdots & \alpha_n(x_2, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \begin{array}{c}\alpha_2(x_n, u(x)) - \\ \sum_{k=1}^{m} r_k \alpha_2(\eta_k, u(x))\end{array} & \begin{array}{c}\alpha_3(x_n, u(x)) - \\ \sum_{k=1}^{m} r_k \alpha_3(\eta_k, u(x))\end{array} & \cdots & \begin{array}{c}\alpha_n(x_n, u(x)) - \\ \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))\end{array} \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist

$p_2, p_3, \ldots, p_n \in \mathbb{R}$ not all zero such that

$$p_2 \begin{pmatrix} \alpha_2(x_1, u(x)) \\ \alpha_2(x_2, u(x)) \\ \vdots \\ \alpha_2(x_n, u(x)) - \\ \sum r\alpha_2(\eta, u(x)) \end{pmatrix} + p_3 \begin{pmatrix} \alpha_3(x_1, u(x)) \\ \alpha_3(x_2, u(x)) \\ \vdots \\ \alpha_3(x_n, u(x)) - \\ \sum r\alpha_3(\eta, u(x)) \end{pmatrix} + \cdots + p_n \begin{pmatrix} \alpha_n(x_1, u(x)) \\ \alpha_n(x_2, u(x)) \\ \vdots \\ \alpha_n(x_n, u(x)) - \\ \sum r\alpha_n(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix},$$

where the subscripts of r and η and the limits of summation have been suppressed. Let

$$w(x, u(x)) := p_2\alpha_2(x, u(x)) + p_3\alpha_3(x, u(x)) + \cdots + p_n\alpha_n(x, u(x)).$$

Then

$$w(x_i, u(x)) = 0, \ 1 \leq i \leq n-1,$$

and

$$w(x_n, u(x)) - \sum_{k=1}^{m} r_k\alpha_n(\eta_k, u(x)) = 0,$$

which when coupled with hypothesis (v) gives $p_2 = p_3 = \cdots = p_n = 0$. This is a contradiction to the choice of p_i's. Hence $\det(M) \neq 0$ which means M has an inverse. Thus, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse, and therefore we can solve $\alpha = M(h)\epsilon$ by finding $[M(h)]^{-1}$ using Crammer's rule. So, we have $[M(h)]^{-1} \cdot \alpha = \epsilon$ implying as a result consequence of continuous dependence, $\lim_{h \to 0} \epsilon_i/h := C_i$ exists for $2 \leq i \leq n$.

Now let $w_j(x) = \lim_{h \to 0} w_{jh}(x)$, and note by construction of $w_{jh}(x)$,

$$w_j(x) = \frac{\partial u}{\partial \eta_j}(x).$$

Furthermore,

$$\begin{aligned} w_j(x) &= \lim_{h \to 0} w_{jh}(x) \\ &= \sum_{i=2}^{n} C_i\alpha_i(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \end{aligned}$$

$$= \sum_{i=2}^{n} C_i \alpha_i(x, u(x)),$$

which is a solution of the variational equation (3.12) along $u(x)$.

In addition, from above observations, $w_j(x)$ satisfies the boundary conditions

$$w_j(x_i) = \lim_{h \to 0} w_{jh}(x_i) = 0, \ 1 \leq i \leq n-1,$$

and

$$w_j(x_n) - \sum_{k=1}^{m} r_k w_j(\eta_k)) = \lim_{h \to 0} \left[w_{jh}(x_n) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k) \right] = r_j u'(\eta_j).$$

This completes the proof for $\frac{\partial u}{\partial \eta_j}$.

It remains to verify part (d). Fix $1 \leq j \leq m$ as before and consider $\frac{\partial u}{\partial r_j}$. Again, let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, denote $u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, r_j)$, and define

$$v_{jh}(x) = \frac{1}{h}[u(x, r_j + h) - u(x, r_j)].$$

Note that for $h \neq 0$,

$$\begin{aligned}
v_{jh}(x_j) \quad &- \quad \sum_{k=1}^{m} r_k v_{jh}(\eta_k) \\
&= \frac{1}{h}\Big[u(x_j, r_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, r_j + h) \\
&\quad - u(x_j, r_j) + \sum_{k=1}^{m} r_k u(\eta_k, r_j)\Big] \\
&= \frac{1}{h}\Big[u(x_j, r_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, r_j + h) \\
&\quad - hu(\eta_j, r_j + h) + hu(\eta_j, r_j + h) - u_n\Big] \\
&= u(\eta_j, r_j + h).
\end{aligned}$$

Also, for $1 \leq i \leq n-1$ and $h \neq 0$

$$v_{jh}(x_i) = \frac{1}{h}[u(x_i, r_j + h) - u(x_i, r_j)]$$

$$= \frac{1}{h}[u_i - u_i]$$
$$= 0.$$

Now, for $2 \leq i \leq n$, let

$$\beta_i = u^{(i-1)}(x_j, r_j),$$

and

$$\epsilon_i = \epsilon_i(h) = u^{(i-1)}(x_j, r_j + h) - \beta_i.$$

By Theorem 3.2, for $2 \leq i \leq n$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. We employ the notation of Theorem 3.1 for solutions of initial value problems for (3.10). Viewing the solution $u(x)$ as the solution of an initial value problem, $u(x) = y(x, x_j, u_j, \beta_2, \beta_3, \ldots, \beta_n)$, and using a telescoping sum, we have

$$\begin{aligned} v_{jh}(x) &= \frac{1}{h}[y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad -y(x, x_j, u_j, \beta_2, \ldots, \beta_n)] \\ &= \frac{1}{h}[y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &\quad -y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &\quad +y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &\quad - + \cdots \\ &\quad -y(x, x_j, u_j, \beta_2, \ldots, \beta_n)]. \end{aligned}$$

By the Mean Value Theorem and Theorem 3.1,

$$\begin{aligned} v_{jh}(x) &= \frac{1}{h}[\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n))(\beta_2 + \epsilon_2 - \beta_2) \\ &\quad + \cdots \\ &\quad +\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n))(\beta_n + \epsilon_n - \beta_n)] \\ &= \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n)) \\ &\quad + \cdots \end{aligned}$$

$$+\frac{\epsilon_n}{h}\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n)),$$

where for $2 \leq i \leq n$, $\beta_i + \bar{\epsilon}_i$ lies between β_i and $\beta_i + \epsilon_i$ and, for $1 \leq k \leq n$, $\alpha_k(x, y(\cdot))$ is the solution of (3.12) along $y(\cdot)$ satisfying

$$\alpha_k^{(i-1)}(x_j) = \delta_{ik},\ 1 \leq i \leq n.$$

Therefore, to show $\lim_{h \to 0} v_{jh}(x)$ exists, it suffices to show, for $2 \leq i \leq n$, $\lim_{h \to 0} \frac{\epsilon_i}{h}$ exists.

Now for $1 \leq i \leq n-1$, $i \neq j$,

$$0 = v_{jh}(x_i) = \frac{\epsilon_2}{h}\alpha_2(x_i, y(\cdot)) + \frac{\epsilon_3}{h}\alpha_3(x_i, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_i, y(\cdot)),$$

and

$$\begin{aligned} u(\eta_j, r_j + h) &= v_{jh}(x_n) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) \\ &= \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\ &\quad + \frac{\epsilon_3}{h}\left[\alpha_3(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_3(\eta_k, y(\cdot))\right] \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right]. \end{aligned}$$

Hence, we have a system of $n-1$ equations with $n-1$ unknowns (note the x_jth equation is omitted):

$$0 = \frac{\epsilon_2}{h}\alpha_2(x_1, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_1, y(\cdot)),$$

$$0 = \frac{\epsilon_2}{h}\alpha_2(x_2, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_2, y(\cdot)),$$

$$\vdots$$

$$u(\eta_j, r_j + h) = \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right]$$

$$+\cdots$$
$$+\frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right].$$

Now, define the following matrices:

$$\alpha := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ u(\eta_j, r_j + h) \end{pmatrix}, \quad \epsilon := \begin{pmatrix} \frac{\epsilon_2}{h} \\ \frac{\epsilon_3}{h} \\ \vdots \\ \frac{\epsilon_n}{h} \end{pmatrix},$$

and

$$M(h) := \begin{pmatrix} \alpha_2(x_1, y(\cdot)) & \alpha_3(x_1, y(\cdot)) & \cdots & \alpha_n(x_1, y(\cdot)) \\ \alpha_2(x_2, y(\cdot)) & \alpha_3(x_2, y(\cdot)) & \cdots & \alpha_n(x_2, y(\cdot)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n, y(\cdot)) - & \alpha_3(x_n, y(\cdot)) - & & \alpha_n(x_n, y(\cdot)) - \\ \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot)) & \sum_{k=1}^{m} r_k \alpha_3(\eta_k, y(\cdot)) & \cdots & \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot)) \end{pmatrix}.$$

Then our system of equations written in its matrix form is

$$\alpha = M(h)\epsilon.$$

As $y(\cdot)$ does not have to be the consistant throughout $M(h)$, we instead, consider the matrix

$$M := \begin{pmatrix} \alpha_2(x_1, u(x)) & \alpha_3(x_1, u(x)) & \cdots & \alpha_n(x_1, u(x)) \\ \alpha_2(x_2, u(x)) & \alpha_3(x_2, u(x)) & \cdots & \alpha_n(x_2, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n, u(x)) - & \alpha_3(x_n, u(x)) - & & \alpha_n(x_n, u(x)) - \\ \sum_{k=1}^{m} r_k \alpha_2(\eta_k, u(x)) & \sum_{k=1}^{m} r_k \alpha_3(\eta_k, u(x)) & \cdots & \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_2, p_3, \ldots, p_n \in \mathbb{R}$ not all zero such that

$$p_2 \begin{pmatrix} \alpha_2(x_1, u(x)) \\ \alpha_2(x_2, u(x)) \\ \vdots \\ \alpha_2(x_n, u(x)) - \\ \sum r\alpha_2(\eta, u(x)) \end{pmatrix} + p_3 \begin{pmatrix} \alpha_3(x_1, u(x)) \\ \alpha_3(x_2, u(x)) \\ \vdots \\ \alpha_3(x_n, u(x)) - \\ \sum r\alpha_3(\eta, u(x)) \end{pmatrix} + \cdots + p_n \begin{pmatrix} \alpha_n(x_1, u(x)) \\ \alpha_n(x_2, u(x)) \\ \vdots \\ \alpha_n(x_n, u(x)) - \\ \sum r\alpha_n(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix},$$

where we have suppressed the subscripts of r and η and the limits of summation. Let

$$w(x, u(x)) := p_2\alpha_2(x, u(x)) + p_3\alpha_3(x, u(x)) + \cdots + p_n\alpha_n(x, u(x)).$$

Then

$$w(x_i, u(x)) = 0, \ 1 \le i \le n-1,$$

and

$$w(x_n, u(x)) - \sum_{k=1}^{m} r_k\alpha_n(\eta_k, u(x)) = 0,$$

which when coupled with hypothesis (v) yields $p_2 = p_3 = \cdots = p_n = 0$. This is a contradiction to the choice of p_i's. Hence $\det(M) \neq 0$ which means M has an inverse. Thus, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse, and therefore we can solve $\alpha = M(h)\epsilon$ by finding $[M(h)]^{-1}$ using Crammer's rule. So, we have $[M(h)]^{-1} \cdot \alpha = \epsilon$ implying, again, as a result consequence of continuous dependence, $\lim_{h \to 0} \epsilon_i/h := D_i$ exists for $2 \le i \le n$.

Now let $v_j(x) = \lim_{h \to 0} v_{jh}(x)$, and note by construction of $v_{jh}(x)$,

$$v_j(x) = \frac{\partial u}{\partial r_j}(x).$$

Furthermore,

$$v_j(x) \ = \ \lim_{h \to 0} v_{jh}(x)$$

$$
\begin{aligned}
&= \sum_{i=2}^{n} D_i\alpha_i(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\
&= \sum_{i=2}^{n} D_i\alpha_i(x, u(x)),
\end{aligned}
$$

which is a solution of the variational equation (3.12) along $u(x)$.

In addition, from above observations, $w_j(x)$ satisfies the boundary conditions

$$
v_j(x_i) = \lim_{h\to 0} v_{jh}(x_i) = 0,\ 1 \le i \le n-1,
$$

and

$$
v_j(x_n) - \sum_{k=1}^{m} r_k v_j(\eta_k)) = \lim_{h\to 0}\left[v_{jh}(x_n) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k)\right] = u(\eta_j).
$$

This completes the proof for $\frac{\partial u}{\partial r_j}$. □

3.4.2 Boundary Value Problem Proof

Our goal in this section is to apply the following theorem of Henderson:

Theorem 3.6. *[Henderson] Assume that* (3.10) *and the variational equation* (3.12) *along solutions $y(x)$ of* (3.10) *are disconjugate on (a, b). Let $u(x)$ be a solution of* (3.10) *on (a, b). Let $a < x_1 < \cdots < x_n < b$ and $u_1, \ldots, u_n \in \mathbb{R}$ be given, so that*

$$
u(x) = u(x, x_1, \ldots, x_n, u_1, \ldots, u_n),
$$

where

$$
u(x_j) = u_j,\ 1 \le j \le n.
$$

Then,

(a) *for each $1 \le j \le n$, $\dfrac{\partial u}{\partial u_j}(x)$ exists on (a, b) and $\alpha_j := \dfrac{\partial u}{\partial u_j}(x)$ is the solution of the variational equation* (3.12) *along $u(x)$ satisfying the boundary conditions*

$$
\alpha_j(x_i) = \delta_{ij},\ 1 \le i \le n.
$$

(b) for each $1 \le j \le n$, $\dfrac{\partial u}{\partial x_j}(x)$ *exists on* (a,b) *and* $\beta_j := \dfrac{\partial u}{\partial x_j}(x)$ *is the solution of the variational equation* (3.12) *along* $u(x)$ *satisfying the boundary conditions*

$$\beta_j(x_i) = -\delta_{ij}u'(x_i),\ 1 \le i \le n.$$

Theorem 3.6 requires disconjugacy on (a,b) for (3.10) and (3.12) along all solutions $y(x)$ of (3.10), but recall from the preliminaries section that if we assume conditions (iv) and (v), we automatically have disconjugacy for (3.10) and (3.12) along solutions $y(x)$ of (3.10). Therefore, Theorem 3.6 may be applied.

Now we begin the proof of Theorem 3.5 using the BVP method.

Proof. For part (a), first let $1 \le j \le n-1$, and consider $\frac{\partial u}{\partial u_j}$. Let $\delta > 0$ be as in Theorem 3.2 and $0 < |h| < \delta$ be given. Define

$$\begin{aligned} y_{jh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad - u(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)]. \end{aligned}$$

Note that $u(x_j, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) = u_j + h$, and $u(x_j, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) = u_j$, so that, for every $h \ne 0$,

$$\begin{aligned} y_{jh}(x_j) &= \frac{1}{h}[u(x_j, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad - u(x_j, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)] \\ &= \frac{1}{h}[u_j + h - u_j] \\ &= 1. \end{aligned}$$

Also, for every $h \ne 0,\ 1 \le i \le n-1$, and $i \ne j$,

$$\begin{aligned} y_{jh}(x_i) &= \frac{1}{h}[u(x_i, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad - u(x_i, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}[u_i - u_i] \\
&= 0,
\end{aligned}
$$

and for $h \neq 0$,

$$
\begin{aligned}
y_{jh}(x_n) &- \sum_{k=1}^{m} r_k y_{jh}(\eta_k) \\
&= \frac{1}{h}[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad -u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)] \\
&\quad -\sum_{k=1}^{m} \frac{r_k}{h}\big[u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad -u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)\big] \\
&= \frac{1}{h}[u_n - u_n] \\
&= 0.
\end{aligned}
$$

Let

$$\beta = u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

and

$$\epsilon = u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) - \beta.$$

By Theorem 3.2, $\epsilon = \epsilon(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.6 for solutions of conjugate boundary value problems for (3.10), viewing the solution $u(x)$ as the solution of a conjugate boundary value problem, and denoting the solution $u(x) = z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, \beta)$, we have

$$\begin{aligned} y_{jh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad -z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta)]. \end{aligned}$$

Then, by utilizing a telescoping sum, the Mean Value Theorem, and Theorem 3.6, we obtain

$$\begin{aligned} y_{jh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad -z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta)] \\ &= \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + h, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad -z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad +z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad -z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta)] \\ &= \frac{1}{h}[\alpha_j(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + \bar{h}, \ldots, u_{n-1}, \beta))(u_j + h - u_j) \\ &\quad +\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta + \bar{\epsilon}))(\beta + \epsilon - \beta)] \\ &= \alpha_j(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + \bar{h}, \ldots, u_{n-1}, \beta + \epsilon)) \\ &\quad +\frac{\epsilon}{h}\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta + \bar{\epsilon})), \end{aligned}$$

where $\alpha_j(x, z(\cdot))$ and $\alpha_n(x, z(\cdot))$ are the solutions of the variational equation (3.12) along $z(\cdot)$ satisfying, respectively,

$$\alpha_j(x_i, z(\cdot)) = \delta_{ij},\ 1 \leq i \leq n,$$

$$\alpha_n(x_i, z(\cdot)) = \delta_{in},\ 1 \leq i \leq n.$$

Furthermore, $u_j + \bar{h}$ lies between u_j and $u_j + h$, and $\beta + \bar{\epsilon}$ lies between β and $\beta + \epsilon$.

Therefore, to show $\lim_{h \to 0} y_{jh}(x)$ exists, we must show $\lim_{h \to 0} \epsilon/h$ exists. Well, we have

$$\begin{aligned}
0 &= y_{jh}(x_n) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k) \\
&= \Big[\alpha_j(x_n, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + \bar{h}, \ldots, u_{n-1}, \beta + \epsilon)) \\
&\quad - \sum_{k=1}^{m} r_k \alpha_j(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + \bar{h}, \ldots, u_{n-1}, \beta + \epsilon))\Big]. \\
&\quad + \frac{\epsilon}{h}\Big[\alpha_n(x_n, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta + \bar{\epsilon})) \\
&\quad - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big] \\
&= 0 - \sum_{k=1}^{m} r_k \alpha_j(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + \bar{h}, \ldots, u_{n-1}, \beta + \epsilon)) \\
&\quad + \frac{\epsilon}{h}\Big[1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big].
\end{aligned}$$

When we solve for ϵ/h and take the limit by applying Theorem 3.2, we find

$$\begin{aligned}
\lim_{h \to 0} \frac{\epsilon}{h} &= \lim_{h \to 0} \frac{\sum_{k=1}^{m} r_k \alpha_j(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j + \bar{h}, \ldots, u_{n-1}, \beta + \epsilon))}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta + \bar{\epsilon}))} \\
&= \frac{\sum_{k=1}^{m} r_k \alpha_j(\eta_k, z(\cdot))}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(\cdot))} \\
&= \frac{\sum_{k=1}^{m} r_k \alpha_j(\eta_k, u(x))}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))} := A,
\end{aligned}$$

provided the denominator is nonzero which we are assured by condition (v).

So we have

$$\begin{aligned} y_j(x) &= \lim_{h\to 0} y_{jh}(x) \\ &= \alpha_j(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, \beta)) \\ &\quad + A\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_j, \ldots, \beta)) \\ &= \alpha_j(x, u(x)) + A\alpha_n(x, u(x)), \end{aligned}$$

which is a solution of the variational equation (3.12) along $u(x)$. In addition,

$$y_j(x_i) = \lim_{h\to 0} y_{jh}(x_i) = \delta_{ij},\ 1 \le i \le n-1,$$

$$y_j(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = \lim_{h\to 0} \left[y_{jh}(x_n) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k) \right] = 0.$$

This completes the argument for $\frac{\partial u}{\partial u_j}$, $1 \le i \le n-1$.

Now consider, $\frac{\partial u}{\partial u_n}$. Let $\delta > 0$ be as in Theorem 3.2 and $0 < |h| < \delta$ be given. Define

$$\begin{aligned} y_{nh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, u_n + h, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad - u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)]. \end{aligned}$$

Note that for every $h \neq 0$ and $1 \le i \le n-1$,

$$\begin{aligned} y_{nh}(x_i) &= \frac{1}{h}[u(x_i, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, u_n + h, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad - u(x_i, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)] \\ &= \frac{1}{h}[u_i - u_i] \\ &= 0. \end{aligned}$$

Also for every $h \neq 0$,

$$\begin{aligned} y_{nh}(x_n) &- \sum_{k=1}^{m} r_k y_{nh}(\eta_k) \\ &= \frac{1}{h}\Big[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, u_n + h, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \end{aligned}$$

$$- \sum_{k=1}^{m} r_k u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, u_n + h, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)\Big]$$

$$- \frac{1}{h}\Big[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$$

$$- \sum_{k=1}^{m} r_k u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)\Big]$$

$$= \frac{1}{h}[u_n + h - u_n]$$

$$= 1.$$

Let

$$\beta = u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

and

$$\epsilon = u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, u_n + h, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) - \beta.$$

By Theorem 3.2, $\epsilon = \epsilon(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.6 for solutions of conjugate boundary value problems, viewing the solution $u(x)$ as the solution of a conjugate boundary value problem, and denoting the solution $u(x) = z(x, x_1, \ldots, x_n, u_1, \ldots, \beta)$, we have

$$\begin{aligned} y_{nh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad - z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)]. \end{aligned}$$

Then, by the Mean Value Theorem and Theorem 3.6, we obtain,

$$\begin{aligned} y_{nh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots u_{n-1}, \beta + \epsilon) \\ &\quad - z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)] \\ &= \frac{1}{h}\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))(\beta + \epsilon - \beta) \\ &= \frac{\epsilon}{h}\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon})), \end{aligned}$$

where $\alpha_n(x, z(\cdot))$ is the solution of the variational equation along $z(\cdot)$ satisfying

$$\alpha_n(x_i, z(\cdot)) = \delta_{in},\ 1 \le i \le n.$$

Furthermore, $\beta + \bar{\epsilon}$ lies between β and $\beta + \epsilon$.

Thus, to show $\lim_{h\to 0} y_{nh}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\epsilon}{h}$ exists. Now,

$$\begin{aligned} 1 &= y_{nh}(x_n) - \sum_{k=1}^{m} r_k y_{nh}(\eta_k) \\ &= \frac{\epsilon}{h}\Big[\alpha_n(x_n, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon})) \\ &\quad - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big] \\ &= \frac{\epsilon}{h}\Big[1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big] \end{aligned}$$

Hence, we find, when solving for ϵ/h and taking the limit by Theorem 3.2,

$$\begin{aligned} \lim_{h\to 0} \frac{\epsilon}{h} &= \lim_{h\to 0} \frac{1}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))} \\ &= \frac{1}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(\cdot))} \\ &= \frac{1}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))} := A, \end{aligned}$$

provided the denominator is nonzero which we are guaranteed by hypothesis (v).

Now let $y_n(x) = \lim_{h\to 0} y_{nh}(x)$, and note by construction of $y_{nh}(x)$,

$$y_n(x) = \frac{\partial u}{\partial u_n}(x).$$

So we have

$$\begin{aligned} y_n(x) &= \lim_{h\to 0} y_{nh}(x) \\ &= A\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)) \\ &= A\alpha_n(x, u(x)), \end{aligned}$$

which is a solution of the variational equation (3.12) along $u(x)$. In addition,

$$y_n(x_i) = \lim_{h\to 0} y_{nh}(x_i) = 0,\ 1 \le i \le n-1,$$

and

$$y_n(x_n) - \sum_{k=1}^{m} r_k y_n(\eta_k) = \lim_{h\to 0}\left[y_{nh}(x_n) - \sum_{k=1}^{m} r_k y_{nh}(\eta_k)\right] = 1.$$

This completes the argument for $\frac{\partial u}{\partial u_n}$, and hence, part (a).

For part (b), let $1 \le j \le n-1$ and consider $\frac{\partial u}{\partial x_j}$. Let $\delta > 0$ be as in Theorem 3.2 and $0 < |h| < \delta$ be given. Define

$$\begin{aligned} z_{jh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad -u(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)]. \end{aligned}$$

Then, for every $h \neq 0$, we have

$$\begin{aligned} z_{jh}(x_j) &= \frac{1}{h}[u(x_j, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad -u(x_j, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)] \\ &= \frac{1}{h}[u(x_j, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad -u(x_j + h, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad +u(x_j + h, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \\ &\qquad \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) - u_j] \\ &= -\frac{1}{h}[u'(c_{x_j,h}, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \\ &\qquad \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \cdot h + u_j - u_j] \\ &= -u'(c_{x_j,h}, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m), \end{aligned}$$

where $c_{x_j,h}$ lies between x_j and $x_j + h$.

In addition, for every $h \neq 0,\ 1 \le i \le n-1$, and $i \neq j$,

$$\begin{aligned} z_{jh}(x_i) &= \frac{1}{h}[u(x_i, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad -u(x_i, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)] \\ &= \frac{1}{h}[u_i - u_i] \\ &= 0, \end{aligned}$$

and for $h \neq 0$,

$$\begin{aligned} z_{jh}(x_n) &- \sum_{k=1}^{m} r_k z_{jh}(\eta_k) \\ &= \frac{1}{h}[u(x_n, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad -u(x_n, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)] \\ &\quad -\sum_{k=1}^{m} \frac{r_k}{h}\Big[u(\eta_k, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad -u(\eta_k, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)\Big] \\ &= \frac{1}{h}[u_n - u_n] \\ &= 0. \end{aligned}$$

Now define

$$\beta = u(x_n, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

and

$$\epsilon = u(x_n, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) - \beta.$$

Note that $\epsilon = \epsilon(h) \to 0$ as $h \to 0$ by Theorem 3.2. Using the notation of Theorem 3.6 for solutions of the conjugate boundary value problems for (3.10), viewing the solution $u(x)$ as the solution of a conjugate boundary value problem, and denoting the solution $u(x) = z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)$, we have

$$\begin{aligned} z_{jh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad -z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)]. \end{aligned}$$

Then, by utilizing a telescoping sum, the Mean Value Theorem, and Theorem 3.6, we obtain

$$\begin{aligned} z_{jh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad -z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h}[z(x, x_1, \ldots, x_j + h, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\
&\quad -z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\
&\quad +z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\
&\quad -z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)] \\
&= \frac{1}{h}[\beta_j(x, z(x, x_1, \ldots, x_j + \bar{h}, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon))(x_j + h - x_j) \\
&\quad +\alpha_n(x, z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))(\beta + \epsilon - \beta)] \\
&= \beta_j(x, z(x, x_1, \ldots, x_j + \bar{h}, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\
&\quad +\frac{\epsilon}{h}\alpha_n(x, z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}),
\end{aligned}$$

where $\beta_j(x, z(\cdot))$ and $\alpha_n(x, z(\cdot))$ are the solutions of the variational equation (3.12) along $z(\cdot)$ satisfying

$$\beta_j(x_i, z(\cdot)) = -\delta_{ij}u'(x_i),\ 1 \le i \le n,$$

and

$$\alpha_n(x_i, z(\cdot)) = \delta_{in},\ 1 \le i \le n.$$

Furthermore, $x_j + \bar{h}$ is between x_j and $x_j + h$ and $\beta + \bar{\epsilon}$ is between β and $\beta + \epsilon$.

Therefore, to show $\lim_{h\to 0} z_{jh}(x)$ exists, we must show $\lim_{h\to 0} \epsilon/h$ exists. Well,

$$\begin{aligned}
0 &= z_{jh}(x_n) - \sum_{k=1}^{m} r_k z_{jh}(\eta_k) \\
&= \Big[\beta_j(x_n, z(x, x_1, \ldots, x_j + \bar{h}, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon)) \\
&\quad -\sum_{k=1}^{m} \beta_j(\eta_k, z(x, x_1, \ldots, x_j + \bar{h}, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon))\Big] \\
&\quad +\frac{\epsilon}{h}\Big[\alpha_n(x_n, z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon})) \\
&\quad -\sum_{k=1}^{m} \alpha_n(\eta_k, z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big] \\
&= 0 - \sum_{k=1}^{m} \beta_j(\eta_k, z(x, x_1, \ldots, x_j + \bar{h}, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon))
\end{aligned}$$

$$+\frac{\epsilon}{h}\Big[1-\sum_{k=1}^{m}\alpha_n(\eta_k, z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta+\bar{\epsilon}))\Big].$$

Hence when we solve for ϵ/h, we have

$$\begin{aligned}\lim_{h\to 0}\frac{\epsilon}{h} &= \lim_{h\to 0}\frac{\sum_{k=1}^{m}\beta_j(\eta_k, z(x, x_1, \ldots, x_j+\bar{h}, \ldots, x_n, u_1, \ldots, \beta+\epsilon))}{1-\sum_{k=1}^{m}\alpha_n(\eta_k, z(x, x_1, \ldots, x_j, \ldots, x_n, u_1, \ldots, \beta+\bar{\epsilon}))} \\ &= \frac{\sum_{k=1}^{m}\beta_j(\eta_k, z(\cdot))}{1-\sum_{k=1}^{m}\alpha_n(\eta_k, z(\cdot))} \\ &= \frac{\sum_{k=1}^{m}\beta_j(\eta_k, u(x))}{1-\sum_{k=1}^{m}\alpha_n(\eta_k, u(x))} := B,\end{aligned}$$

where the denominator is nonzero by hypothesis (v). So, we have

$$\begin{aligned}z_j(x) &= \lim_{h\to 0} z_{jh}(x) \\ &= \beta_j(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)) \\ &\quad +B\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)) \\ &= \beta_j(x, u(x)) + B\alpha_n(x, u(x)),\end{aligned}$$

which is a solution of the variational equation (3.12) along $u(x)$. In addition,

$$z_j(x_i) = \lim_{h\to 0} z_{jh}(x_i) = -\delta_{ij}u'(x_i),\ 1 \le i \le n-1,$$

and

$$z_j(x_n) - \sum_{k=1}^{m} r_k z_j(\eta_k) = \lim_{h\to 0}\left[z_{jh}(x_n) - \sum_{k=1}^{m} r_k z_{jh}(\eta_k)\right] = 0.$$

This completes the argument for $\frac{\partial u}{\partial x_j}$, $1 \le i \le n-1$.

Next consider $\frac{\partial u}{\partial x_n}$. Let $\delta > 0$ be as in Theorem 3.2 and $0 < |h| < \delta$ be given. Define

$$\begin{aligned}z_{nh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_{n-1}, x_n+h, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad -u(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)].\end{aligned}$$

Then for every $h \neq 0$, we have

$$\begin{aligned}
z_{nh}(x_n) &- \sum_{k=1}^{m} r_k z_{nh}(\eta_k) \\
&= \frac{1}{h}\Big[u(x_n, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad - \sum_{k=1}^{m} r_k u(\eta_k, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \\
&\qquad\qquad \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad - u(x_n, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad - \sum_{k=1}^{m} r_k u(\eta_k, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)\Big] \\
&= \frac{1}{h}\Big[u(x_n, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad - \sum_{k=1}^{m} r_k u(\eta_k, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \\
&\qquad\qquad \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad - u(x_n + h, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad + u(x_n + h, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \\
&\qquad \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) - u_n\Big] \\
&= -\frac{1}{h}[u'(c_{x_n,h}, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \\
&\qquad \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \cdot h + u_n - u_n] \\
&= -u'(c_{x_n,h}, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),
\end{aligned}$$

where $c_{x_n,h}$ lies between x_n and $x_n + h$.

In addition, for every $h \neq 0$ and $1 \leq i \leq n-1$,

$$\begin{aligned}
z_{nh}(x_i) &= \frac{1}{h}[u(x_i, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) \\
&\quad - u(x_i, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)] \\
&= \frac{1}{h}[u_i - u_i] \\
&= 0,
\end{aligned}$$

Let

$$\beta = u(x_n, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

$$\epsilon \;=\; \epsilon(h) = u(x_n, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m) - \beta.$$

By Theorem 3.2, $\epsilon_n = \epsilon_n(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.6 for solutions of conjugate boundary value problems for (3.10), viewing the solution $u(x)$ as the solution of a conjugate boundary value problem, and denoting the solution $u(x) = z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta)$, we have

$$\begin{aligned} z_{nh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad - z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta)]. \end{aligned}$$

Then, by utilizing a telescoping sum, the Mean Value Theorem, and Theorem 3.6, we obtain

$$\begin{aligned} z_{nh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad - z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta)] \\ &= \frac{1}{h}[z(x, x_1, \ldots, x_{n-1}, x_n + h, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad - z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad + z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad - z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta)] \\ &= \frac{1}{h}\beta_n(x, z(x, x_1, \ldots, x_{n-1}, x_n + \bar{h}, u_1, \ldots, u_{n-1}, \beta + \epsilon)(x_n + h - x_n) \\ &\quad + \frac{1}{h}\alpha_n(x, z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, u_n + \bar{\epsilon})(\beta + \epsilon - \beta) \\ &= \beta_n(x, z(x, x_1, \ldots, x_{n-1}, x_n + \bar{h}, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\ &\quad + \frac{\epsilon}{h}\alpha_n(x, z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}), \end{aligned}$$

where $\beta_n(x, z(\cdot))$ and $\alpha_n(x, z(\cdot))$ are solutions of the variational equation (3.12) along $z(\cdot)$ satisfying

$$\beta_n(x_i, z(\cdot)) = -\delta_{in} u'(x_n),\ 1 \le i \le n,$$

and

$$\alpha_n(x_i, z(\cdot)) = \delta_{in},\ 1 \le i \le n.$$

Furthermore, $x_n + \bar{h}$ lies between x_n and $x_n + h$, and $\beta + \bar{\epsilon}$ lies between β and $\beta + \epsilon$.

Thus, to show $\lim_{h \to 0} z_{nh}(x)$ exists, it suffices to show $\lim_{h \to 0} \dfrac{\epsilon}{h}$ exists. Now,

$$\begin{aligned}
-u'(c_{x_n,h}, x_n + h) &= z_{nh}(x_n) - \sum_{k=1}^{m} r_k z_{nh}(\eta_k) \\
&= \beta_n(x_n, z(x, x_1, \ldots, x_{n-1}, x_n + \bar{h}, u_1, \ldots, u_{n-1}, \beta + \epsilon)) \\
&\quad - \sum_{k=1}^{m} r_k \beta_n(\eta_k, z(x, x_1, \ldots, x_{n-1}, x_n + \bar{h}, \\
&\qquad\qquad u_1, \ldots, u_{n-1}, \beta + \epsilon)) \\
&\quad + \frac{\epsilon}{h}\Big[\alpha_n(x_n, z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon})) \\
&\quad - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_{n-1}, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big] \\
&= -u'(c_{x_n,h}, x_n + h) \\
&\quad - \sum_{k=1}^{m} r_k \beta_n(\eta_k, z(x, x_1, \ldots, x_{n-1}, x_n + \bar{h}, u_1, \ldots, u_{n-1}, \beta + \epsilon)) \\
&\quad + \frac{\epsilon}{h}\Big[1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_{n-1}, x_n, \\
&\qquad\qquad u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big].
\end{aligned}$$

Hence, when we solve for ϵ/h and take the limit using Theorem 3.2, we have

$$\begin{aligned}
\lim_{h \to 0} \frac{\epsilon}{h} &= \lim_{h \to 0} \frac{\sum_{k=1}^{m} r_k \beta_n(\eta_k, z(x, x_1, \ldots, x_{n-1}, x_n + \bar{h}, u_1, \ldots, u_{n-1}, \beta + \epsilon))}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_{x-1}, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))} \\
&= \frac{\sum_{k=1}^{m} r_k \beta_n(\eta_k, z(\cdot))}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(\cdot))} \\
&= \frac{\sum_{k=1}^{m} r_k \beta_n(\eta_k, u(x))}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))} := B,
\end{aligned}$$

provided the denominator is nonzero which we are assured by hypothesis (v).

Now let $z_n(x) = \lim_{h \to 0} z_{nh}(x)$, and note by construction of $z_{nh}(x)$,

$$z_n(x) = \frac{\partial u}{\partial x_n}(x).$$

Furthermore,

$$\begin{aligned} z_{nh}(x) &= \lim_{h\to 0} z_{nh}(x) = \beta_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)) \\ &\quad + B\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)) \\ &= \beta_n(x, u(x)) + B\alpha_n(x, u(x)) \end{aligned}$$

which is a solution of the variational equation along $u(x)$. In addition,

$$z_n(x_i) = \lim_{h\to 0} z_{nh}(x_i) = 0,\ 1 \le i \le n-1,$$

$$z_n(x_n) - \sum_{k=1}^{m} r_k z_n(\eta_k) = \lim_{h\to 0}\left[z_{nh}(x_n) - \sum_{k=1}^{m} r_k z_{nh}(\eta_k)\right] = -u'(x_n).$$

This completes the argument for $\frac{\partial u}{\partial x_j}$, $1 \le i \le n$.

Now on to part (c). Let $1 \le j \le m$ and consider $\frac{\partial u}{\partial \eta_j}$. Let $\delta > 0$ be as in Theorem 3.2 and $0 < |h| < \delta$ be given. Define

$$\begin{aligned} w_{jh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad - u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j, \ldots, \eta_m, r_1, \ldots, r_m)]. \end{aligned}$$

Note that for every $h \neq 0$ and $1 \le i \le n-1$,

$$\begin{aligned} w_{jh}(x_i) &= \frac{1}{h}[u(x_i, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad - u(x_i, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j, \ldots, \eta_m, r_1, \ldots, r_m)] \\ &= \frac{1}{h}[u_i - u_i] \\ &= 0. \end{aligned}$$

Also for every $h \neq 0$,

$$\begin{aligned} w_{jh}(x_n) &- \sum_{k=1}^{m} r_k w_{jh}(\eta_k) \\ &= \frac{1}{h}\Big[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m) \\ &\quad - \sum_{k=1}^{m} r_k u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m)\Big] \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{h}\Big[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j, \ldots, \eta_m, r_1, \ldots, r_m) \\
&-\sum_{k=1}^{m} u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j, \ldots, \eta_m, r_1, \ldots, r_m)\Big] \\
= \; & \frac{1}{h}\Big[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m) \\
&-\sum_{k=1}^{m} u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j, \ldots, \eta_m, r_1, \ldots, r_m) - u_n \\
&-r_j u(\eta_j + h, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m) \\
&+r_j u(\eta_j + h, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m)\Big] \\
= \; & \frac{1}{h}[r_j u(\eta_j + h, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m) \\
&-r_j u(\eta_j, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m) \\
&+u_n - u_n] \\
= \; & \frac{r_j}{h} u'(c_{\eta_j,h}, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, \\
&\qquad r_1, \ldots, r_m)(\eta_j + h - \eta_j) \\
= \; & r_j u'(c_{\eta_j,h}, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m),
\end{aligned}
$$

where $c_{\eta_j,h}$ lies between η_j and $\eta_j + h$.

Let

$$
\beta = u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j, \ldots, \eta_m, r_1, \ldots, r_m),
$$

and

$$
\epsilon = \epsilon(h) = u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_j + h, \ldots, \eta_m, r_1, \ldots, r_m) - \beta.
$$

By Theorem 3.2, $\epsilon = \epsilon(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.6 for solutions of conjugate boundary value problems, viewing the solution $u(x)$ as the solution of a conjugate boundary value problem, and denoting the solution $u(x) = z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)$, we have

$$
w_{jh}(x) \;=\; \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon)
$$

$$-z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)].$$

Then, by the Mean Value Theorem and Theorem 3.6, we obtain,

$$\begin{aligned} w_{jh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots u_{n-1}, \beta + \epsilon) \\ &\quad -z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)] \\ &= \frac{1}{h}\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))(\beta + \epsilon - \beta) \\ &= \frac{\epsilon}{h}\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon})), \end{aligned}$$

where $\alpha_n(x, z(\cdot))$ is the solution of the variational equation (3.12) along $z(\cdot)$ satisfying

$$\alpha_n(x_i, z(\cdot)) = 0,\ 1 \le i \le n.$$

Furthermore, $\beta + \bar{\epsilon}$ lies between β and $\beta + \epsilon$.

Thus, to show $\lim_{h \to 0} w_{jh}(x)$ exists, it suffices to show $\lim_{h \to 0} \frac{\epsilon}{h}$ exists. Now,

$$\begin{aligned} r_j u'(c_{\eta_j, h}, \eta_j + h) &= w_{jh}(x_n) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k) \\ &= \frac{\epsilon}{h}\Big[\alpha_n(x_n, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon})) \\ &\quad -\sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big] \\ &= \frac{\epsilon}{h}\Big[1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))\Big] \end{aligned}$$

Hence, by Theorem 3.2, we solve for ϵ/h and are able to take the limit, yielding

$$\begin{aligned} \lim_{h \to 0} \frac{\epsilon}{h} &= \lim_{h \to 0} \frac{r_j u'(c_{\eta_j, h}, \eta_j + h)}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \bar{\epsilon}))} \\ &= \frac{r_j u'(\eta_j)}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, z(\cdot))} \\ &= \frac{r_j u'(\eta_j)}{1 - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x))} := C, \end{aligned}$$

provided the denominator is nonzero which we are guaranteed by hypothesis (v).

Now let $w_j(x) = \lim_{h\to 0} w_{jh}(x)$, and note by construction of $w_{jh}(x)$,

$$w_j(x) = \frac{\partial u}{\partial \eta_j}(x).$$

Furthermore,

$$\begin{aligned} w_j(x) &= \lim_{h\to 0} w_{jh}(x) \\ &= C\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)) \\ &= C\alpha_n(x, u(x)), \end{aligned}$$

which is a solution of the variational equation (3.12) along $u(x)$. In addition,

$$w_j(x_i) = \lim_{h\to 0} w_{jh}(x_i) = 0,\ 1 \le i \le n-1,$$

$$w_j(x_n) - \sum_{k=1}^{m} r_k w_j(\eta_k) = \lim_{h\to 0}\left[w_{jh}(x_n) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k)\right] = r_j u'(\eta_j).$$

This completes the argument for $\frac{\partial u}{\partial \eta_j}$, $1 \le j \le m$.

Lasty, we show part (d). Let $1 \le j \le m$ and consider $\frac{\partial u}{\partial r_j}$. Let $\delta > 0$ be as in Theorem 3.2 and $0 < |h| < \delta$ be given. Define

$$\begin{aligned} v_{jh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m) \\ &\quad -u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j, \ldots, r_m)]. \end{aligned}$$

Note that for every $h \neq 0$ and $1 \le i \le n-1$,

$$\begin{aligned} v_{jh}(x_i) &= \frac{1}{h}[u(x_i, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m) \\ &\quad -u(x_i, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j, \ldots, r_m)] \\ &= \frac{1}{h}[u_i - u_i] \\ &= 0. \end{aligned}$$

Also for every $h \neq 0$,

$$v_{jh}(x_n) \quad - \quad \sum_{k=1}^{m} r_k v_{jh}(\eta_k)$$

$$
\begin{aligned}
&= \frac{1}{h}\Big[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m) \\
&\quad -\sum_{k=1}^{m} r_k u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m)\Big] \\
&\quad -\frac{1}{h}\Big[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j, \ldots, r_m) \\
&\quad -\sum_{k=1}^{m} u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j, \ldots, r_m)\Big] \\
&= \frac{1}{h}[u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m) \\
&\quad -\sum_{k=1}^{m} u(\eta_k, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + j, \ldots, r_m) - u_n \\
&\quad -hu(\eta_j, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m) \\
&\quad +hu(\eta_j, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m)] \\
&= \frac{1}{h}[hu(\eta_j, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m) \\
&\quad +u_n - u_n] \\
&= u(\eta_j, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m).
\end{aligned}
$$

Let

$$\beta = u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j, \ldots, r_m),$$

and

$$\epsilon = \epsilon(h) = u(x_n, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_j + h, \ldots, r_m) - \beta.$$

By Theorem 3.2, $\epsilon = \epsilon(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.6 for solutions of conjugate boundary value problems, viewing the solution $u(x)$ as the solution of a conjugate boundary value problem, and denoting the solution $u(x) = z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)$, we have

$$
\begin{aligned}
v_{jh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta + \epsilon) \\
&\quad -z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)].
\end{aligned}
$$

Then, by the Mean Value Theorem and Theorem 3.6, we obtain,

$$\begin{aligned} v_{jh}(x) &= \frac{1}{h}[z(x, x_1, \ldots, x_n, u_1, \ldots u_{n-1}, \beta+\epsilon) \\ &\quad -z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)] \\ &= \frac{1}{h}\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta+\bar{\epsilon}))(\beta+\epsilon-\beta) \\ &= \frac{\epsilon}{h}\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta+\bar{\epsilon})), \end{aligned}$$

where $\alpha_n(x, z(\cdot))$ is the solution of the variational equation (3.12) along $z(\cdot)$ satisfying

$$\alpha_n(x_i, z(\cdot)) = \delta_{in},\ 1 \leq i \leq n.$$

Furthermore, $\beta + \bar{\epsilon}$ lies between β and $\beta + \epsilon$.

Thus, to show $\lim_{h\to 0} v_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\epsilon}{h}$ exists. Now,

$$\begin{aligned} u(\eta_j, r_j + h) &= v_{jh}(x_n) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) \\ &= \frac{\epsilon}{h}\Big[\alpha_n(x_n, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta+\bar{\epsilon})) \\ &\quad -\sum_{k=1}^{m} r_k\alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta+\bar{\epsilon}))\Big] \\ &= \frac{\epsilon}{h}\Big[1 - \sum_{k=1}^{m} r_k\alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta+\bar{\epsilon}))\Big]. \end{aligned}$$

Hence, we solve for ϵ/h and take the limit using Theorem 3.2,

$$\begin{aligned} \lim_{h\to 0}\frac{\epsilon}{h} &= \lim_{h\to 0}\frac{u(\eta_j, r_j + h)}{1 - \sum_{k=1}^{m} r_k\alpha_n(\eta_k, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta+\bar{\epsilon}))} \\ &= \frac{u(\eta_j)}{1 - \sum_{k=1}^{m} r_k\alpha_n(\eta_k, z(\cdot))} \\ &= \frac{u(\eta_j)}{1 - \sum_{k=1}^{m} r_k\alpha_n(\eta_k, u(x))} := D, \end{aligned}$$

provided the denominator is nonzero which we are assured by hypothesis (v).

Now let $v_j(x) = \lim_{h\to 0} v_{jh}(x)$, and note by construction of $v_{jh}(x)$,

$$v_j(x) = \frac{\partial u}{\partial r_j}(x).$$

Furthermore,

$$\begin{aligned} v_j(x) &= \lim_{h\to 0} v_{jh}(x) \\ &= D\alpha_n(x, z(x, x_1, \ldots, x_n, u_1, \ldots, u_{n-1}, \beta)) \\ &= D\alpha_n(x, u(x)), \end{aligned}$$

which is a solution of the variational equation (3.12) along $u(x)$. In addition,

$$v_j(x_i) = \lim_{h\to 0} v_{jh}(x_i) = 0,\ 1 \le i \le n-1,$$

$$v_j(x_n) - \sum_{k=1}^{m} r_k v_j(\eta_k) = \lim_{h\to 0} \left[v_{jh}(x_n) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) \right] = u(\eta_j).$$

This completes the argument for $\frac{\partial u}{\partial r_j}$, $1 \le j \le m$. □

3.5 General nth Order Problem

With the ideas and strategies of the previous sections, we are now able to turn to the general nth order differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ x \in (a,b),\ n \ge 2, \tag{3.13}$$

satisfying

$$\begin{aligned} &y^{(i)}(x_j) = y_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \\ &y^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y(\eta_{ip}) = y_{ik},\ 0 \le i \le m_k - 1, \end{aligned} \tag{3.14}$$

where $2 \le k \le n,\ m \in \mathbb{N},\ m_1, \ldots, m_k$ are positive integers such that $\sum_{i=1}^{k} m_i = n,\ a < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$, and $y_{01}, \ldots, y_{m_k-1,k}, r_{01}, \ldots, r_{m_k-1,m} \in \mathbb{R}$. The work found in this section was recently submitted to the Electronic Journal of Qualitative Theory of Differential Equations.

Remark 3.2. Note we need not always be given m values for η and r as is done in this section. This is done simply to ease the burdensome notation.

Once again, we deal with the same variational equation for (3.13) given a solution $y(x)$ of (3.13):

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(x, y(x), y'(x), \dots, y^{(n-1)}(x)) z^{(i-1)}. \tag{3.15}$$

Now we present the analogue of Theorem 3.1 as it pertains to (3.13), (3.14).

Theorem 3.7. *Assume conditions* (i)-(v) *are satisfied. Let* $n \geq 2$, $m \in \mathbb{N}$, *and* $2 \leq k \leq n$ *be given and* $m_1, \dots, m_k$ *be positive integers such that* $\sum_{i=1}^{k} m_i = n$. *Let* $u(x)$ *be a solution of* (3.13) *on* (a, b). *Let* $a < x_1 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$ *and* $u_{01}, \dots, u_{m_k-1,k}, r_{01}, \dots, r_{m_k-1,m} \in \mathbb{R}$ *be given so that*

$$u(x) = u(x, x_1, \dots, x_k, u_{01}, \dots, u_{m_k-1,k}, \eta_{01}, \dots, \eta_{m_k-1,m}, r_{01}, \dots, r_{m_k-1,m}),$$

where

$$u^{(i)}(x_j) = u_{ij},\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

and

$$u^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} u(\eta_{ip}) = u_{ik},\ 0 \leq i \leq m_k - 1.$$

Then,

(*a*) *for each* $1 \leq l \leq k-1$ *and* $0 \leq r \leq m_l - 1$, $\dfrac{\partial u}{\partial u_{rl}}(x)$ *exists on* (a, b), *and* $y_{rl}(x) := \dfrac{\partial u}{\partial u_{rl}}(x)$ *is the solution of the variational equation* (3.15) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned}
y_{rl}^{(i)}(x_j) &= 0, && 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,\ j \neq l,\\
y_{rl}^{(i)}(x_l) &= 0, && 0 \leq i \leq m_j - 1,\ i \neq r,\\
y_{rl}^{(r)}(x_l) &= 1, &&\\
y_{rl}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} y_{rl}(\eta_{ip}) = 0, && 0 \leq i \leq m_k - 1,
\end{aligned}$$

and for $0 \le r \le m_k - 1$, $y_{rk}(x) := \dfrac{\partial u}{\partial u_{rk}}(x)$ *exists on* (a, b) *and solves* (3.15) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned}
y_{rk}^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\\
y_{rk}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} y_{rk}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1,\ i \ne r,\\
y_{rk}^{(r)}(x_k) &- \sum_{p=1}^{m} r_{rp} y_{rk}(\eta_{rp}) = 1.
\end{aligned}$$

(b) *for each* $1 \le l \le k-1$, $\dfrac{\partial u}{\partial x_l}(x)$ *exists on* (a, b), *and* $z_l(x) := \dfrac{\partial u}{\partial x_l}(x)$ *is the solution of the variational equation* (3.15) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned}
z_l^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l,\\
z_l^{(i)}(x_l) &= -u^{(i+1)}(x_l), \qquad 0 \le i \le m_l - 1,\\
z_l^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} z_l(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1,
\end{aligned}$$

and $z_k(x) := \dfrac{\partial u}{\partial x_k}(x)$ *exists on* (a, b) *and solves* (3.15) *along u(x) satisfying the boundary conditions*

$$\begin{aligned}
z_k^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\\
z_k^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} z_k(\eta_{ip}) = -u^{(i+1)}(x_k), \qquad 0 \le i \le m_k - 1.
\end{aligned}$$

(c) *for* $0 \le r \le m_k - 1$ *and* $1 \le s \le m$, $\dfrac{\partial u}{\partial \eta_{rs}}(x)$ *exists on* (a, b), *and* $w_{rs}(x) := \dfrac{\partial u}{\partial \eta_{rs}}(x)$ *is the solution of* (3.15) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned}
w_{rs}^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\\
w_{rs}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} w_{rs}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1,\ i \ne r,
\end{aligned}$$

$$w_{rs}^{(r)}(x_k) \quad - \quad \sum_{p=1}^{m} r_{rp} w_{rs}(\eta_{rs}) = r_{rs} u'(\eta_{rs}).$$

(*d*) *for* $0 \le r \le m_k - 1$ *and* $1 \le s \le m$, $\dfrac{\partial u}{\partial r_{rs}}(x)$ *exists on* (a, b), *and* $v_{rs}(x) := \dfrac{\partial u}{\partial r_{rs}}(x)$ *is the solution of* (3.15) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned} v_{rs}^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \\ v_{rs}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} v_{rs}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1,\ i \ne r, \\ v_{rs}^{(r)}(x_k) &- \sum_{p=1}^{m} r_{rp} v_{rs}(\eta_{rp}) = u(\eta_{rs}). \end{aligned}$$

Proof. For the first piece of part (a), let $0 \le r \le m_l - 1,\ 1 \le l \le k-1$, and consider $\frac{\partial u}{\partial u_{rl}}$. Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define

$$\begin{aligned} y_{rlh}(x) = \frac{1}{h}[&u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl} + h, \ldots, u_{m_k-1,k}, \\ &\eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &- u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl}, \ldots, u_{m_k-1,k}, \\ &\eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})]. \end{aligned}$$

Note that

$$\begin{aligned} &u^{(r)}(x_l, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl} + h, \ldots, u_{m_k-1,k}, \\ &\eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) = u_{rl} + h, \end{aligned}$$

and

$$\begin{aligned} &u^{(r)}(x_l, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl}, \ldots, u_{m_k-1,k}, \\ &\eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) = u_{rl}, \end{aligned}$$

so that, for every $h \ne 0$,

$$y_{rlh}^{(r)}(x_l) = \frac{1}{h}[u^{(r)}(x_l, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl} + h, \ldots, u_{m_k-1,k},$$

$$\eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})$$

$$-u^{(r)}(x_l, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl}, \ldots, u_{m_k-1,k},$$

$$\eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})]$$

$$= \frac{1}{h}[u_{rl} + h - u_{rl}]$$

$$= 1.$$

Also, for every $h \neq 0,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1$, and $j \neq l$,

$$\begin{aligned} y_{rlh}^{(i)}(x_j) &= \frac{1}{h}[u^{(i)}(x_j, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl} + h, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u^{(i)}(x_j, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\ &= \frac{1}{h}[u_{ij} - u_{ij}] \\ &= 0, \end{aligned}$$

for every $h \neq 0,\ 0 \leq i \leq m_l - 1$, and $i \neq r$,

$$\begin{aligned} y_{rlh}^{(i)}(x_l) &= \frac{1}{h}[u^{(i)}(x_l, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl} + h, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u^{(i)}(x_l, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\ &= \frac{1}{h}[u_{il} - u_{il}] \\ &= 0, \end{aligned}$$

and for every $h \neq 0$, and $0 \leq i \leq m_k - 1$,

$$\begin{aligned} y_{rlh}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} y_{rlh}(\eta_{ip}) \\ &= \frac{1}{h}\Big[u^{(i)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{rl} + h, \ldots, u_{m_k-1,k}, \end{aligned}$$

$$\begin{aligned}
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&-u^{(i)}(x_k,x_1,\ldots,x_k,u_{01},\ldots,u_{rl},\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\Big]\\
&-\sum_{p=1}^{m}\frac{r_{ip}}{h}\Big[u(\eta_{ip},x_1,\ldots,x_k,u_{01},\ldots,u_{rl}+h,\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&-u(\eta_{ip},x_1,\ldots,x_k,u_{01},\ldots,u_{rl},\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\Big]\\
=\ &\frac{1}{h}[u_{ik}-u_{ik}]=0.
\end{aligned}$$

For $m_l \le i \le n-1$, let

$$\beta_i = u^{(i)}(x_l,x_1,\ldots,x_k,u_{01},\ldots,u_{rl},\ldots,u_{m_k-1,k},\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m}),$$

and

$$\begin{aligned}
\epsilon_i=\epsilon_i(h) \;=\;& u^{(i)}(x_l,x_1,\ldots,x_k,u_{01},\ldots,u_{rl}+h,\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})-\beta_i.
\end{aligned}$$

By Theorem 3.2, for $m_l \le i \le n-1$, $\epsilon_i=\epsilon_i(h)\to 0$ as $h\to 0$. Using the notation of Theorem 3.1 for solutions of initial value problems for (3.13), viewing the solution $u(x)$ as the solution of an initial value problem, and denoting the solution $u(x)=y(x,x_l,u_{0l},\ldots,u_{m_l-1,l},\beta_{m_l},\ldots,\beta_{n-1})$, we have

$$\begin{aligned}
y_{rlh}(x) \;=\;& \frac{1}{h}[y(x,x_l,u_{0l},\ldots,u_{rl}+h,\ldots,u_{m_l-1,l},\beta_{m_l}+\epsilon_{m_l},\\
&\beta_{m_l+1}+\epsilon_{m_l+1},\ldots,\beta_{n-1}+\epsilon_{n-1})\\
&-y(x,x_l,u_{0l},\ldots,u_{rl},\ldots,u_{m_l-1,l},\\
&\beta_{m_l},\beta_{m_l+1},\ldots,\beta_{n-1})].
\end{aligned}$$

Then, by utilizing a telescoping sum, we have

$$y_{rlh}(x) \;=\; \frac{1}{h}\Big\{\big[y(x,x_l,u_{0l},\ldots,u_{rl}+h,\ldots,u_{m_l-1,l},\beta_{m_l}+\epsilon_{m_l},$$

$$\beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})$$

$$-y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l},$$

$$\beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})\big]$$

$$+\big[y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l},$$

$$\beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})$$

$$-y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})\big]$$

$$+\big[y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})$$

$$-y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})\big]$$

$$+ - \cdots$$

$$+\big[y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})$$

$$-y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1})\big]\Big\}.$$

By Theorem 3.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} y_{rlh}(x) \;=\;& \alpha_r(x, y(x, x_l, u_{0l}, \ldots, u_{rl} + \bar{h}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & + \frac{\epsilon_{m_l}}{h}\alpha_{m_l}(x, y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & + \frac{\epsilon_{m_l+1}}{h}\alpha_{m_l+1}(x, y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ & \qquad \beta_{m_l+1} + \bar{\epsilon}_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & + \cdots \\ & + \frac{\epsilon_{n-1}}{h}\alpha_{n-1}(x, y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ & \qquad \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})), \end{aligned}$$

where for $0 \le j \le n-1$, $\alpha_j(x, y(\cdot))$ is the solution of the variational equation (3.15) along $y(\cdot)$ satisfying

$$\alpha_j^{(i)}(x_l) = \delta_{ij}, \quad 0 \le i \le n-1.$$

Furthermore, $u_{rl} + \bar{h}$ is between u_{rl} and $u_{rl} + h$, and for $m_l \le i \le n-1$, $\beta_i + \bar{\epsilon}_i$ is between β_i and $\beta_i + \epsilon_i$.

Thus, to show $\lim_{h \to 0} y_{rlh}(x)$ exists, it suffices to show, for $m_l \le i \le n-1$, $\lim_{h \to 0} \frac{\epsilon_i}{h}$ exists.

Now, from the construction of $y_{rlh}(x)$, we have,

$$y_{rlh}^{(i)}(x_j) = 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l,$$

and

$$y_{rlh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y_{rlh}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1.$$

Hence, we have a system of $n - m_l$ linear equations with $n - m_l$ unknowns:

$$\begin{aligned}
- \quad & \alpha_r^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{rl} + \bar{h}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\
& \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
= \quad & \frac{\epsilon_{m_l}}{h} \alpha_{m_l}^{(i)}(x_j, y(x_j, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\
& \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
& + \cdots \\
& + \frac{\epsilon_{n-1}}{h} \alpha_{n-1}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\
& \qquad \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})), \\
& 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l,
\end{aligned}$$

and

$$\begin{aligned}
- \quad & \alpha_r^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{rl} + \bar{h}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\
& \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
& - \sum_{p=1}^{m} r_{ip} \alpha_r(\eta_{ip}, y(x, x_l, u_{0l}, \ldots, u_{rl} + \bar{h}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\
& \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
= \quad & \frac{\epsilon_{m_l}}{h} \Big[\alpha_{m_l}^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l},
\end{aligned}$$

$$\beta_{m_l+1}+\epsilon_{m_l+1},\ldots,\beta_{n-1}+\epsilon_{n-1}))$$

$$-\sum_{p=1}^{m} r_{ip}\alpha_{m_l}(\eta_{ip}, y(x, x_l, u_{0l},\ldots,u_{rl},\ldots,u_{m_l-1,l},\beta_{m_l}+\bar{\epsilon}_{m_l},$$

$$\beta_{m_l+1}+\epsilon_{m_l+1},\ldots,\beta_{n-1}+\epsilon_{n-1}))\Big]$$

$$+\cdots$$

$$+\frac{\epsilon_{n-1}}{h}\Big[\alpha_{n-1}^{(i)}(x_k, y(x, x_l, u_{0l},\ldots,u_{rl},\ldots,u_{m_l-1,l},\beta_{m_l},$$

$$\beta_{m_l+1},\ldots,\beta_{n-1}+\bar{\epsilon}_{n-1}))$$

$$-\sum_{p=1}^{m} r_{ip}\alpha_{n-1}(\eta_{ip}, y(x, x_l, u_{0l},\ldots,u_{rl},\ldots,u_{m_l-1,l},\beta_{m_l},$$

$$\beta_{m_l+1},\ldots,\beta_{n-1}+\bar{\epsilon}_{n-1}))\Big],$$

$$0\le i\le m_k-1.$$

At this point in the proof, we will occassionally suppress the arguments of α, the subscripts of r and η, and limits of the summation. As $y(\cdot)$ is not necessarily the same in every α, we consider the matrix

$$M := \begin{pmatrix}
\alpha_{m_l}(x_1,u(x)) & \alpha_{m_l+1}(x_1,u(x)) & \cdots & \alpha_{n-1}(x_1,u(x)) \\
\alpha'_{m_l}(x_1,u(x)) & \alpha'_{m_l+1}(x_1,u(x)) & \cdots & \alpha'_{n-1}(x_1,u(x)) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m_l}^{(m_1-1)}(x_1,u(x)) & \alpha_{m_l+1}^{(m_1-1)}(x_1,u(x)) & \cdots & \alpha_{n-1}^{(m_1-1)}(x_1,u(x)) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \alpha_{m_l+1}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \cdots & \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\
\alpha_{m_l}(x_{l+1},u(x)) & \alpha_{m_l+1}(x_{l+1},u(x)) & \cdots & \alpha_{n-1}(x_{l+1},u(x)) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m_l}(x_k,u(x))- & \alpha_{m_l+1}(x_k,u(x))- & & \alpha_{n-1}(x_k,u(x))- \\
\sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x)) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m_l}^{(m_k-1)}(x_k,u(x))- & \alpha_{m_l+1}^{(m_k-1)}(x_k,u(x))- & & \alpha_{n-1}^{(m_k-1)}(x_k,u(x))- \\
\sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x))
\end{pmatrix}.$$

We claim $\det(M) \neq 0$ so suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R},\ m_l \leq i \leq n-1$, not all zero such that

$$p_{m_l}\begin{pmatrix} \alpha_{m_l}(x_1, u(x)) \\ \alpha'_{m_l}(x_1, u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{m_l}(x_{l+1}, u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{m_l}(\eta, u(x)) \end{pmatrix} + \cdots + p_{n-1}\begin{pmatrix} \alpha_{n-1}(x_1, u(x)) \\ \alpha'_{n-1}(x_1, u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{n-1}(x_{l+1}, u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{n-1}(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$w(x, u(x)) := p_{m_l}\alpha_{m_l}(x, u(x)) + \cdots + p_{n-1}\alpha_{n-1}(x, u(x)).$$

Then, $w(x, u(x))$ is a nontrivial solution of (3.15), but

$$w^{(i)}(x_j, u(x)) = 0,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

and

$$w^{(i)}(x_k, u(x)) - \sum_{p=1}^{m} r_{ip} w(\eta_{ip}, u(x)) = 0,\ 0 \leq i \leq m_k - 1,$$

which when coupled with hypothesis (v) implies $w(x, u(x)) = 0$. Thus $p_{m_l} = p_{m_l+1} = \cdots = p_{n-1} = 0$ contradicting our supposition and consequently forcing $\det(M) \neq 0$. So, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore we can solve $\epsilon_i(h)/h$, for $m_l \leq i \leq n-1$, by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|} \times$$

$$\begin{vmatrix} \alpha_{m_l} & \cdots & \alpha_{i-2} & -\alpha_r & \alpha_i & \cdots & \alpha_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_k-1)} - & & \alpha_{i-2}^{(m_k-1)} - & -\alpha_r^{(m_k-1)} + & \alpha_i^{(m_k-1)} - & & \alpha_{n-1}^{(m_k-1)} - \\ \sum r\alpha_{m_l} & \cdots & \sum r\alpha_{i-2} & \sum r\alpha_r & \sum r\alpha_i & \cdots & \sum r\alpha_{n-l} \end{vmatrix}$$

Note as $h \to 0,\ \det(M(h)) \to \det(M)$, and so for $m_l \le i \le n-1,\ \epsilon_i(h)/h \to \det(M_i)/\det(M) := A_i$ as $h \to 0$, where M_i is the $n - m_l \times n - m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\begin{aligned} \text{col}\Big[& -\alpha_r(x_1, u(x)), \dots, -\alpha_r^{(m_l-1)}(x_1, u(x)), \dots, -\alpha_r(x_{l-1}, u(x)), \dots, \\ & -\alpha_r^{(m_{l-1}-1)}(x_{l-1}, u(x)), -\alpha_r(x_{l+1}, u(x)), \dots, -\alpha_r^{(m_{l+1}-1)}(x_{l+1}, u(x)), \dots, \\ & -\alpha_r(x_k, u(x)) + \sum_{p=1}^{m} r_{0p}\alpha_r(\eta_{0p}, u(x)), \dots, \\ & -\alpha_r^{(m_k-1)}(x_k, u(x)) + \sum_{p=1}^{m} r_{m_k-1,p}\alpha_r(\eta_{m_k-1,p}, u(x))\Big] \end{aligned}$$

Now let $y_{rl}(x) = \lim_{h\to 0} y_{rlh}(x)$, and note by the construction of $y_{rlh}(x)$,

$$y_{rl}(x) = \frac{\partial u}{\partial u_{rl}}(x).$$

Furthermore,

$$y_{rl}(x) = \lim_{h\to 0} y_{rlh}(x) = \alpha_r(x, u(x)) + \sum_{i=m_l}^{n-1} A_i\alpha_i(x, u(x)),$$

which is a solution of the variational equation (3.15) along $u(x)$. In addition,

$$y_{rl}^{(i)}(x_j) = \lim_{h\to 0} y_{rlh}^{(i)}(x_j) = \delta_{ir}\delta_{jl},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$y_{rl}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip}y_{rl}(\eta_{ip}) = \lim_{h\to 0}\left[y_{rlh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip}y_{rlh}(\eta_{ip})\right] = 0,\ 0 \le i \le m_k - 1.$$

This completes the argument for $\frac{\partial u}{\partial u_{rl}}$.

To conclude part (a), let $0 \le r \le m_k - 1$ and consider $\frac{\partial u}{\partial u_{rk}}$. Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define

$$\begin{aligned} y_{rkh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk} + h, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})]. \end{aligned}$$

Note that for every $h \neq 0$,

$$\begin{aligned} y_{rkh}^{(r)}(x_k) &- \sum_{p=1}^{m} r_{rp} y_{rkh}(\eta_{rp}) \\ &= \frac{1}{h}[u^{(r)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk} + h, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u^{(r)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\ &- \sum_{p=1}^{m} \frac{r_{rp}}{h}[u(\eta_{rp}, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk} + h, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u(\eta_{rp}, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\ &= \frac{1}{h}[u_{rk} + h - u_{rk}] \\ &= 1. \end{aligned}$$

Also, for every $h \neq 0,\ 0 \le i \le m_j - 1$, and $1 \le j \le k-1$,

$$\begin{aligned} y_{rkh}^{(i)}(x_j) &= \frac{1}{h}[u^{(i)}(x_j, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk} + h, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u^{(i)}(x_j, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk}, \ldots, u_{m_k-1,k}, \end{aligned}$$

$$
\begin{aligned}
&\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\
&= \frac{1}{h}[u_{ij} - u_{ij}] \\
&= 0,
\end{aligned}
$$

and for every $h \neq 0,\ 0 \leq i \leq m_k - 1$, and $i \neq r$,

$$
\begin{aligned}
y_{rkh}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} y_{rkh}(\eta_{ip}) \\
&= \frac{1}{h}[u^{(i)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk} + h, \ldots, u_{m_k-1,k}, \\
&\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\
&\quad -u^{(i)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk}, \ldots, u_{m_k-1,k}, \\
&\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\
&- \sum_{p=1}^{m} \frac{r_{ip}}{h}[u(\eta_{ip}, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk} + h, \ldots, u_{m_k-1,k}, \\
&\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\
&\quad -u(\eta_{ip}, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk}, \ldots, u_{m_k-1,k}, \\
&\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\
&= \frac{1}{h}[u_{ik} - u_{ik}] \\
&= 0.
\end{aligned}
$$

Let $1 \leq l \leq k - 1$, and for $m_l \leq i \leq n - 1$, let

$$
\beta_i = u^{(i)}(x_l, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})
$$

and

$$
\begin{aligned}
\epsilon_i = \epsilon_i(h) &= u^{(i)}(x_l, x_1, \ldots, x_k, u_{01}, \ldots, u_{rk} + h, \ldots, u_{m_k-1,k}, \\
&\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) - \beta_i.
\end{aligned}
$$

By Theorem 3.2, for $m_l \leq i \leq n - 1$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.1 for solutions of initial value problems for (3.13), viewing

$u(x)$ as the solution of an initial value problem, and denoting a solution $u(x) = y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \ldots, \beta_{n-1})$, we have

$$\begin{aligned} y_{rkh}(x) &= \frac{1}{h}[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ &\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1})]. \end{aligned}$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} y_{rkh}(x) &= \frac{1}{h}\{[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ &\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})] \\ &\quad + [y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ &\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})] \\ &\quad + - \cdots \\ &\quad + [y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ &\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1})]\}. \end{aligned}$$

Applying Theorem 3.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} y_{rkh}(x) = &\frac{\epsilon_{m_l}}{h}\alpha_{m_l}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ &\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ &+ \frac{\epsilon_{m_l+1}}{h}\alpha_{m_l+1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ &\qquad \beta_{m_l+1} + \bar{\epsilon}_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) + \cdots \\ &+ \frac{\epsilon_{n-1}}{h}\alpha_{n-1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1}), \end{aligned}$$

where for $0 \le j \le n-1,\ \alpha_j(x, y(\cdot))$ is the solution of the variational equation (3.15) along $y(\cdot)$ satisfying

$$\alpha_j^{(i)}(x_l) = \delta_{ij}, \quad 0 \le i \le n-1.$$

Furthermore, for $m_l \le i \le n-1,\ \beta_i + \bar{\epsilon}_i$ is between β_i and $\beta_i + \epsilon_i$.

Thus, to show $\lim_{h\to 0} y_{rkh}(x)$ exists, it suffices to show, for $m_l \le i \le n-1$, $\lim_{h\to 0} \dfrac{\epsilon_i}{h}$ exists.

Now, from the construction of $y_{rkh}(x)$, we have

$$y_{rkh}^{(i)}(x_j) = 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l$$

and

$$y_{rkh}^{(i)} - \sum_{p=1}^{m} r_{ip} y_{rkh}(\eta_{ip}) = \delta_{ir},\ 0 \le i \le m_k - 1.$$

Hence, we have a system of $n - m_l$ linear equations with $n - m_l$ unknowns:

$$\begin{aligned} 0 \;=\; & \frac{\epsilon_{m_l}}{h}\alpha_{m_l}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) + \cdots \\ & + \frac{\epsilon_{n-1}}{h}\alpha_{n-1}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{rl}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ & \qquad \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})), \\ & 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \end{aligned}$$

and

$$\begin{aligned} \delta_{ir} \;=\; & \frac{\epsilon_{m_l}}{h}\Big[\alpha_{m_l}^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & - \sum_{p=1}^{m} r_{ip}\alpha_{m_l}(\eta_{ip}, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\ & \qquad \beta_{m_l+1} + \bar{\epsilon}_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}))\Big] + \cdots \\ & + \frac{\epsilon_{n-1}}{h}\Big[\alpha_{n-1}^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})) \\ & - \sum_{p=1}^{m} r_{ip}\alpha_{n-1}(\eta_{ip}, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1}))\Big], \\ & 0 \le i \le m_k - 1. \end{aligned}$$

At this point in the proof, we will occassionally suppress the arguments of α, the subscripts of r and η, and limits of the summation. In the system of equations above, the $y(\cdot)$ is not the same in every case. Therefore, we consider the matrix

$$M := \begin{pmatrix} \alpha_{m_l}(x_1,u(x)) & \alpha_{m_l+1}(x_1,u(x)) & \cdots & \alpha_{n-1}(x_1,u(x)) \\ \alpha'_{m_l}(x_1,u(x)) & \alpha'_{m_l+1}(x_1,u(x)) & \cdots & \alpha'_{n-1}(x_1,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_1-1)}(x_1,u(x)) & \alpha_{m_l+1}^{(m_1-1)}(x_1,u(x)) & \cdots & \alpha_{n-1}^{(m_1-1)}(x_1,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \alpha_{m_l+1}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \cdots & \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{m_l}(x_{l+1},u(x)) & \alpha_{m_l+1}(x_{l+1},u(x)) & \cdots & \alpha_{n-1}(x_{l+1},u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}(x_k,u(x))- & \alpha_{m_l+1}(x_k,u(x))- & & \alpha_{n-1}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k,u(x))- & \alpha_{m_l+1}^{(m_k-1)}(x_k,u(x))- & & \alpha_{n-1}^{(m_k-1)}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R}$, $m_l \leq i \leq n-1$, not all zero, such that

$$p_{m_l} \begin{pmatrix} \alpha_{m_l}(x_1,u(x)) \\ \alpha'_{m_l}(x_1,u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{m_l}(x_{l+1},u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \alpha_{n-1}(x_1,u(x)) \\ \alpha'_{n-1}(x_1,u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{n-1}(x_{l+1},u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_k-1)}(x_k,u(x))- \\ \sum r\alpha_{n-1}(\eta,u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$w(x,u(x)) := p_{m_l}\alpha_{m_l}(x,u(x)) + \cdots + p_{n-1}\alpha_{n-1}(x,u(x)).$$

Then, $w(x, u(x))$ is a nontrivial solution of (3.15), but

$$w^{(i)}(x_j, u(x)) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$w^{(i)}(x_k, u(x)) - \sum_{p=1}^{m} r_{ip} w(\eta_{ip}, u(x)) = 0,\ 0 \le i \le m_k - 1,$$

which when coupled with hypothesis (v) implies $w(x, u(x)) = 0$. Thus, $p_{m_l} = p_{m_l+1} = \cdots = p_{n-1} = 0$ which is a contradiction. Hence $\det(M) \neq 0$. So, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, for each $m_l \le i \le n-1$, we can solve $\epsilon_i(h)/h$ by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|} \times \begin{vmatrix} \alpha_{m_l} & \cdots & \alpha_{i-2} & 0 & \alpha_i & \cdots & \alpha_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{matrix}\alpha_{m_l}- \\ \sum r\alpha_{m_l}\end{matrix} & \cdots & \begin{matrix}\alpha_{i-2}- \\ \sum r\alpha_{i-2}\end{matrix} & 0 & \begin{matrix}\alpha_i- \\ \sum r\alpha_i\end{matrix} & \cdots & \begin{matrix}\alpha_{n-1}^{(r)}- \\ \sum r\alpha_{n-1}\end{matrix} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{matrix}\alpha_{m_l}^{(r)}- \\ \sum r\alpha_{m_l}\end{matrix} & \cdots & \begin{matrix}\alpha_{i-2}^{(r)}- \\ \sum r\alpha_{i-2}\end{matrix} & 1 & \begin{matrix}\alpha_i^{(r)}- \\ \sum r\alpha_i\end{matrix} & \cdots & \begin{matrix}\alpha_{n-1}^{(r)}- \\ \sum r\alpha_{n-1}\end{matrix} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{matrix}\alpha_{m_l}^{(m_k-1)}- \\ \sum r\alpha_{m_l}\end{matrix} & \cdots & \begin{matrix}\alpha_{i-2}^{(m_k-1)} \\ \sum r\alpha_{i-2}\end{matrix} & 0 & \begin{matrix}\alpha_i^{(m_k-1)}- \\ \sum r\alpha_i\end{matrix} & \cdots & \begin{matrix}\alpha_{n-1}^{(m_k-1)}- \\ \sum r\alpha_{n-l}\end{matrix} \end{vmatrix}$$

Note as $h \to 0$, $\det(M(h)) \to \det(M)$, and so for $m_l \le i \le n-1$, $\epsilon_i(h)/h \to \det(M_i)/\det(M) := A_i$ as $h \to 0$, where M_i is the $n - m_l \times n - m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\text{col}[0, \ldots, 0, 1, 0, \ldots, 0].$$

Now let $y_{rk}(x) = \lim_{h\to 0} y_{rkh}(x)$, and note by construction of $y_{rkh}(x)$,

$$y_{rk}(x) = \frac{\partial u}{\partial u_{rk}}(x).$$

Furthermore,

$$y_{rk}(x) = \lim_{h\to 0} y_{rkh}(x) = \sum_{i=m_l}^{n-1} A_i \alpha_i(x, u(x)),$$

which is a solution of the variational equation (3.15) along $u(x)$. In addition,

$$y_{rk}^{(i)}(x_j) = \lim_{h\to 0} y_{rkh}^{(i)}(x_j) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$y_{rk}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y_{rk}(\eta_{ip}) = \lim_{h\to 0}\left[y_{rk}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y_{rk}(\eta_{ip})\right] = \delta_{ir},\ 0 \le i \le m_k - 1.$$

This completes the argument for $\frac{\partial u}{\partial u_{rk}}$, and consequently, part (a).

For the first piece of part (b), let $1 \le l \le k-1$, and consider $\frac{\partial u}{\partial x_l}$. Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define

$$\begin{aligned} z_{lh}(x) \quad = \quad & \frac{1}{h}[u(x, x_1, \ldots, x_l + h, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ & \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ & -u(x, x_1, \ldots, x_l, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ & \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})]. \end{aligned}$$

Note that for every $h \ne 0$ and $1 \le i \le m_l - 1$,

$$\begin{aligned} z_{lh}^{(i)}(x_l) \quad = \quad & \frac{1}{h}[u^{(i)}(x_l, x_1, \ldots, x_l + h, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ & \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ & -u^{(i)}(x_l, x_1, \ldots, x_l, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ & \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\ = \quad & \frac{1}{h}[u^{(i)}(x_l, x_1, \ldots, x_l + h, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \end{aligned}$$

$$
\begin{aligned}
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&-u^{(i)}(x_l+h,x_1,\ldots,x_l+h,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&+u^{(i)}(x_l+h,x_1,\ldots,x_l+h,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&-u^{(i)}(x_l,x_1,\ldots,x_l,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})]\\
=\ & -\frac{1}{h}[u^{(i+1)}(c_{x_l,h},x_1,\ldots,x_l+h,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\cdot h+u_{il}-u_{il}]\\
=\ & -u^{(i+1)}(c_{x_l,h},x_1,\ldots,x_l+h,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m}),
\end{aligned}
$$

where $c_{x_l,h}$ lies between x_l and x_l+h.

Also, for every $h\neq 0,\ 0\leq i\leq m_j-1,\ 1\leq j\leq k-1$, and $j\neq l$,

$$
\begin{aligned}
z_{lh}^{(i)}(x_j) &= \frac{1}{h}[u^{(i)}(x_j,x_1,\ldots,x_l+h,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&\quad -u^{(i)}(x_j,x_1,\ldots,x_l,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})]\\
&= \frac{1}{h}[u_{ij}-u_{ij}]\\
&= 0,
\end{aligned}
$$

and for every $h\neq 0$ and $0\leq i\leq m_k-1$,

$$
\begin{aligned}
z_{lh}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip}z_{lh}(\eta_{ip})\\
&= \frac{1}{h}[u^{(i)}(x_k,x_1,\ldots,x_l+h,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})
\end{aligned}
$$

$$
\begin{aligned}
&-u^{(i)}(x_k, x_1, \ldots, x_l, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\
&\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\
&-\sum_{p=1}^{m} \frac{r_{ip}}{h}[u(\eta_{ip}, x_1, \ldots, x_l + h, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\
&\qquad\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\
&-u(\eta_{ip}, x_1, \ldots, x_l, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\
&\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\
= \ & \frac{1}{h}[u_{ik} - u_{ik}] \\
= \ & 0.
\end{aligned}
$$

Now that we have established the boundary conditions, for $m_l \le i \le n-1$, let

$$\beta_i = u^{(i)}(x_l, x_1, \ldots, x_l, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}),$$

and

$$
\begin{aligned}
\epsilon_i = \epsilon_i(h) \ = \ & u^{(i)}(x_l, x_1, \ldots, x_l + h, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\
& \qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) - \beta_i.
\end{aligned}
$$

By Theorem 3.2, for $m_l \le i \le n-1$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.1 for solutions of initial value problems for (3.13), viewing $u(x)$ as the solution of an initial value problem, and denoting a solution $u(x) = y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \ldots, \beta_{n-1})$, we have

$$
\begin{aligned}
z_{lh}(x) \ = \ & \frac{1}{h}[y(x, x_l + h, u_{0l}, \ldots, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\
& \qquad \beta_{m_l+1} + \epsilon_{m_l+}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\
& -y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} \ldots, \beta_{n-1})].
\end{aligned}
$$

Then, by utilizing a telescoping sum, we have

$$z_{lh}(x) \ = \ \frac{1}{h}\{[y(x, x_l + h, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l},$$

$$\beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})$$

$$-y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})]$$

$$+[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})$$

$$-y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})]$$

$$+[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})$$

$$-y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})]$$

$$+ - \cdots$$

$$+[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})$$

$$-y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1})]\big\}.$$

By Theorem 3.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} z_{lh}(x) &= \beta(x, y(x, x_l + \bar{h}, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\ &\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ &\quad + \frac{\epsilon_{m_l}}{h} \alpha_{m_l}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ &\qquad\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ &\quad + \frac{\epsilon_{m_l+1}}{h} \alpha_{m_l+1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ &\qquad\qquad \beta_{m_l+1} + \bar{\epsilon}_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_{n-1}}{h} \alpha_{n-1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ &\qquad\qquad \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1}), \end{aligned}$$

where $\beta(x, y(\cdot))$ is the solution of the variational equation (3.15) along $y(\cdot)$ satisfying

$$\beta^{(i)}(x_l, y(\cdot)) = -y^{(i+1)}(x_l),\ 0 \le i \le n-1,$$

and where, for $0 \le j \le n-1$, $\alpha_j(x, y(\cdot))$ is the solution of the variational equation (3.15) along $y(\cdot)$ satisfying

$$\alpha_j^{(i)}(x_l) = \delta_{ij}, \quad 0 \le i \le n-1.$$

Furthermore, $x_l + \bar{h}$ is between x_l and $x_l + h$, and for $m_l \leq i \leq n-1$, $\beta_i + \bar{\epsilon}_i$ is between β_i and $\beta_i + \epsilon_i$.

Thus, to show $\lim_{h\to 0} z_{lh}(x)$ exists, it suffices to show, for $m_l \leq i \leq n-1$, $\lim_{h\to 0} \dfrac{\epsilon_i}{h}$ exists.

Now, from the construction of $z_{lh}(x)$, we have

$$z_{lh}^{(i)}(x_j) = 0, \qquad 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,\ j \neq l,$$

and

$$z_{lh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z_{lh}(\eta_{ip}) = 0, \qquad 0 \leq i \leq m_k - 1.$$

Hence, we have a system of $n - m_l$ linear equations with $n - m_l$ unknowns:

$$\begin{aligned}
- \quad & \beta^{(i)}(x_j, y(x, x_l + \bar{h}, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
= \quad & \frac{\epsilon_{m_l}}{h} \alpha_{m_l}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
& + \cdots \\
& + \frac{\epsilon_{n-1}}{h} \alpha_{n-1}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})),
\end{aligned}$$

$$0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,\ j \neq l,$$

and

$$\begin{aligned}
- \quad & \beta^{(i)}(x_k, y(x, x_l + \bar{h}, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
& + \sum_{p=1}^{m} r_{ip} \beta(\eta_{ip}, y(x, x_l + \bar{h}, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\
& \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
= \quad & \frac{\epsilon_{m_l}}{h} \Big[\alpha_{m_l}^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
& - \sum_{p=1}^{m} r_{ip} \alpha_{m_l}(\eta_{ip}, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l},
\end{aligned}$$

$$\beta_{m_l+1}+\epsilon_{m_l+1},\ldots,\beta_{n-1}+\epsilon_{n-1}))\Big]$$

$$+\cdots$$

$$+\frac{\epsilon_{n-1}}{h}\Big[\alpha_{n-1}^{(i)}(x_k,y(x,x_l,u_{0l},\ldots,u_{m_l-1,l},\beta_{m_l},\beta_{m_l+1},\ldots,\beta_{n-1}+\bar{\epsilon}_{n-1}))$$

$$-\sum_{p=1}^{m} r_{ip}\alpha_{n-1}(\eta_{ip},y(x,x_l,u_{0l},\ldots,u_{m_l-1,l},\beta_{m_l},\beta_{m_l+1},\ldots,\beta_{n-1}+\bar{\epsilon}_{n-1}))\Big],$$

$$0\le i\le m_k-1.$$

At this point in the proof, we will occassionally suppress the arguments of α and β as well as the subscripts of r and η, and limits of the summation. In the system of equations above, we notice that $y(\cdot)$ is not always the same. Therefore, we must consider the matrix

$$M:=\begin{pmatrix}
\alpha_{m_l}(x_1,u(x)) & \alpha_{m_l+1}(x_1,u(x)) & \cdots & \alpha_{n-1}(x_1,u(x))\\
\alpha'_{m_l}(x_1,u(x)) & \alpha'_{m_l+1}(x_1,u(x)) & \cdots & \alpha'_{n-1}(x_1,u(x))\\
\vdots & \vdots & \ddots & \vdots\\
\alpha_{m_l}^{(m_1-1)}(x_1,u(x)) & \alpha_{m_l+1}^{(m_1-1)}(x_1,u(x)) & \cdots & \alpha_{n-1}^{(m_1-1)}(x_1,u(x))\\
\vdots & \vdots & \ddots & \vdots\\
\alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \alpha_{m_l+1}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \cdots & \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1},u(x))\\
\alpha_{m_l}(x_{l+1},u(x)) & \alpha_{m_l+1}(x_{l+1},u(x)) & \cdots & \alpha_{n-1}(x_{l+1},u(x))\\
\vdots & \vdots & \ddots & \vdots\\
\alpha_{m_l}(x_k,u(x))- & \alpha_{m_l+1}(x_k,u(x))- & & \alpha_{n-1}(x_k,u(x))-\\
\sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x))\\
\vdots & \vdots & \ddots & \vdots\\
\alpha_{m_l}^{(m_k-1)}(x_k,u(x))- & \alpha_{m_l+1}^{(m_k-1)}(x_k,u(x))- & & \alpha_{n-1}^{(m_k-1)}(x_k,u(x))-\\
\sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x))
\end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R}$, $m_l \leq i \leq n-1$, not all zero such that

$$p_{m_l} \begin{pmatrix} \alpha_{m_l}(x_1, u(x)) \\ \alpha'_{m_l}(x_1, u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{m_l}(x_{l+1}, u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{m_l}(\eta, u(x)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \alpha_{n-1}(x_1, u(x)) \\ \alpha'_{n-1}(x_1, u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{n-1}(x_{l+1}, u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{n-1}(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$w(x, u(x)) := p_{m_l}\alpha_{m_l}(x, u(x)) + \cdots + p_{n-1}\alpha_{n-1}(x, u(x)).$$

Then, $w(x, u(x))$ is a nontrivial solution of (3.15), but

$$w^{(i)}(x_j, u(x)) = 0,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

and

$$w^{(i)}(x_k, u(x)) - \sum_{p=1}^{m} r_{ip} w(\eta_{ip}, u(x)) = 0,\ 0 \leq i \leq m_k - 1,$$

which when coupled with hypothesis (v) implies $w(x, u(x)) = 0$. Thus, $p_{m_l} = p_{m_{l+1}} = \cdots = p_{n-1} = 0$ which is a contradiction to the choice of the $p_i's$. Hence $\det(M) \neq 0$. Thus, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, for each $m_l \leq i \leq n-1$, we can solve $\epsilon_i(h)/h$ by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|} \times$$
$$\begin{vmatrix} \alpha_{m_l} & \cdots & \alpha_{i-2} & -\beta & \alpha_i & \cdots & \alpha_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_k-1)}- & & \alpha_{i-2}^{(m_k-1)} & -\beta^{(m_k-1)}+ & \alpha_i^{(m_k-1)}- & & \alpha_{n-1}^{(m_k-1)}- \\ \sum r\alpha_{m_l} & \cdots & \sum r\alpha_{i-2} & \sum r\beta & \sum r\alpha_i & \cdots & \sum r\alpha_{n-l} \end{vmatrix}$$

Note as $h \to 0$, $\det(M(h)) \to \det(M)$, and so for $m_l \le i \le n-1$, $\epsilon_i(h)/h \to \det(M_i)/\det M := B_i$ as $h \to 0$, where M_i is the $n - m_l \times n - m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\begin{aligned} \text{col}\Big[& -\beta(x_1, u(x)), \ldots, -\beta^{(m_1-1)}(x_1, u(x)), \ldots, \\ & -\beta(x_{l-1}, u(x)), \ldots, -\beta^{(m_{l-1}-1)}(x_{l-1}, u(x)), \\ & -\beta(x_{l+1}, u(x)), \ldots, -\beta^{(m_{l+1}-1)}(x_{l+1}, u(x)), \ldots, \\ & -\beta(x_k, u(x)) - \sum_{p=1}^{m} r_{0p}\beta(\eta_{0p}, u(x)), \ldots, \\ & -\beta^{(m_k-1)}(x_k, u(x)) - \sum_{p=1}^{m} r_{m_k-1,p}\beta(\eta_{m_k-1,p}, u(x))\Big]. \end{aligned}$$

Now let $z_l(x) = \lim_{h\to 0} z_{lh}(x)$, and note by construction of $z_{lh}(x)$,

$$z_l(x) = \frac{\partial u}{\partial x_l}(x).$$

Furthermore,

$$z_l(x) = \lim_{h\to 0} z_{lh}(x) = \beta(x, u(x)) + \sum_{i=m_l}^{n-1} B_i \alpha_i(x, u(x))$$

which is a solution of the variational equation (3.15) along $u(x)$. In addition,

$$z_l^{(i)}(x_j) = \lim_{h\to} z_{lh}^{(i)}(x_j) = -u^{(i+1)}(x_j)\delta_{jl},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$z_l^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z_l(\eta_{ip}) = \lim_{h\to 0} \left[z_{lh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z_{lh}(\eta_{ip}) \right] = 0,\ 0 \le i \le m_k - 1.$$

This completes the argument for $\frac{\partial u}{\partial x_l}$.

Next consider $\frac{\partial u}{\partial x_k}$. Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define

$$\begin{aligned} z_{kh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_k + h, \ldots, x_n, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u(x, x_1, \ldots, x_k, \ldots, x_n, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})]. \end{aligned}$$

Note that for every $h \neq 0$ and $0 \leq i \leq m_k - 1$,

$$\begin{aligned} z_{kh}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} z(\eta_{ip}) \\ &= \frac{1}{h}\Big[u^{(i)}(x_k, x_1, \ldots, x_k + h, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u^{(i)}(x_k, x_1, \ldots, x_k, x_n, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\ &\quad -\sum_{p=1}^{m} \frac{r_{ip}}{h}[u(\eta_{ip}, x_1, \ldots, x_k + h, x_n, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u(\eta_{ip}, x_1, \ldots, x_k, , u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})\Big] \\ &= \frac{1}{h}\Big[u^{(i)}(x_k, x_1, \ldots, x_k + h, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u^{(i)}(x_k + h, x_1, \ldots, x_k + h, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad +u^{(i)}(x_k + h, x_1, \ldots, x_k + h, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \end{aligned}$$

$$
\begin{aligned}
& -\sum_{p=1}^{m}\frac{r_{ip}}{h}[u(\eta_{ip},x_1,\ldots,x_k+h,x_n,u_{01},\ldots,u_{m_k-1,k}, \\
& \qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m}) \\
& -u(\eta_{ip},x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k}, \\
& \qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})] \\
& -u^{(i)}(x_k,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k}, \\
& \qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\Big] \\
= \; & -\frac{1}{h}[u^{(i+1)}(c_{x_k,h},x_1,\ldots,x_k+h,u_{01},\ldots,u_{m_k-1,k}, \\
& \qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\cdot h+u_{ik}-u_{ik}] \\
= \; & -u^{(i+1)}(c_{x_k,h},x_1,\ldots,x_k+h,u_{01},\ldots,u_{m_k-1,k}, \\
& \qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m}),
\end{aligned}
$$

where $c_{x_k,h}$ lies between x_k and x_k+h.

Also, for every $h\neq 0,\ 0\leq i\leq m_j-1$, and $1\leq j\leq k-1$,

$$
\begin{aligned}
z_{kh}^{(i)}(x_j) &= \frac{1}{h}[u^{(i)}(x_j,x_1,\ldots,x_k+h,u_{01},\ldots,u_{m_k-1,k}, \\
&\qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m}) \\
&\quad -u^{(i)}(x_j,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k}, \\
&\qquad \eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})] \\
&= \frac{1}{h}[u_{ij}-u_{ij}] \\
&= 0,
\end{aligned}
$$

For $m_l\leq i\leq n-1$, let

$$
\beta_i=u^{(i)}(x_l,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})
$$

and

$$\begin{aligned}\epsilon_i = \epsilon_i(h) &= u^{(i)}(x_l, x_1, \ldots, x_k + h, u_{01}, \ldots\ldots, u_{m_k-1,k},\\ &\qquad \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) - \beta_i.\end{aligned}$$

By Theorem 3.2, for $m_l \le i \le n-1$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.1 for solutions of initial value problems for (3.13), viewing the solution $u(x)$ as the solution of an initial value problem, and denoting a solution $u(x) = y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \ldots, \beta_{n-1})$, we have

$$\begin{aligned}z_{kh}(x) &= \frac{1}{h}[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})\\ &\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1})].\end{aligned}$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned}z_{kh}(x) &= \frac{1}{h}\big\{[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})\\ &\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})]\\ &\quad + [y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})\\ &\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})]\\ &\quad + - \cdots\\ &\quad + [y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})\\ &\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1})]\big\}.\end{aligned}$$

By Theorem 3.1 and the Mean Value Theorem, we obtain

$$\begin{aligned}z_{kh}(x) &= \frac{\epsilon_{m_l}}{h}\alpha_{m_l}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l},\\ &\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}))\\ &\quad + \frac{\epsilon_{m_l+1}}{h}\alpha_{m_l+1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l},\\ &\qquad \beta_{m_l+1} + \bar{\epsilon}_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}))\\ &\quad + \cdots\end{aligned}$$

$$+\frac{\epsilon_{n-1}}{h}\alpha_{n-1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l},$$
$$\beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1}),$$

where $\alpha_j(x, y(\cdot)),\ 0 \leq j \leq n-1$, is the solution of the variational equation (3.15) along $y(\cdot)$ satisfying

$$\alpha_j^{(i)}(x_l) = \delta_{ij}, \quad 0 \leq i \leq n-1.$$

Furthermore, for $m_l \leq i \leq n-1,\ \beta_i + \bar{\epsilon}_i$ is between β_i and $\beta_i + \epsilon_i$.

Thus, to show $\lim_{h\to 0} z_{kh}(x)$ exists, it suffices to show, for $m_l \leq i \leq n-1,\ \lim_{h\to 0}\frac{\epsilon_i}{h}$ exists.

Now, from the construction of $z_{kh}(x)$, for $1 \leq j \leq k-1,\ j \neq l$,

$$z_{kh}^{(i)}(x_j) = 0, \qquad\qquad 0 \leq i \leq m_j - 1,$$

and

$$z_{kh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z_{kh}(\eta_{ip}) = -u^{(i+1)}(c_{x_k,h}, \cdot, x_k + h, \cdot),\ 0 \leq i \leq m_k - 1.$$

Hence, we have a system of $n - m_l$ linear equations with $n - m_l$ unknowns:

$$\begin{aligned} 0 \quad = \quad & \frac{\epsilon_{m_l}}{h}\alpha_{m_l}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & + \cdots \\ & + \frac{\epsilon_{n-1}}{h}\alpha_{n-1}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ & \qquad \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})), \\ & 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,\ j \neq l, \end{aligned}$$

and

$$
\begin{aligned}
-u^{(i+1)}(c_{x_k,h},\cdot,x_k+h,\cdot) \;=\;& \frac{\epsilon_{m_l}}{h}\Big[\alpha^{(i)}_{m_l}(x_k,y(x,x_l,u_{0l},\ldots,u_{m_l-1,l},\beta_{m_l}+\epsilon_{m_l},\\
&\qquad \beta_{m_l+1}+\epsilon_{m_l+1},\ldots,\beta_{n-1}+\epsilon_{n-1}))\\
&-\sum_{p=1}^{m} r_{ip}\alpha_{m_l}(\eta_{ip},y(x,x_l,u_{0l},\ldots,u_{m_l-1,l},\beta_{m_l}+\epsilon_{m_l},\\
&\qquad \beta_{m_l+1}+\epsilon_{m_l+1},\ldots,\beta_{n-1}+\epsilon_{n-1}))\Big]\\
&+\cdots\\
&+\frac{\epsilon_{n-1}}{h}\Big[\alpha^{(i)}_{n-1}(x_k,y(x,x_l,u_{0l},\ldots,u_{m_l-1,l},\beta_{m_l},\\
&\qquad \beta_{m_l+1},\ldots,\beta_{n-1}+\bar{\epsilon}_{n-1}))\\
&-\sum_{p=1}^{m} r_{ip}\alpha_{n-1}(\eta_{ip},y(x,x_l,u_{0l},\ldots,u_{m_l-1,l},\beta_{m_l},\\
&\qquad \beta_{m_l+1},\ldots,\beta_{n-1}+\bar{\epsilon}_{n-1}))\Big],\\
&0\le i\le m_k-1.
\end{aligned}
$$

At this point in the proof, we will occassionally suppress the arguments of α, the subscripts of r and η, and limits of the summation. Notice that in the above system, most of the $y(\cdot)$ are different. Thus, we consider the matrix

$$
M := \begin{pmatrix}
\alpha_{m_l}(x_1,u(x)) & \alpha_{m_l+1}(x_1,u(x)) & \cdots & \alpha_{n-1}(x_1,u(x))\\
\alpha'_{m_l}(x_1,u(x)) & \alpha'_{m_l+1}(x_1,u(x)) & \cdots & \alpha'_{n-1}(x_1,u(x))\\
\vdots & \vdots & \ddots & \vdots\\
\alpha^{(m_1-1)}_{m_l}(x_1,u(x)) & \alpha^{(m_1-1)}_{m_l+1}(x_1,u(x)) & \cdots & \alpha^{(m_1-1)}_{n-1}(x_1,u(x))\\
\vdots & \vdots & \ddots & \vdots\\
\alpha^{(m_{l-1}-1)}_{ml}(x_{l-1},u(x)) & \alpha^{(m_{l-1}-1)}_{m_l+1}(x_{l-1},u(x)) & \cdots & \alpha^{(m_{l-1}-1)}_{n-1}(x_{l-1},u(x))\\
\alpha_{m_l}(x_{l+1},u(x)) & \alpha_{m_l+1}(x_{l+1},u(x)) & \cdots & \alpha_{n-1}(x_{l+1},u(x))\\
\vdots & \vdots & \ddots & \vdots\\
\alpha_{m_l}(x_k,u(x))- & \alpha_{m_l+1}(x_k,u(x))- & & \alpha_{n-1}(x_k,u(x))-\\
\sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x))\\
\vdots & \vdots & \ddots & \vdots\\
\alpha^{(m_k-1)}_{m_l}(x_k,u(x))- & \alpha^{(m_k-1)}_{m_l+1}(x_k,u(x))- & & \alpha^{(m_k-1)}_{n-1}(x_k,u(x))-\\
\sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x))
\end{pmatrix}.
$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R},\ m_l \leq i \leq n-1$, not all zero, such that

$$p_{m_l} \begin{pmatrix} \alpha_{m_l}(x_1, u(x)) \\ \alpha'_{m_l}(x_1, u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{m_l}(x_{l+1}, u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{m_l}(\eta, u(x)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \alpha_{n-1}(x_1, u(x)) \\ \alpha'_{n-1}(x_1, u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{n-1}(x_{l+1}, u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{n-1}(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$w(x, u(x)) := p_{m_l}\alpha_{m_l}(x, u(x)) + \cdots + p_{n-1}\alpha_{n-1}(x, u(x)).$$

Then, $w(x, u(x))$ is a nontrivial solution of (3.15), but

$$w^{(i)}(x_j, u(x)) = 0,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

and

$$w^{(i)}(x_k, u(x)) - \sum_{p=1}^{m} r_{ip} w(\eta_{ip}, u(x)) = 0,\ 0 \leq i \leq m_k - 1,$$

which when coupled with hypothesis (v) implies $w(x, u(x)) = 0$. Thus, $p_{m_l} = p_{m_{l+1}} = \cdots = p_{n-1} = 0$ which is a contradition to the choice of each $p_i,\ m_l \leq i \leq n-1$. Hence $\det(M) \neq 0$. Thus, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, we can solve $\epsilon_i(h)/h$, for $m_l \leq i \leq n-1$, by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|}\times$$

$$\begin{vmatrix} \alpha_{m_l} & \cdots & \alpha_{i-2} & 0 & \alpha_i & \cdots & \alpha_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{array}{c}\alpha_{m_l}- \\ \sum r\alpha_{m_l}\end{array} & \cdots & \begin{array}{c}\alpha_{i-2}- \\ \sum r\alpha_{i-2}\end{array} & -u'(c_{x_j,h}, x_k+h) & \begin{array}{c}\alpha_{i}- \\ \sum r\alpha_{i}\end{array} & \cdots & \begin{array}{c}\alpha_{n-1}- \\ \sum r\alpha_{n-l}\end{array} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{array}{c}\alpha_{m_l}^{(m_k-1)}- \\ \sum r\alpha_{m_l}\end{array} & \cdots & \begin{array}{c}\alpha_{i-2}^{(m_k-1)}- \\ \sum r\alpha_{i-2}\end{array} & -u^{(m_k)}(c_{x_j,h}, x_k+h) & \begin{array}{c}\alpha_{i}^{(m_k-1)}- \\ \sum r\alpha_{i}\end{array} & \cdots & \begin{array}{c}\alpha_{n-1}^{(m_k-1)}- \\ \sum r\alpha_{n-l}\end{array} \end{vmatrix}$$

Note as $h \to 0$, $\det(M(h)) \to \det(M)$ and so for $m_l \le i \le n-1$, $\epsilon_i(h)/h \to \det(M_i)\det(M) := B_i$ as $h \to 0$, where M_i is the $n - m_l \times n - m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\mathrm{col}[0, \ldots, 0, -u'(x_k), \ldots, -u^{(m_k)}(x_k)].$$

Now let $z_k(x) = \lim_{h\to 0} z_{kh}(x)$, and note by construction of $z_{kh}(x)$,

$$z_k(x) = \frac{\partial u}{\partial x_k}(x).$$

Furthermore,

$$z_k(x) = \lim_{h\to 0} z_{kh}(x) = \beta(x, u(x)) + \sum_{i=m_l}^{n-1} B_i\alpha_i(x, u(x)),$$

which is a solution of the variational equation (3.15) along $u(x)$. In addition,

$$z_k^{(i)}(x_j) = \lim_{h\to 0} z_{kh}^{(i)}(x_j) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

$$\begin{aligned} z_k^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z_k(\eta_{ip}) &= \lim_{h\to 0}\left[z_{hk}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z_{kh}(\eta_{ip})\right] \\ &= -u^{(i+1)}(x_k),\ 0 \le i \le m_k - 1. \end{aligned}$$

This completes the argument for $\frac{\partial u}{\partial x_k}$, and hence the proof of part (b).

Now on to part (c). Let $0 \leq r \leq m_k - 1,\ 1 \leq s \leq m$, and consider $\frac{\partial u}{\partial \eta_{rs}}$. Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define

$$\begin{aligned} w_{rsh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs} + h, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})]. \end{aligned}$$

Note that for every $h \neq 0$,

$$\begin{aligned} w_{rsh}^{(r)}(x_k) &- \sum_{p=1}^{m} r_{rp} w_{rsh}(\eta_{rp}) \\ &= \frac{1}{h}[u^{(r)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs} + h, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u^{(r)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\ &\quad -\sum_{p=1}^{m} \frac{r_{rp}}{h} \Big[u(\eta_{rp}, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs} + h, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u(\eta_{rp}, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})\Big] \\ &= \frac{1}{h}[u^{(r)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs} + h, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -r_{rs} u(\eta_{rs} + h, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs} + h, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad +r_{rs} u(\eta_{rs} + h, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs} + h, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -\sum_{p=1}^{m} \frac{r_{rp}}{h} u(\eta_{rp}, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \end{aligned}$$

$$
\begin{aligned}
&\qquad \eta_{01},\ldots,\eta_{rs}+h,\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m}) - u_{ik}\\
&= \frac{1}{h}[u_{ik} + r_{rs}u(\eta_{rs}+h,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{rs}+h,\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&\quad -r_{rs}u(\eta_{rs},x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{rs}+h,\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m}) - u_{rk}]\\
&= \frac{r_{rs}}{h}[u'(c_{\eta_{rs},h},x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{rs}+h,\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})(\eta_{rs}+h-\eta_{rs})\\
&= r_{rs}u'(c_{\eta_{rs},h},x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{rs}+h,\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m}),
\end{aligned}
$$

where $c_{\eta_{rs},h}$ lies between η_{rs} and $\eta_{rs}+h$.

Also, for every $h \neq 0$ and $0 \leq i \leq m_k - 1,\ i \neq r$,

$$
\begin{aligned}
w_{rsh}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} w_{rsh}(\eta_{ip})\\
&= \frac{1}{h}[u^{(i)}(x_k,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{rs}+h,\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&\quad -u^{(i)}(x_k,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{rs},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})]\\
&\quad -\sum_{p=1}^{m}\frac{r_{ip}}{h}\Big[u(\eta_{ip},x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{rs}+h,\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\\
&\quad -u(\eta_{ip},x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\\
&\qquad \eta_{01},\ldots,\eta_{rs},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{m_k-1,m})\Big]\\
&= [u_{ik} - u_{ik}]\\
&= 0,
\end{aligned}
$$

and for every $h \neq 0,\ 0 \leq i \leq m_j - 1$, and $1 \leq j \leq k-1$,

$$\begin{aligned} w_{rsh}^{(i)}(x_j) &= \frac{1}{h}[u^{(i)}(x_j, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs} + h, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) \\ &\quad -u^{(i)}(x_j, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})] \\ &= \frac{1}{h}[u_{ij} - u_{ij}] \\ &= 0. \end{aligned}$$

Let $1 \leq l \leq k-1$, and for $m_1 \leq i \leq n-1$, let

$$\beta_i = u^{(i)}(x_1, x_1, \ldots, x_n, u_{01}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{rs}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}),$$

and

$$\begin{aligned} \epsilon_i = \epsilon_i(h) &= u^{(i)}(x_l, x_1, \ldots, x_n, u_{01}, \ldots, u_{m_k-1,k}, \\ &\qquad \eta_{01}, \ldots, \eta_{rs} + h, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}) - \beta_i. \end{aligned}$$

By Theorem 3.2, for $m_1 \leq i \leq n-1,\ \epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.1 for solutions of initial value problems for (3.13), viewing the solution $u(x)$ as the solution of an initial value problem, and denoting the solution $u(x) = y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \ldots, \beta_{n-1})$, we have

$$\begin{aligned} w_{rsh}(x) &= \frac{1}{h}[y(x, x_l, u_{0l}, \ldots, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\ &\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ &\quad -y(x, x_l, u_{01}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} \ldots, \beta_{n-1})]. \end{aligned}$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} w_{rsh}(x) &= \frac{1}{h}\{[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ &\quad -y(x, x_l, u_{01}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})] \end{aligned}$$

$$+[y(x,x_l,u_{01},\dots,u_{m_l-1,l},\beta_{m_l},\beta_{m_l+1}+\epsilon_{m_l+1},\dots,\beta_{n-1}+\epsilon_{n-1})$$
$$-y(x,x_l,u_{0l},\dots,u_{m_l-1,l},\beta_{m_l},\beta_{m_l+1},\dots,\beta_{n-1}+\epsilon_{n-1})]$$
$$+-\cdots$$
$$+[y(x,x_l,u_{01},\dots,u_{m_l-1,l},\beta_{m_l},\beta_{m_l+1},\dots,\beta_{n-1}+\epsilon_{n-1})$$
$$-y(x,x_l,u_{0l},\dots,u_{m_l-1,l},\beta_{m_l},\beta_{m_l+1},\dots,\beta_{n-1})]\big\}.$$

By Theorem 3.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} w_{rsh}(x) &= \frac{\epsilon_{m_l}}{h}\alpha_{m_l}(x,y(x,x_l,u_{0l},\dots,u_{m_l-1,l},\beta_{m_l}+\bar{\epsilon}_{m_l},\\ &\qquad \beta_{m_l+1}+\epsilon_{m_l+1},\dots,\beta_{n-1}+\epsilon_{n-1}))\\ &\quad +\frac{\epsilon_{m_l+1}}{h}\alpha_{m_l+1}(x,y(x,x_l,u_{0l},\dots,u_{m_l-1,l},\beta_{m_l},\\ &\qquad \beta_{m_l+1}+\bar{\epsilon}_{m_l+1},\dots,\beta_{n-1}+\epsilon_{n-1}))\\ &\quad +\cdots\\ &\quad +\frac{\epsilon_{n-1}}{h}\alpha_{n-1}(x,y(x,x_l,u_{0l},\dots,u_{m_l-1,l},\beta_{m_l},\\ &\qquad \beta_{m_l+1},\dots,\beta_{n-1}+\bar{\epsilon}_{n-1}), \end{aligned}$$

where $\alpha_j(x,y(\cdot)),\ 0\le j\le n-1$, is the solution of the variational equation (3.15) along $y(\cdot)$ satisfying, in each case,

$$\alpha_j^{(i)}(x_l)=\delta_{ij},\quad 0\le i\le n-1,$$

Furthermore, for $m_l\le i\le n-1,\ \beta_i+\bar{\epsilon}_i$ is between β_i and $\beta_i+\epsilon_i$.

Thus, to show $\lim_{h\to 0} w_{rsh}(x)$ exists, it suffices to show for $m_l\le i\le n-1,\ \lim_{h\to 0}\frac{\epsilon_i}{h}$ exists.

Now, from the construction of $w_{rsh}(x)$,

$$w_{rsh}^{(i)}(x_j)=0,\qquad 0\le i\le m_j-1,\ 1\le j\le k-1,\ j\ne l,$$

and

$$w_{rsh}^{(i)}(x_k)-\sum_{p=1}^{m} r_{ip}w_{rsh}(\eta_{ip})=r_{is}u'(c_{\eta_{is},h},\cdot,\eta_{rs}+h,\cdot)\delta_{ir},\ 0\le i\le m_k-1.$$

Hence, we have a system of $n - m_l$ linear equations with $n - m_l$ unknowns:

$$\begin{aligned}0 \;=\; & \frac{\epsilon_{m_l}}{h}\alpha_{m_l}^{(i)}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) + \cdots \\ & + \frac{\epsilon_{n-1}}{h}\alpha_{n-1}^{(i)}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})), \\ & 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l,\end{aligned}$$

and

$$\begin{aligned}& r_{is}u'(c_{\eta_{is},h}, \cdot, \eta_{rs} + h, \cdot)\delta_{ir} \\ & \quad = \frac{\epsilon_{m_l}}{h}\Big[\alpha_{m_l}^{(i)}(x_k, y(x, x_1, u_{01}, \ldots, u_{m_1-1,1}, \\ & \qquad\qquad \beta_{m_l} + \bar{\epsilon}_{m_1}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & \qquad - \sum_{p=1}^{m} r_{ip}\alpha_{m_l}(\eta_{ip}, y(x, x_1, u_{01}, \ldots, u_{m_l-1,1}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad\qquad\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}))\Big] + \cdots \\ & \qquad + \frac{\epsilon_{n-1}}{h}\Big[\alpha_{n-1}^{(i)}(x_k, y(x, x_1, u_{01}, \ldots, u_{m_l-1,1}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})) \\ & \qquad - \sum_{p=1}^{m} r_{ip}\alpha_{n-1}(\eta_{ip}, y(x, x_1, u_{01}, \ldots, u_{m_l-1,1}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1}))\Big], \\ & \qquad 0 \le i \le m_k - 1.\end{aligned}$$

At this point in the proof, we will occassionally suppress the arguments of α, the subscripts of r and η, and limits of the summation.

As in the other parts, we must consider the following matrix as $y(\cdot)$ is not consistent within the above system of equations.

$$M := \begin{pmatrix} \alpha_{m_l}(x_1,u(x)) & \alpha_{m_l+1}(x_1,u(x)) & \cdots & \alpha_{n-1}(x_1,u(x)) \\ \alpha'_{m_l}(x_1,u(x)) & \alpha'_{m_l+1}(x_1,u(x)) & \cdots & \alpha'_{n-1}(x_1,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{ml}^{(m_1-1)}(x_1,u(x)) & \alpha_{m_l+1}^{(m_1-1)}(x_1,u(x)) & \cdots & \alpha_{n-1}^{(m_1-1)}(x_1,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{ml}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \alpha_{m_l+1}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \cdots & \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{m_l}(x_{l+1},u(x)) & \alpha_{m_l+1}(x_{l+1},u(x)) & \cdots & \alpha_{n-1}(x_{l+1},u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}(x_k,u(x))- & \alpha_{m_l+1}(x_k,u(x))- & \cdots & \alpha_{n-1}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k,u(x))- & \alpha_{m_l+1}^{(m_k-1)}(x_k,u(x))- & \cdots & \alpha_{n-1}^{(m_k-1)}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R},\ m_l \leq i \leq n-1$ not all zero such that

$$p_{m_l}\begin{pmatrix} \alpha_{m_l}(x_1,u(x)) \\ \alpha'_{m_l}(x_1,u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{m_l}(x_{l+1},u(x)) \\ \vdots \\ \alpha_{m_l}^{m_k-1}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) \end{pmatrix} + \cdots + p_{n-1}\begin{pmatrix} \alpha_{n-1}(x_1,u(x)) \\ \alpha'_{n-1}(x_1,u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{n-1}(x_{l+1},u(x)) \\ \vdots \\ \alpha_{n-1}^{m_k-1}(x_k,u(x))- \\ \sum r\alpha_{n-1}(\eta,u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$\omega(x,u(x)) := p_{m_l}\alpha_{m_l}(x,u(x)) + \cdots + p_{n-1}\alpha_{n-1}(x,u(x)).$$

Then, $\omega(x, u(x))$ is a nontrivial solution of (3.15), but

$$\omega^{(i)}(x_j, u(x)) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$\omega^{(i)}(x_k, u(x)) - \sum_{p=1}^{m} r_{ip}\omega(\eta_{ip}, u(x)) = 0,\ 0 \le i \le m_k - 1,$$

which when coupled with hypothesis (v) implies $\omega(x, u(x)) = 0$ forcing each $p_i = 0,\ m_l \le i \le n-2$. This is a contradiction to the choice of the $p_i's$. Hence $\det(M) \neq 0$. Thus, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, we can solve $\epsilon_i(h)/h$, for $m_l \le i \le n-1$, by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|}\times$$

$$\begin{vmatrix}
\alpha_{m_l} & \cdots & \alpha_{i-2} & 0 & \alpha_i & \cdots & \alpha_{n-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\begin{matrix}\alpha_{m_l}- \\ \sum r\alpha_{m_l}\end{matrix} & \cdots & \begin{matrix}\alpha_{i-2}- \\ \sum r\alpha_{i-2}\end{matrix} & 0 & \begin{matrix}\alpha_{i}- \\ \sum r\alpha_{i}\end{matrix} & \cdots & \begin{matrix}\alpha_{n-1}- \\ \sum r\alpha_{n-l}\end{matrix} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\begin{matrix}\alpha_{m_l}^{(r)}- \\ \sum r\alpha_{m_l}\end{matrix} & \cdots & \begin{matrix}\alpha_{i-2}^{(r)}- \\ \sum r\alpha_{i-2}\end{matrix} & r_{rs}u'(c_{\eta_{rs}}, \eta_{rs}+h) & \begin{matrix}\alpha_{i}^{(r)}- \\ \sum r\alpha_{i}\end{matrix} & \cdots & \begin{matrix}\alpha_{n-1}^{(r)}- \\ \sum r\alpha_{n-l}\end{matrix} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\begin{matrix}\alpha_{m_l}^{(m_k-1)}- \\ \sum r\alpha_{m_l}\end{matrix} & \cdots & \begin{matrix}\alpha_{i-2}^{(m_k-1)}- \\ \sum r\alpha_{i-2}\end{matrix} & 0 & \begin{matrix}\alpha_{i}^{(m_k-1)}- \\ \sum r\alpha_{i}\end{matrix} & \cdots & \begin{matrix}\alpha_{n-1}^{(m_k-1)}- \\ \sum r\alpha_{n-l}\end{matrix}
\end{vmatrix}.$$

Note as $h \to 0$, $\det(M(h)) \to \det(M)$ and so for $m_l \le i \le n-1$, $\epsilon_i(h)/h \to \det(M_i)/\det(M) := C_i$ as $h \to 0$, where M_i is the $n - m_l \times n - m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\text{col}[0, \ldots, 0, r_{rs}u'(\eta_{rs}), 0, \ldots, 0].$$

Now let $w_{rs}(x) = \lim_{h\to 0} w_{rsh}(x)$, and note by construction of $w_{rsh}(x)$,

$$w_{rs}(x) = \frac{\partial u}{\partial \eta_{rs}}(x).$$

Furthermore,

$$w_{rs}(x) = \lim_{h\to 0} w_{rsh}(x) = \sum_{p=m_l}^{n-1} C_i \alpha_i(x, u(x)),$$

which is a solution of the variational equation (3.15) along $u(x)$. In addition,

$$w_{rs}^{(i)}(x_j) = \lim_{h\to 0} w_{rsh}^{(i)}(x_j) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$\begin{aligned} w_{rs}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} w_{rs}(\eta_{ip}) &= \lim_{h\to 0}\left[w_{rsh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} w_{rsh}(\eta_{ip})\right] \\ &= r_{is} u'(\eta_{is})\delta_{ir},\ 0 \le i \le m_k - 1. \end{aligned}$$

This completes the argument for $\frac{\partial u}{\partial \eta_{rs}}$.

Lastly, the proof of part (d). Let $0 \le r \le m_k - 1,\ 1 \le s \le m$, and consider $\frac{\partial u}{\partial r_{rs}}$. Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define

$$\begin{aligned} v_{rsh}(x) &= \frac{1}{h}[u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad r_{01}, \ldots, r_{rs} + h, \ldots, r_{m_k-1,m}) \\ &\quad - u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad r_{01}, \ldots, r_{rs}, \ldots, r_{m_k-1,m})]. \end{aligned}$$

Note that for every $h \neq 0$,

$$\begin{aligned} v_{rsh}^{(r)}(x_k) &- \sum_{p=1}^{m} r_{rp} v_{rsh}(\eta_{rp}) \\ &= \frac{1}{h}[u^{(r)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad r_{01}, \ldots, r_{rs} + h, \ldots, r_{m_k-1,m}) \\ &\quad - u^{(r)}(x_k, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{m_k-1,m}, \end{aligned}$$

$$r_{01},\dots,r_{rs},\dots,r_{m_k-1,m})]$$

$$-\sum_{p=1}^{m}\frac{r_{rp}}{h}\Big[u(\eta_{rp},x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs}+h,\dots,r_{m_k-1,m})$$

$$-u(\eta_{rp},x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs},\dots,r_{m_k-1,m})\Big]$$

$$= \frac{1}{h}[u^{(r)}(x_k,x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs}+h,\dots,r_{m_k-1,m})$$

$$-hu(\eta_{rs},x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs}+h,\dots,r_{m_k-1,m})$$

$$+hu(\eta_{rs},x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs}+h,\dots,r_{m_k-1,m})$$

$$-\sum_{p=1}^{m}\frac{r_{rp}}{h}u(\eta_{rp},x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs}+h,\dots,r_{m_k-1,m})-u_{rk}$$

$$= \frac{1}{h}[u_{rk}+hu(\eta_{rs},x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs}+h,\dots,r_{m_k-1,m})-u_{rk}]$$

$$= u(\eta_{rs},x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs}+h,\dots,r_{m_k-1,m}).$$

Also, for every $h\neq 0$ and $0\leq i\leq m_k-1,\ i\neq r$,

$$v^{(i)}_{rsh}(x_k) - \sum_{p=1}^{m}r_{ip}v_{rsh}(\eta_{ip})$$

$$= \frac{1}{h}[u^{(i)}(x_k,x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs}+h,\dots,r_{m_k-1,m})$$

$$-u^{(i)}(x_k,x_1,\dots,x_k,u_{01},\dots,u_{m_k-1,k},\eta_{01},\dots,\eta_{m_k-1,m},$$

$$r_{01},\dots,r_{rs},\dots,r_{m_k-1,m})]$$

$$-\sum_{p=1}^{m}\frac{r_{ip}}{h}\Big[u(\eta_{ip},x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\eta_{01},\ldots,\eta_{m_k-1,m},$$
$$r_{01},\ldots,r_{rs}+h,\ldots,r_{m_k-1,m})$$
$$-u(\eta_{ip},x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\eta_{01},\ldots,\eta_{m_k-1,m},$$
$$r_{01},\ldots,r_{rs},\ldots,r_{m_k-1,m})\Big],$$
$$= [u_{ik}-u_{ik}]$$
$$= 0,$$

and for every $h \neq 0,\ 0 \le i \le m_j - 1$, and $1 \le j \le k-1$,

$$\begin{aligned} v_{rsh}^{(i)}(x_j) &= \frac{1}{h}[u^{(i)}(x_j,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\eta_{01},\ldots,\eta_{m_k-1,m},\\ &\qquad r_{01},\ldots,r_{rs}+h,\ldots,r_{m_k-1,m})\\ &\quad -u^{(i)}(x_j,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\eta_{01},\ldots,\eta_{m_k-1,m},\\ &\qquad r_{01},\ldots,r_{rs},\ldots,r_{m_k-1,m})]\\ &= \frac{1}{h}[u_{ij}-u_{ij}]\\ &= 0. \end{aligned}$$

Let $1 \le l \le k-1$, and for $m_1 \le i \le n-1$, let

$$\beta_i = u^{(i)}(x_l,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\eta_{01},\ldots,\eta_{m_k-1,m},r_{01},\ldots,r_{rs},\ldots,r_{m_k-1,m})$$

and

$$\begin{aligned} \epsilon_i = \epsilon_i(h) &= u^{(i)}(x_l,x_1,\ldots,x_k,u_{01},\ldots,u_{m_k-1,k},\eta_{01},\ldots,\eta_{m_k-1,m},\\ &\qquad r_{01},\ldots,r_{rs}+h,\ldots,r_{m_k-1,m}) - \beta_i. \end{aligned}$$

By Theorem 3.2, for $m_l \le i \le n-1,\ \epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 3.1 for solutions of initial value problems for (3.13), viewing the solution $u(x)$ as the solution of an initial value problem, and denoting the solution $u(x) = y(x,x_l,u_{0l},\ldots,u_{m_l-1,l},\beta_{m_l},\ldots,\beta_{n-1})$, we have

$$
\begin{aligned}
v_{rsh}(x) &= \frac{1}{h}[y(x, x_l, u_{0l}, \ldots, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\
&\quad - y(x, x_l, u_{01}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} \ldots, \beta_{n-1})].
\end{aligned}
$$

Then, by utilizing a telescoping sum, we have

$$
\begin{aligned}
v_{rsh}(x) &= \frac{1}{h}\big\{[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\
&\quad - y(x, x_l, u_{01}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})] \\
&\quad + [y(x, x_l, u_{01}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\
&\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})] \\
&\quad + - \cdots \\
&\quad + [y(x, x_l, u_{01}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\
&\quad - y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1})]\big\}.
\end{aligned}
$$

By Theorem 3.1 and the Mean Value Theorem, we obtain

$$
\begin{aligned}
v_{rsh}(x) &= \frac{\epsilon_{m_l}}{h}\alpha_{m_l}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\
&\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
&\quad + \frac{\epsilon_{m_l+1}}{h}\alpha_{m_l+1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\
&\qquad \beta_{m_l+1} + \bar{\epsilon}_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
&\quad + \cdots \\
&\quad + \frac{\epsilon_{n-1}}{h}\alpha_{n-1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1}),
\end{aligned}
$$

where $\alpha_j(x, y(\cdot)),\ 0 \le j \le n-1$, is the solution of the variational equation (3.15) along $y(\cdot)$ satisfying, in each case,

$$
\alpha_j^{(i)}(x_l) = \delta_{ij}, \quad 0 \le i \le n-1,
$$

Furthermore, for $m_l \le i \le n-1,\ \beta_i + \bar{\epsilon}_i$ is between β_i and $\beta_i + \epsilon_i$.

Thus, to show $\lim_{h\to 0} v_{rsh}(x)$ exists, it suffices to show for $m_l \le i \le n-1$, $\lim_{h\to 0} \frac{\epsilon_i}{h}$ exists.

Now, from the construction of $v_{rsh}(x)$,

$$v_{rsh}^{(i)}(x_j) = 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l.$$

$$\begin{aligned} v_{rsh}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} v_{rsh}(\eta_{ip}) \\ &= u(\eta_{is}, \cdot, r_{rs} + h, \cdot)\delta_{ir},\ 0 \le i \le m_k - 1. \end{aligned}$$

Hence, we have a system of $n - m_l$ linear equations with $n - m_l$ unknowns:

$$\begin{aligned} 0 \ = \ & \frac{\epsilon_{m_l}}{h}\alpha_{m_l}^{(i)}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & + \cdots \\ & + \frac{\epsilon_{n-1}}{h}\alpha_{n-1}^{(i)}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})), \\ & 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l, \end{aligned}$$

and

$$\begin{aligned} u(\eta_{rs}, \cdot, r_{rs} + h, \cdot)\delta_{ir} \ = \ & \frac{\epsilon_{m_l}}{h}\Big[\alpha_{m_l}^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & - \sum_{p=1}^{m} r_{ip}\alpha_{m_l}(\eta_{ip}, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}))\Big] \\ & + \cdots \\ & + \frac{\epsilon_{n-1}}{h}\Big[\alpha_{n-1}^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ & \qquad \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})) \\ & - \sum_{p=1}^{m} r_{ip}\alpha_{n-1}(\eta_{ip}, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ & \qquad \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1}))\Big], \\ & 0 \le i \le m_k - 1. \end{aligned}$$

At this point in the proof, we will occassionally suppress the arguments of α, the subscripts of r and η, and limits of the summation.

Since in the above system we have varying $y(\cdot)$, we consider the matrix

$$M := \begin{pmatrix} \alpha_{m_l}(x_1, u(x)) & \alpha_{m_l+1}(x_1, u(x)) & \cdots & \alpha_{n-1}(x_1, u(x)) \\ \alpha'_{m_l}(x_1, u(x)) & \alpha'_{m_l+1}(x_1, u(x)) & \cdots & \alpha'_{n-1}(x_1, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_1-1)}(x_1, u(x)) & \alpha_{m_l+1}^{(m_1-1)}(x_1, u(x)) & \cdots & \alpha_{n-1}^{(m_1-1)}(x_1, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1}, u(x)) & \alpha_{m_l+1}^{(m_{l-1}-1)}(x_{l-1}, u(x)) & \cdots & \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{m_l}(x_{l+1}, u(x)) & \alpha_{m_l+1}(x_{l+1}, u(x)) & \cdots & \alpha_{n-1}(x_{l+1}, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}(x_k, u(x)) - & \alpha_{m_l+1}(x_k, u(x)) - & & \alpha_{n-1}(x_k, u(x)) - \\ \sum r\alpha_{m_l}(\eta, u(x)) & \sum r\alpha_{m_l+1}(\eta, u(x)) & \cdots & \sum r\alpha_{n-1}(\eta, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k, u(x)) - & \alpha_{m_l+1}^{(m_k-1)}(x_k, u(x)) - & & \alpha_{n-1}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{m_l}(\eta, u(x)) & \sum r\alpha_{m_l+1}(\eta, u(x)) & \cdots & \sum r\alpha_{n-1}(\eta, u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R},\ m_l \leq i \leq n-1$ not all zero such that

$$p_{m_l} \begin{pmatrix} \alpha_{m_l}(x_1, u(x)) \\ \alpha'_{m_l}(x_1, u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{m_l}(x_{l+1}, u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{m_l}(\eta, u(x)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \alpha_{n-1}(x_1, u(x)) \\ \alpha'_{n-1}(x_1, u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1}, u(x)) \\ \alpha_{n-1}(x_{l+1}, u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_k-1)}(x_k, u(x)) - \\ \sum r\alpha_{n-1}(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$w(x,u(x)) := p_{m_l}\alpha_{m_l}(x,u(x)) + \cdots + p_{n-1}\alpha_{n-1}(x,u(x)).$$

Then, $w(x,u(x))$ is a nontrivial solution of (3.15), but

$$w^{(i)}(x_j,u(x)) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$w^{(i)}(x_k,u(x)) - \sum_{p=1}^{m} r_{ip} w(\eta_{ip},u(x)) = 0,\ 0 \le i \le m_k - 1,$$

which when coupled with hypothesis (v) implies $w(x,u(x)) = 0$ forcing $p_i = 0,\ m_l \le i \le n-2$. This is a contradtiction to the choice of the $p_i's$. Hence $\det(M) \ne 0$. Thus, as a result of continuous dependence, for $h \ne 0$ and sufficiently small, $\det(M(h)) \ne 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, we can solve $\epsilon_i(h)/h$, for $m_l \le i \le n-1$, by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|}\times$$

$$\begin{vmatrix}
\alpha_{m_l} & \cdots & \alpha_{i-2} & 0 & \alpha_i & \cdots & \alpha_{n-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{m_l} - & & \alpha_{i-2} - & & \alpha_i - & & \alpha_{n-1} - \\
\sum r\alpha_{m_l} & \cdots & \sum r\alpha_{i-2} & 0 & \sum r\alpha_i & \cdots & \sum r\alpha_{n-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{m_l}^{(r)} - & & \alpha_{i-2}^{(r)} - & & \alpha_i^{(r)} - & & \alpha_{n-1}^{(r)} - \\
\sum r\alpha_{m_l} & \cdots & \sum r\alpha_{i-2} & u(\eta_{rs}, r_{rs}+h) & \sum r\alpha_i & \cdots & \sum r\alpha_{n-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{m_l}^{(m_k-1)} - & & \alpha_{i-2}^{(m_k-1)} - & & \alpha_i^{(m_k-1)} - & & \alpha_{n-1}^{(m_k-1)} - \\
\sum r\alpha_{m_l} & \cdots & \sum r\alpha_{i-2} & 0 & \sum r\alpha_i & \cdots & \sum r\alpha_{n-1}
\end{vmatrix}$$

Note as $h \to 0$, $\det(M(h)) \to \det(M)$ and so for $m_l \le i \le n-1$, $\epsilon_i(h)/h \to \det(M_i)/\det(M) := D_i$ as $h \to 0$, where M_i is the $n - m_l \times n - m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\mathrm{col}[0, \ldots, 0, u(\eta_{rs}), 0, \ldots, 0].$$

Now let $v_{rs}(x) = \lim_{h\to 0} v_{rsh}(x)$, and note by construction of $v_{rsh}(x)$,

$$v_{rs}(x) = \frac{\partial u}{\partial r_{rs}}(x).$$

Furthermore,

$$v_{rs}(x) = \lim_{h\to 0} v_{rsh}(x) = \sum_{p=m_l}^{n-1} D_i\alpha_i(x, u(x))$$

which is a solution of the variational equation (3.15) along $u(x)$. In addition,

$$v_{rs}^{(i)}(x_j) = \lim_{h\to 0} v_{rsh}^{(i)}(x_j) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$\begin{aligned} v_{rs}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} v_{rs}(\eta_{ip}) &= \lim_{h\to 0}\left[v_{rsh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} v_{rsh}(\eta_{ip})\right] \\ &= u(\eta_{is})\delta_{ir},\ 0 \le i \le m_k - 1. \end{aligned}$$

This completes the argument for $\frac{\partial u}{\partial r_{rs}}$. □

3.6 A Corollary and Examples

In this section, we present a corollary that follows from Theorem 3.7. The proof is immediate from the n-dimensionality of the solution space for the variational equation (3.15) along solutions of (3.13), and also creates a nice analogue to part (c) of Theorem 3.1 of Peano.

Corollary 3.1. *Assume the conditions of Theorem 3.7. Then,*

(*a*) *for each* $1 \le l \le k$,

$$\frac{\partial u}{\partial x_l}(x) = -\sum_{r=0}^{m_l-1} u^{(r+1)}(x_l)\frac{\partial u}{\partial u_{rl}}(x).$$

(*b*) *for* $0 \leq r \leq m_k - 1$ *and* $1 \leq s \leq m$,

$$\frac{\partial u}{\partial \eta_{rs}}(x) = r_{rs}\frac{u'(\eta_{rs})}{u(\eta_{rs})}\frac{\partial u}{\partial r_{rs}}(x).$$

In conclusion of Chapter 3, we now give two examples illustrating Theorem 3.7.

Example 3.1. Consider the BVP

$$y'' = 0, \tag{3.16}$$

$$y(x_1) = y_1, \qquad y(x_2) - ry(\eta) = y_2, \tag{3.17}$$

where $x_1, x_2, \eta, y_1, y_2, r \in \mathbb{R}$ with $x_1 < \eta < x_2$.

Our first goal is to show this BVP satisfies hypotheses (i)-(v). Then we may apply the main result of this chapter.

As $f(x, y, y') = 0$, it is clear that hypotheses (i) and (ii) are satisfied. Also note, as solutions of (3.16) are linear, the solutions will exist over all of $\mathbb{R}$. Hence, hypothesis (iii) is satisfied. All that remains is to show (3.16), (3.17) satisfies hypotheses (iv) and the associated variational equation along solutions of (3.16) satisfies (v). Now in this example the variational equation associated to a soltuion of (3.16) is simply $y'' = 0$. Hence, if we show solutions of (3.16) are unique using homogeneous boundary conditions, we will satisfy both hypotheses (iv) and (v).

Thus, we consider (3.16) with homogeneous BC's

$$y(x_1) = 0, \qquad y(x_2) - ry(\eta) = 0. \tag{3.18}$$

If (3.16), (3.18) has only the trivial solution $y(x) = 0$, then solutions of (3.16), (3.17) are unique.

A general solution of (3.16), (3.18) is $y(x) = A(x - x_1)$ where A is some constant. Thus we need to find a condition to force $A = 0$. Well,

$$0 = y(x_2) - ry(\eta)$$

$$\begin{aligned} &= A(x_2 - x_1) - rA(\eta - x_1) \\ &= A(x_2 - x_1 - r\eta + rx_1). \end{aligned}$$

Hence, if $x_2 - x_1 - r\eta + rx_1 \neq 0$, i.e. $r \neq \dfrac{x_2 - x_1}{\eta - x_1}$, then $A = 0$ which implies $y(x) = 0$. Thus when $x_2 - x_1 - r\eta + rx_1 \neq 0$, solutions of (3.16), (3.17) satisfy hypothesis (iv) and the variational equation along solutions of (3.16) satisfies hypothesis (v).

Now that hypotheses (i)-(v) are satisfied with respect to (3.16), (3.17), we may apply the main result of this section. The solution of (3.16), (3.17) is

$$y(x) = \frac{y_1 x - ry_1 x - y_2 x - y_1 x_1 + ry_1 x_1 + y_2 x_1}{x_1 - x_2 - rx_1 + r\eta} + y_1,$$

with derivative

$$y'(x) = \frac{y_1 - ry_1 - y_2}{x_1 - x_2 - rx_1 + r\eta}.$$

We now show that the assertion of the theorem is satisfied.

(a) First we consider the partial with respect to y_1,

$$y_1(x) := \frac{\partial y}{\partial y_1}(x) = \frac{x - rx - x_1 + rx_1}{x_1 - x_2 - rx_1 + r\eta} + 1.$$

Clearly, $y_1(x)$ solves the variational equation as it is a linear function,

$$\begin{aligned} y_1(x_1) &= \frac{x_1 - rx_1 - x_1 + rx_1}{x_1 - x_2 - rx_1 + r\eta} + 1 \\ &= 0 + 1 \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} y_1(x_2) - ry(\eta) &= \left[\frac{x_2 - rx_2 - x_1 + rx_1}{x_1 - x_2 - rx_1 + r\eta} + 1\right] \\ &\quad - r\left[\frac{\eta - r\eta - x_1 + rx_1}{x_1 - x_2 - rx_1 + r\eta} + 1\right] \\ &= \frac{x_2 - x_1 + rx_1 - r\eta}{x_1 - rx_1 - x_2 + r\eta} + \frac{-rx_2 - r\eta + rx_1 - rx_1}{x_1 - rx_1 - x_2 + r\eta} + 1 - r \\ &= -1 + r + 1 - r = 0. \end{aligned}$$

Now on to the partial with respect to y_2,

$$y_2(x) := \frac{\partial y}{\partial y_2}(x) = \frac{x_1 - x}{x_1 - rx_1 - x_2 + r\eta}.$$

Again, as $y_2(x)$ is linear it solves the variational equation,

$$y_2(x_1) = \frac{x_1 - x_1}{x_1 - rx_1 - x_2 + r\eta} = 0,$$

and

$$\begin{aligned} y_2(x_2) - ry_2(\eta) &= \left[\frac{x_1 - x_2}{x_1 - rx_1 - x_2 + r\eta}\right] - r\left[\frac{x_1 - \eta}{x_1 - rx_1 - x_2 + r\eta}\right] \\ &= \frac{x_1 - x_2 - rx_1 + r\eta}{x_1 - rx_1 - x_2 + r\eta} \\ &= 1. \end{aligned}$$

(b) Next we look at the partial with respect to x_1,

$$z_1(x) := \frac{\partial y}{\partial x_1}(x) = \frac{y_2 - y_1 + ry_1}{x_1 - rx_1 - x_2 + r\eta} - \frac{(y_1 - ry_1 - y_2)(1-r)(x - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2}.$$

Note $z_1(x)$ solves the variational equation,

$$\begin{aligned} z_1(x_1) &= \frac{y_2 - y_1 + ry_1}{x_1 - rx_1 - x_2 + r\eta} - 0 \\ &= -y'(x_1), \end{aligned}$$

and

$$\begin{aligned} z_1(x_2) - rz_1(\eta) &= \frac{y_2 - y_1 + ry_1}{x_1 - rx_1 - x_2 + r\eta} \\ &\quad - \frac{(y_1 - ry_1 - y_2)(1-r)(x_2 - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\ &\quad - r\frac{y_2 - y_1 + ry_1}{x_1 - rx_1 - x_2 + r\eta} \\ &\quad + r\frac{(y_1 - ry_1 - y_2)(1-r)(\eta - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\ &= \frac{(y_1 - ry_1 - y_2)(1-r)(-x_2 + x_1 + r\eta - rx_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\ &\quad + \frac{(y_1 - ry_1 - y_2)(r-1)}{x_1 - rx_1 - x_2 + r\eta} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(y_1 - ry_1 - y_2)(1-r)}{x_1 - rx_1 - x_2 + r\eta} + \frac{(y_1 - ry_1 - y_2)(r-1)}{x_1 - rx_1 - x_2 + r\eta} \\
&= 0.
\end{aligned}
$$

Now we look at $\partial y/\partial x_2$,

$$
z_2(x) := \frac{\partial y}{\partial x_2}(x) = \frac{(y_1 - ry_1 - y_2)(x - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2}.
$$

We have $z_2(x)$ solves the variational equation,

$$
\begin{aligned}
z_2(x_1) &= \frac{(y_1 - ry_1 - y_2)(x_1 - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
z_2(x_2) - rz_2(\eta) &= \frac{(y_1 - ry_1 - y_2)(x_2 - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} - r\frac{(y_1 - ry_1 - y_2)(\eta - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\
&= \frac{(y_1 - ry_1 - y_2)(x_2 - x_1 - r\eta + rx_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\
&= -y'(x_2).
\end{aligned}
$$

(c) Penultimately, we investigate $\partial y/\partial \eta$,

$$
w(x) := \frac{\partial y}{\partial \eta}(x) = \frac{r(y_1 - ry_1 - y_2)(x_1 - x)}{(x_1 - rx_1 - x_2 + r\eta)^2}.
$$

Our partial derivative solves the variational equation,

$$
\begin{aligned}
w(x_1) &= \frac{r(y_1 - ry_1 - y_2)(x_1 - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
w(x_2) - rw(\eta) &= \frac{r(y_1 - ry_1 - y_2)(x_1 - x_2)}{(x_1 - rx_1 - x_2 + r\eta)^2} - r\frac{r(y_1 - ry_1 - y_2)(x_1 - \eta)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\
&= \frac{r(y_1 - ry_1 - y_2)(x_1 - x_2 - r\eta + rx_1)}{(x_1 - rx_1 - x_2 + r\eta)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{r(y_1 - ry_1 - y_2)}{(x_1 - rx_1 - x_2 + r\eta)} \\
&= ry'(\eta).
\end{aligned}
$$

(d) Finally, we consider $\frac{\partial y}{\partial r}$,

$$
v(x) := \frac{\partial y}{\partial r}(x) = \frac{y_1(x_1 - x)}{x_1 - rx_1 - x_2 + r\eta} + \frac{(y_1 - ry_1 - y_2)(\eta - x_1)(x_1 - x)}{(x_1 - rx_1 - x_2 + r\eta)^2}.
$$

Once again, we have $v(x)$ solves the variational equation,

$$
\begin{aligned}
v(x_1) &= \frac{y_1(x_1 - x_1)}{x_1 - rx_1 - x_2 + r\eta} + \frac{(y_1 - ry_1 - y_2)(\eta - x_1)(x_1 - x_1)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
v(x_2) - rv(\eta) &= \Big[\frac{y_1(x_1 - x_2)}{x_1 - rx_1 - x_2 + r\eta} \\
&\quad + \frac{(y_1 - ry_1 - y_2)(\eta - x_1)(x_1 - x_2)}{(x_1 - rx_1 - x_2 + r\eta)^2}\Big] \\
&\quad - r\Big[\frac{y_1(x_1 - \eta)}{x_1 - rx_1 - x_2 + r\eta} \\
&\quad + \frac{(y_1 - ry_1 - y_2)(\eta - x_1)(x_1 - \eta)}{(x_1 - rx_1 - x_2 + r\eta)^2}\Big] \\
&= \frac{y_1(x_1 - rx_1 - x_2 + r\eta)}{x_1 - rx_1 - x_2 + r\eta} \\
&\quad + \frac{(y_1 - ry_1 - y_2)(\eta - x_1)(x_1 - x_2 - rx_1 + r\eta)}{(x_1 - rx_1 - x_2 + r\eta)^2} \\
&= y_1 + \frac{(y_1 - ry_1 - y_2)(\eta - x_1)}{(x_1 - rx_1 - x_2 + r\eta)} \\
&= y(\eta).
\end{aligned}
$$

Example 3.2. Consider the BVP

$$
y'' - y = 0, \tag{3.19}
$$

$$
y(x_1) = y_1, \qquad y(x_2) - ry(\eta) = y_2, \tag{3.20}
$$

where $x_1, x_2, \eta, y_1, y_2, r \in \mathbb{R}$ with $x_1 < \eta < x_2$.

Our first goal is to show this BVP satisfies hypotheses (i)-(v). Then we may apply the main result of this chapter.

As $f(x, y, y') = y$, it is clear that hypotheses (i) and (ii) are satisfied. Also note, as solutions of (3.19) involve hyperbolic sines and cosines, the solutions will exist over all of $\mathbb{R}$. Hence, hypothesis (iii) is satisfied. All that remains is to show (3.19), satisfies hypotheses (iv) and the associated variational equation along solutions of (3.19) satisfies (v). Now in this example the variational equation associated to a soltuion of (3.19) is simply $y'' - y = 0$. Hence, if we show solutions of (3.19) are unique using homogeneous boundary conditions, we will satisfy both hypotheses (iv) and (v).

Thus, we consider (3.19) with homogeneous BC's

$$y(x_1) = 0, \qquad y(x_2) - ry(\eta) = 0. \tag{3.21}$$

If (3.19), (3.21) has only the trivial solution $y(x) = 0$, then solutions of (3.19), (3.20) are unique.

A general solution of (3.19), (3.21) is $y(x) = A\sinh(x - x_1) + B\sinh(x - x_2)$ where A and B are constant. Thus we need to find a condition to force $A = B = 0$. Well,

$$\begin{aligned} 0 &= y(x_1) \\ &= A\sinh(x_1 - x_1) + B\sinh(x_1 - x_2) \\ &= 0 + B\sinh(x_1 - x_2) \\ &= B\sinh(x_1 - x_2). \end{aligned}$$

Since $x_1 \neq x_2,\ B = 0$. Thus, $y(x) = A\sinh(x - x_1)$.

Also,

$$\begin{aligned} 0 &= y(x_2) - ry(\eta) \\ &= A\sinh(x_2 - x_1) - rA\sinh(\eta - x_1) \end{aligned}$$

$$= A\big[\sinh(x_2 - x_1) - r\sinh(\eta - x_1)\big].$$

Now,

$$\begin{aligned} 0 &= \sinh(x_2 - x_1) - r\sinh(\eta - x_1) \\ \Leftrightarrow &\quad r\sinh(\eta - x_1) = \sinh(x_2 - x_1) \\ \Leftrightarrow &\quad r = \frac{\sinh(x_2 - x_1)}{\sinh(\eta - x_1)}. \end{aligned}$$

Hence, if $r \neq \dfrac{\sinh(x_2 - x_1)}{\sinh(\eta - x_1)}$, then $A = 0$ which implies $y(x) = 0$. Thus when $r \neq \dfrac{\sinh(x_2 - x_1)}{\sinh(\eta - x_1)}$, solutions of (3.19), (3.20) satisfy hypothesis (iv) and the variational equation along solutions of (3.19) satisfies hypothesis (v).

Now that hypotheses (i)-(v) are satisfied, we may apply the main result of this section.

The solution of (3.19), (3.20) is

$$\begin{aligned} y(x) &= \left[\frac{y_2\sinh(x_2 - x_1) + y_1 r\sinh(x_2 - \eta)}{\sinh(x_2 - x_1)^2 - r\sinh(\eta - x_1)\sinh(x_2 - x_1)}\right]\sinh(x - x_1) \\ &\quad + \left[\frac{y_1}{\sinh(x_1 - x_2)}\right]\sinh(x - x_2) \end{aligned}$$

with derivative

$$\begin{aligned} y'(x) &= \left[\frac{y_2\sinh(x_2 - x_1) + y_1 r\sinh(x_2 - \eta)}{\sinh(x_2 - x_1)^2 - r\sinh(\eta - x_1)\sinh(x_2 - x_1)}\right]\cosh(x - x_1) \\ &\quad + \left[\frac{y_1}{\sinh(x_1 - x_2)}\right]\cosh(x - x_2). \end{aligned}$$

(a) First, we take the partial of $y(x)$ with respect to y_1. We have

$$\begin{aligned} \alpha_1(x) &:= \frac{\partial y}{\partial y_1}(x) \\ &= \left[\frac{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)}{\sinh(x_2 - x_1)^2 - r\sinh(x_2 - x_1)\sinh(\eta - x_1)}\right]\sinh(x_2 - x) \end{aligned}$$

$$+\left[\frac{r\sinh(x_2-\eta)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\right]\sinh(x-x_1),$$

which solves the variational equation with

$$\begin{aligned}
\alpha_1(x_1) &= \left[\frac{\sinh(x_2-x_1)-r\sinh(\eta-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\right]\sinh(x_2-x_1)\\
&\quad+\left[\frac{r\sinh(x_2-\eta)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\right]\sinh(x_1-x_1)\\
&= \frac{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1(x_2) &- r\alpha_1(\eta)\\
&= \frac{[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]\sinh(x_2-x_2)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\\
&\quad+\frac{r\sinh(x_2-\eta)\sinh(x_2-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\\
&\quad-\frac{[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]r\sinh(x_2-\eta)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\\
&\quad+\frac{r\sinh(x_2-\eta)r\sinh(\eta-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\\
&= \frac{r\sinh(x_2-\eta)[\sinh(x_2-x_1)-\sinh(x_2-x_1)]}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\\
&\quad+\frac{r\sinh(x_2-\eta)[r\sinh(\eta-x_1)-r\sinh(\eta-x_1)]}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}\\
&= 0.
\end{aligned}$$

Next, we investigate the partial of $y(x)$ with respect to y_2 :

$$\alpha_2(x) := \frac{\partial y}{\partial y_2}(x) = \frac{\sinh(x - x_1)}{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)},$$

which is the solution of the variational equation with

$$\begin{aligned} \alpha_2(x_1) &= \frac{\sinh(x_1 - x_1)}{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \alpha_2(x_2) - r\alpha_2(\eta) &= \frac{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)}{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)} \\ &= 1. \end{aligned}$$

(b) Before taking the partial derviatives, note

$$\begin{aligned} -y'(x_1) &= \frac{y_1\cosh(x_2 - x_1)}{\sinh(x_2 - x_1)} - \frac{y_2}{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)} \\ &\quad - \frac{y_1 r\sinh(x_2 - \eta)}{\sinh(x_2 - x_1)(\sinh(x_2 - x_1) - r\sinh(\eta - x_1))}, \end{aligned}$$

and

$$\begin{aligned} -y'(x_2) &= \frac{y_1}{\sinh(x_2 - x_1)} - \frac{y_2\cosh(x_2 - x_1)}{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)} \\ &\quad - \frac{y_1 r\sinh(x_2 - \eta)\cosh(x_2 - x_1)}{\sinh(x_2 - x_1)(\sinh(x_2 - x_1) - r\sinh(\eta - x_1))}. \end{aligned}$$

Now we look at $\partial y(x)/\partial x_1$:

$$\begin{aligned} z_1(x) &:= \frac{\partial y}{\partial x_1}(x) \\ &= [2\sinh(x_2 - x_1)\cosh(x_2 - x_1) - r\sinh(\eta - x_1)\cosh(x_2 - x_1) \\ &\quad - r\sinh(x_2 - x_1)\cosh(\eta - x_1)] \\ &\quad \times \frac{(y_2\sinh(x_2 - x_1) + y_1 r\sinh(x_2 - \eta))\sinh(x - x_1)}{[\sinh(x_2 - x_1)^2 - r\sinh(x_2 - x_1)\sinh(\eta - x_1)]^2} \end{aligned}$$

$$-\frac{y_2\sinh(x_2-x_1)\cosh(x-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}$$

$$-\frac{y_2\cosh(x_2-x_1)\sinh(x-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}$$

$$-\frac{y_1 r\sinh(x_2-\eta)\cosh(x-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}$$

$$+\left[\frac{y_1\cosh(x_2-x_1)}{\sinh(x_2-x_1)^2}\right]\sinh(x_2-x).$$

We have $z_1(x)$ solves the variational equation along $y(x)$ with

$$\begin{aligned}
z_1(x_1) &= [2\sinh(x_2-x_1)\cosh(x_2-x_1)-r\sinh(\eta-x_1)\cosh(x_2-x_1) \\
&\quad -r\sinh(x_2-x_1)\cosh(\eta-x_1)] \\
&\quad \times\frac{(y_2\sinh(x_2-x_1)+y_1 r\sinh(x_2-\eta))\sinh(x_1-x_1)}{[\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)]^2} \\
&\quad -\frac{y_2\sinh(x_2-x_1)\cosh(x_1-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
&\quad -\frac{y_2\sinh(x_1-x_1)\sinh(x_2-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
&\quad +\frac{y_1\cosh(x_2-x_1)\sinh(x_1-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
&\quad -\frac{y_1 r\sinh(x_2-\eta)\cosh(x_1-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
&\quad +\left[\frac{y_1\cosh(x_2-x_1)}{\sinh(x_2-x_1)^2}\right]\sinh(x_2-x_1) \\
&= -\frac{y_2\cosh(x_1-x_1)}{\sinh(x_2-x_1)-r\sinh(\eta-x_1)}+\frac{y_1\cosh(x_2-x_1)}{\sinh(x_2-x_1)}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{y_1 r \sinh(x_2-\eta)}{\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
= \quad & -y'(x_1),
\end{aligned}
$$

and

$$
\begin{aligned}
z_1(x_2) \quad - \quad & r z_1(\eta) \\
= \quad & [\cosh(x_2-x_1)(2\sinh(x_2-x_1) - r\sinh(\eta-x_1)) \\
& \quad -r\sinh(x_2-x_1)\cosh(\eta-x_1)] \\
& \quad \times \left[\frac{y_2\sinh(x_2-x_1)[\sinh(x_2-x_1) - r\sinh(\eta-x_1)]}{[\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)]^2}\right. \\
& \qquad \left. + \frac{y_1 r\sinh(x_2-\eta)[\sinh(x_2-x_1) - r\sinh(\eta-x_1)]}{[\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)]^2}\right] \\
& -\frac{y_2\sinh(x_2-x_1)[\cosh(x_2-x_1) - r\cosh(\eta-x_1)]}{\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
& -\frac{y_2[\sinh(x_2-x_1) - r\sinh(\eta-x_1)]\cosh(x_2-x_1)}{\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
& -\frac{y_1 r\sinh(x_2-\eta)[\cosh(x_2-x_1) - r\cosh(\eta-x_1)]}{\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
& +\left[\frac{y_1\cosh(x_2-x_1)}{\sinh(x_2-x_1)^2}\right][\sinh(x_2-x_2) - r\sinh(x_2-\eta)] \\
= \quad & \frac{y_2\cosh(x_2-x_1)(2\sinh(x_2-x_1) - r\sinh(\eta-x_1))}{\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
& -\frac{y_2 r\sinh(x_2-x_1)\cosh(\eta-x_1)}{\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)} \\
& -\frac{y_2\sinh(x_2-x_1)[\cosh(x_2-x_1) - r\cosh(\eta-x_1)]}{\sinh(x_2-x_1)^2 - r\sinh(x_2-x_1)\sinh(\eta-x_1)}
\end{aligned}
$$

$$-\frac{y_2\cosh(x_2-x_1)[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}$$

$$+\frac{2y_1r\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}$$

$$-\frac{y_1r^2\sinh(\eta-x_1)\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)^3-r\sinh(x_2-x_1)^2\sinh(\eta-x_1)}$$

$$-\frac{y_1r^2\sinh(x_2-\eta)\cosh(\eta-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}$$

$$-\frac{y_1r\sinh(x_2-\eta)[\cosh(x_2-x_1)-r\cosh(\eta-x_1)]}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}$$

$$-\frac{y_1r\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)^2-r\sinh(x_2-x_1)\sinh(\eta-x_1)}$$

$$+\frac{y_1r^2\sinh(x_2-\eta)\cosh(x_2-x_1)\sinh(\eta-x_1)}{\sinh(x_2-x_1)^3-r\sinh(x_2-x_1)^2\sinh(\eta-x_1)}$$

$$= \quad 0$$

Next, we find the partial of $y(x)$ with respect to x_2 :

$$z_2(x) \quad := \quad \frac{\partial y}{\partial x_2}(x) = -\frac{y_2\cosh(x_2-x_1)\sinh(x-x_1)}{[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]^2}$$

$$+\frac{y_1r\cosh(x_2-\eta)\sinh(x-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))}$$

$$-\frac{2y_1r\sinh(x_2-\eta)\sinh(x_2-x_1)\cosh(x_2-x_1)\sinh(x-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2}$$

$$+\frac{y_1r^2\sinh(x_2-\eta)\sinh(\eta-x_1)\cosh(x_2-x_1)\sinh(x-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2}$$

$$+\frac{y_1 \cosh(x_2 - x)}{\sinh(x_2 - x_1)} - \frac{y_1 \cosh(x_2 - x_1)\sinh(x_2 - x)}{\sinh(x_2 - x_1)^2},$$

which solves the variational equation and has boundary conditions

$$\begin{aligned}
z_2(x_1) &= -\frac{y_2 \cosh(x_2 - x_1)\sinh(x_1 - x_1)}{[\sinh(x_2 - x_1) - r\sinh(\eta - x_1)]^2} \\
&\quad + \frac{y_1 r \cosh(x_2 - \eta)\sinh(x_1 - x_1)}{\sinh(x_2 - x_1)(\sinh(x_2 - x_1) - r\sinh(\eta - x_1))} \\
&\quad - \frac{2y_1 r \sinh(x_2 - \eta)\sinh(x_2 - x_1)\cosh(x_2 - x_1)\sinh(x_1 - x_1)}{[\sinh(x_2 - x_1)(\sinh(x_2 - x_1) - r\sinh(\eta - x_1))]^2} \\
&\quad + \frac{y_1 r^2 \sinh(x_2 - \eta)\sinh(\eta - x_1)\cosh(x_2 - x_1)\sinh(x_1 - x_1)}{[\sinh(x_2 - x_1)(\sinh(x_2 - x_1) - r\sinh(\eta - x_1))]^2} \\
&\quad + \frac{y_1 \cosh(x_2 - x_1)}{\sinh(x_2 - x_1)} - \frac{y_1 \cosh(x_2 - x_1)\sinh(x_2 - x_1)}{\sinh(x_2 - x_1)^2} \\
&= 0 + \frac{y_1 \cosh(x_2 - x_1)}{\sinh(x_2 - x_1)} - \frac{y_1 \cosh(x_2 - x_1)}{\sinh(x_2 - x_1)} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
z_2(x_2) - r z_2(\eta) &= -\frac{y_2 \cosh(x_2 - x_1)[\sinh(x_2 - x_1) - r\sinh(\eta - x_1)]}{[\sinh(x_2 - x_1) - r\sinh(\eta - x_1)]^2} \\
&\quad + \frac{y_1 r \cosh(x_2 - \eta)[\sinh(x_2 - x_1) - r\sinh(\eta - x_1)]}{\sinh(x_2 - x_1)(\sinh(x_2 - x_1) - r\sinh(\eta - x_1))} \\
&\quad - \Bigg[\frac{2y_1 r \sinh(x_2 - \eta)\sinh(x_2 - x_1)\cosh(x_2 - x_1)}{[\sinh(x_2 - x_1)(\sinh(x_2 - x_1) - r\sinh(\eta - x_1))]^2} \\
&\qquad + \frac{y_1 r^2 \sinh(x_2 - \eta)\sinh(\eta - x_1)\cosh(x_2 - x_1)}{[\sinh(x_2 - x_1)(\sinh(x_2 - x_1) - r\sinh(\eta - x_1))]^2}\Bigg] \\
&\quad \times [\sinh(x_2 - x_1) - r\sinh(\eta - x_1)]
\end{aligned}$$

$$+\frac{y_1[\cosh(x_2-x_1)-r\cosh(x_2-\eta)]}{\sinh(x_2-x_1)}$$

$$-\frac{y_1\cosh(x_2-x_1)[\sinh(x_2-x_2)-r\sinh(x_2-\eta)]}{\sinh(x_2-x_1)^2}$$

$$= -\frac{y_2\cosh(x_2-x_1)}{\sinh(x_2-x_1)-r\sinh(\eta-x_1)}+\frac{y_1 r\cosh(x_2-\eta)}{\sinh(x_2-x_1)}$$

$$-\frac{2y_1 r\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))}$$

$$+\frac{y_1 r^2\sinh(x_2-\eta)\sinh(\eta-x_1)\cosh(x_2-x_1)}{\sinh(x_2-x_1)^2(\sinh(x_2-x_1)-r\sinh(\eta-x_1))}$$

$$+\frac{y_1[\cosh(x_2-x_1)-r\cosh(x_2-\eta)]}{\sinh(x_2-x_1)}$$

$$-\frac{y_1\cosh(x_2-x_1)[\sinh(x_2-x_2)-r\sinh(x_2-\eta)]}{\sinh(x_2-x_1)^2}$$

$$= -\frac{y_2\cosh(x_2-x_1)}{\sinh(x_2-x_1)-r\sinh(\eta-x_1)}+\frac{y_1}{\sinh(x_2-x_1)} \times\left[1+\frac{r\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)}-r\cosh(x_2-\eta)\right]$$

$$-\frac{y_1 r\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \times\left[1-\frac{r\sinh(\eta-x_1)}{\sinh(x_2-x_1)}-\frac{r\sinh(\eta-x_1)\cosh(x_2-\eta)}{\sinh(x_2-\eta)\cosh(x_2-x_1)}\right]$$

$$= -\frac{y_2\cosh(x_2-x_1)}{\sinh(x_2-x_1)-r\sinh(\eta-x_1)}$$

$$+\frac{y_1}{\sinh(x_2-x_1)}$$

$$\times \left[1 + \frac{r\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)} - \frac{r\cosh(x_2-\eta)\sinh(x_2-x_1)}{\sinh(x_2-x_1)}\right]$$

$$-\frac{y_1 r\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \times \left[1 + \frac{r\sinh(\eta-x_1)\cosh(x_2-\eta)\sinh(x_2-x_1)}{\sinh(x_2-x_1)\cosh(x_2-x_1)\sinh(x_2-\eta)} - \frac{r\sinh(\eta-x_1)\cosh(x_2-x_1)\sinh(x_2-\eta)}{\sinh(x_2-x_1)\cosh(x_2-x_1)\sinh(x_2-\eta)}\right]$$

$$\begin{aligned} &= -\frac{y_2\cosh(x_2-x_1)}{\sinh(x_2-x_1)-r\sinh(\eta-x_1)} + \frac{y_1}{\sinh(x_2-x_1)}(1+0) \\ &\quad -\frac{y_1 r\sinh(x_2-\eta)\cosh(x_2-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))}(1+0) \\ &= -y'(x_2). \end{aligned}$$

(c) For part (c), first note that

$$\begin{aligned} ry'(\eta) &= \frac{y_2 r\cosh(\eta-x_1)}{\sinh(x_2-x_1)-r\sinh(\eta-x_1)} - \frac{y_1 r\cosh(x_2-\eta)}{\sinh(x_2-x_1)} \\ &\quad + \frac{y_1 r^2\sinh(x_2-\eta)\cosh(\eta-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))}. \end{aligned}$$

Now we look at the partial of $y(x)$ with respect to η :

$$\begin{aligned} w(x) &:= \frac{\partial y}{\partial \eta}(x) \\ &= \frac{y_2 r^2\sinh(x_2-\eta)\sinh(x_2-x_1)\cosh(\eta-x_1)\sinh(x-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2} \\ &\quad -\frac{y_1 r\cosh(x_2-\eta)\sinh(x-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \\ &\quad -\frac{y_2 r\cosh(\eta-x_1)\sinh(x-x_1)}{[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]^2}, \end{aligned}$$

which is the solution of the variational equation with

$$
\begin{aligned}
w(x_1) &= \frac{y_2 r^2 \sinh(x_2-\eta)\sinh(x_2-x_1)\cosh(\eta-x_1)\sinh(x_1-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2} \\
&\quad -\frac{y_1 r\cosh(x_2-\eta)\sinh(x_1-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \\
&\quad -\frac{y_2 r\sinh(x_2-x_1)^2\cosh(\eta-x_1)\sinh(x_1-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2} \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
w(x_2) &- rw(\eta) \\
&= \frac{y_2 r^2 \sinh(x_2-\eta)\sinh(x_2-x_1)\cosh(\eta-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2} \\
&\qquad \times[\sinh(x_2-x_1)-r\sinh(\eta-x_1)] \\
&\quad -\frac{y_1 r\cosh(x_2-\eta)[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \\
&\quad -\frac{y_2 r\cosh(\eta-x_1)[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]}{[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]^2} \\
&= \frac{y_2 r\cosh(\eta-x_1)}{\sinh(x_2-x_1)-r\sinh(\eta-x_1)} - \frac{y_2 r\cosh(x_2-\eta)}{\sinh(x_2-x_1)} \\
&\quad +\frac{y_1 r^2\sinh(x_2-\eta)\cosh(\eta-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \\
&= ry'(\eta).
\end{aligned}
$$

(d) Before exploring part (d), note

$$
y(\eta) = \frac{y_2\sinh(\eta-x_1)}{\sinh(x_2-x_1)-r\sinh(\eta-x_1)} + \frac{y_1\sinh(x_2-\eta)}{\sinh(x_2-x_1)}
$$

$$+\frac{y_1 r \sinh(x_2-\eta)\sinh(\eta-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))}.$$

We have

$$\begin{aligned}
v(x) &:= \frac{\partial y}{\partial r}(x) \\
&= \frac{y_2 \sinh(x_2-x_1)^2\sinh(\eta-x_1)\sinh(x-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2} \\
&\quad +\frac{y_1 \sinh(x_2-\eta)\sinh(x-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \\
&\quad +\frac{y_1 r \sinh(x_2-\eta)\sinh(x_2-x_1)\sinh(\eta-x_1)\sinh(x-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2},
\end{aligned}$$

which solves the variational equation with

$$\begin{aligned}
v(x_1) &= \frac{y_2 \sinh(x_2-x_1)^2\sinh(\eta-x_1)\sinh(x_1-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2} \\
&\quad +\frac{y_1 \sinh(x_2-\eta)\sinh(x_1-x_1)}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \\
&\quad +\frac{y_1 r \sinh(x_2-\eta)\sinh(x_2-x_1)\sinh(\eta-x_1)\sinh(x_1-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
v(x_2) &- rv(\eta) \\
&= \frac{y_2 \sinh(x_2-x_1)^2\sinh(\eta-x_1)[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2} \\
&\quad +\frac{y_1 \sinh(x_2-\eta)[\sinh(x_2-x_1)-r\sinh(\eta-x_1)]}{\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))} \\
&\quad +\frac{y_1 r \sinh(x_2-\eta)\sinh(x_2-x_1)\sinh(\eta-x_1)}{[\sinh(x_2-x_1)(\sinh(x_2-x_1)-r\sinh(\eta-x_1))]^2}
\end{aligned}$$

$$\times[\sinh(x_2 - x_1) - r\sinh(\eta - x_1)]$$

$$\begin{aligned}
&= \frac{y_2 \sinh(\eta - x_1)}{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)} + \frac{y_1 \sinh(x_2 - \eta)}{\sinh(x_2 - x_1)} \\
&\quad + \frac{y_1 r \sinh(\eta - x_1)}{\sinh(x_2 - x_1) - r\sinh(\eta - x_1)} \\
&= y(\eta).
\end{aligned}$$

CHAPTER FOUR

Difference Equations with Nonlocal Boundary Conditions

In this chapter, we move on to nonlocal boundary value problems over the discrete domain $\mathbb{Z}$. As was done in the previous chapter, we begin by introducing several definitions, conditions, and theorems that will be used throughout the entire chapter. Afterwards, we present results for the second order nonlocal boundary value problem. These results recently appeared in a paper by Hopkins, Kim, Lyons, and Speer, [30]. Then we use an extension of the techniques to generalize the problem to the nth order general nonlocal boundary value problem. In conclusion, we provide an example.

4.1 Preliminary Definitions, Conditions, and Theorems

As the title confers, this section will provide the background information necessary to the remainder of the chapter. In this chapter, we will be working with the nth order difference equation

$$w(t+n) = f(t, w(t), \ldots, w(t+n-1)),\ t \in \mathbb{Z},\ n \geq 2, \tag{4.1}$$

satisfying

$$\begin{aligned} &w(t_j + i) = w_{ij},\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1, \\ &w(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}) = w_{ik},\ 0 \leq i \leq m_k - 1, \end{aligned} \tag{4.2}$$

where $2 \leq k \leq n,\ m \in \mathbb{N},\ m_1, \ldots, m_k$ are positive integers such that $\sum_{i=1}^{k} m_i = n,\ t_1 < t_1 + m_1 - 1 < t_2 < t_2 + m_2 - 1 < \cdots < t_{k-1} < t_{k-1} + m_{k-1} - 1 < \eta_{01} < \eta_{01} + 1 < \cdots < \eta_{m_k-1,m} < \eta_{m_k-1,m} + 1 < t_k$ in $\mathbb{Z}$, and $\alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k} \in \mathbb{R}$.

Now, we state a few conditions we place upon (4.1):

(i) $f(t, d_1, d_2, \ldots, d_n) : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

(ii) $\dfrac{\partial f}{\partial d_i}(t, d_1, d_2, \ldots, d_n) : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}$ are continuous, $i = 1, 2, \ldots, n$, and

(iii) the equation $d_{n+1} = f(t, d_1, d_2, \ldots, d_n)$ can be solved for d_1 as a continuous function of $d_2, d_3, \ldots, d_{n+1}$, for all $t \in \mathbb{Z}$.

Remark 4.1. We observe conditions (i) and (iii) imply that solutions of initial value problems for (4.1) exist and are unique on all $\mathbb{Z}$.

Definition 4.1. Given a solution $w(t)$ of (4.1), we define the *variational equation along* $w(t)$ by

$$z(t+n) = \sum_{i=1}^{n} \frac{\partial f}{\partial d_i}(t, w(t), \ldots, w(t+n-1))z(t+i-1). \tag{4.3}$$

The results found next are related to continuous dependence, differentiation, and differences as they pertain to initial value problems of difference equations. The proofs are very similar to those regarding initial value problems for differential equations. Therefore, we will omit the proofs. Lastly, we present a theorem involving differences of solutions of (4.1), (4.2) with respect to initial points whose proof is comparable to the those given in [9] and [10].

We denote the solution of (4.1) satisfying the initial conditions

$$u(t_0 + i - 1) = c_i,\ 1 \leq i \leq n, \tag{4.4}$$

where $t_0 \in \mathbb{Z}$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$, by

$$u(t) = u(t, t_0, c_1, c_2, \ldots, c_n). \tag{4.5}$$

Theorem 4.1. *[Continuous Dependence with Respect to Initial Values] Assume conditions* (i) *and* (iii) *hold. Let* $t_0 \in \mathbb{Z}$ *and* $c_1, c_2, \ldots, c_n \in \mathbb{R}$ *be given. Then for all* $\epsilon > 0$ *and for all* $k \in \mathbb{N}$, *there exists* $\delta(\epsilon, k, t_0, c_1, \ldots, c_n) > 0$ *such that* $|c_i - e_i| < \delta,\ 1 \leq i \leq n$, *implies* $|u(t, t_0, c_1, \ldots, c_n) - u(t, t_0, e_1, \ldots, e_n)| < \epsilon$ *for* $m \in [t_0 - k, t_0 + k]$ *and* $e_1, e_2, \ldots, e_n \in \mathbb{R}$.

Theorem 4.2. *[Differentiation with Respect to Initial Values] Assume* (i), (ii), *and* (iii) *hold. Let* $t_0 \in \mathbb{Z}$ *and* $c_1, c_2, \ldots, c_n \in \mathbb{R}$ *be given. Then, for* $j = 1, 2, \ldots, n$, $\beta_j := \frac{\partial u}{\partial c_j}(t, t_0, c_1, \ldots, c_n)$ *exists and is the solution of the variational equation* (4.3) *along* $u(t, t_0, c_1, \ldots, c_n)$*; i.e.,*

$$\beta_j(t+n) = \sum_{i=1}^{n} \frac{\partial f}{\partial d_i}(t, u(t), u(t+1), \ldots, u(t+n-1))\beta_j(t+i-1)$$

satisfying the initial conditions

$$\beta_j(t_0 + i - 1) = \delta_{ij},\ 1 \le i \le n.$$

Theorem 4.3. *[Differences with Respect to Initial Points] Assume* (i), (ii), *and* (iii) *hold. Let* $t_0 \in \mathbb{Z}$ *and* $c_1, c_2, \ldots, c_n \in \mathbb{R}$ *be given. Then*

$$\gamma(t) := \Delta_{t_0} u(t, t_0, c_1, \ldots, c_n) = u(t, t_0 + 1, c_1, \ldots, c_n) - u(t, t_0, c_1, \ldots, c_n)$$

is the solution of the nth order linear difference equation

$$\gamma(t+n) = \sum_{r=1}^{n} A_r(t)\gamma(t+r-1),$$

satisfying the initial conditions

$$\begin{aligned} \gamma(t_0) &= -\Delta_t u(t, t_0+1, c_1, \ldots, c_n)|_{t=t_0}, \\ \gamma(t_0+i) &= -\Delta_t u(t, t_0, c_1, \ldots, c_n)|_{t=t_0+i-1},\ 0 \le i \le n-1, \end{aligned}$$

where

$$\begin{aligned} A_r(t) = \int_0^1 & \frac{\partial f}{\partial d_r}(t, w(t, t_0+1, c_1, \ldots, c_n), w(t+1, t_0+1, c_1, \ldots, c_n), \ldots, \\ & sw(t+r-1, t_0+1, c_1, \ldots, c_n) + (1-s)w(t+r-1, t_0, c_1, \ldots, c_n), \ldots, \\ & w(t+n-1, t_0, c_1, \ldots, c_n))ds. \end{aligned}$$

In order to establish certain properties of nonlocal boundary value problems for difference equations, we must first establish that solutions of (4.1) are unique. To accomplish this, we use Hartman's definition of a generalized zero in [21].

Definition 4.2. Let $v : \mathbb{Z} \to \mathbb{R}$. We say v has a *generalized zero at* $n_0 \in \mathbb{Z}$ provided either $v(n_0) = 0$ or there exists $k \in \mathbb{N}$ such that $(-1)^k v(n_0 - k)v(n_0) > 0$ and if $k > 1,\ v(n_0 - k + 1) = \cdots = v(n_0 - 1) = 0$.

Definition 4.3. Let $n \geq 2,\ 2 \leq k \leq n,\ m \in \mathbb{N}$, and $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$. The nonlinear difference equation (4.1) is said to *satisfy Property (U) on* $\mathbb{Z}$ if, whenever $w_1(t)$ and $w_2(t)$ are solutions of (4.1) such that $w_1(t) - w_2(t)$ has a generalized zero at $t_j + i$ for each $0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1$, and $[w_1(t) - w_2(t)] - \sum_{p=1}^{m} \alpha_{ip}[w_1(\eta_{ip}) - w_2(\eta_{ip})]$ has a generalized zero at $t_k + i$ for each $0 \leq i \leq m_k - 1$, where $t_1 < t_1 + m_1 - 1 < t_2 < t_2 + m_2 - 1 < \cdots < t_{k-1} < t_{k-1} + m_{k-1} - 1 < \eta_{01} < \eta_{01} + 1 < \cdots < \eta_{m_k-1,m} < \eta_{m_k-1,m} + 1 < t_k$ in $\mathbb{Z}$ and $\alpha_{01}, \ldots, \alpha_{m_k-1,m} \in \mathbb{R}$, then $w_1(t) \equiv w_2(t)$ on $\mathbb{Z}$.

Remark 4.2. If (4.1) satisfies Property (U), then solutions of the boundary value problem (4.1), (4.2) are unique.

Since the results of this chapter also require the uniqueness of solutions for the variational equation (4.3), we include the following definition for Property (U) relative to linear difference equations.

Definition 4.4. Let $n \geq 2,\ 2 \leq k \leq n,\ m \in \mathbb{N}$, and $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$. The linear difference equation

$$s(t+n) = \sum_{i=1}^{n} N_i(t)s(t+i-1) \tag{4.6}$$

is said to *satisfy Property (U) on* $\mathbb{Z}$, provided there is no nontrivial solution $s(t)$ of (4.1) such that $s(t)$ has a generalized zero at $t_j + i$ for each $0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1$, and $s(t) - \sum_{p=1}^{m} \alpha_{ip} s(\eta_{ip})$ has a generalized zero at $t_k + i$ for each $0 \leq i \leq m_k - 1$ where $t_1 < t_1 + m_1 - 1 < t_2 < t_2 + m_2 - 1 < \cdots < t_{k-1} < t_{k-1} + m_{k-1} - 1 < \eta_{01} < \eta_{01} + 1 < \cdots < \eta_{m_k-1,m} < \eta_{m_k-1,m} + 1 < t_k$ in $\mathbb{Z}$, and $\alpha_{01}, \ldots, \alpha_{m_k-1,m} \in \mathbb{R}$.

We now present a result establishing, under Property (U), the continuous dependence of solutions with respect to boundary values as well as parameters. A typical proof of the argument can be found in [10] and [27].

Theorem 4.4. *[Continuous Dependence with Respect to Boundary Values and Parameters] Assume conditions* (i) *and* (iii) *hold and that* (4.1) *satisfies Property (U) on* $\mathbb{Z}$. *Let* $y(t)$ *be a solution of* (4.1) *and let* $n \geq 2,\ 2 \leq k \leq n,\ m \in \mathbb{N}$, *and* $m_1, \ldots, m_k$ *such that* $\sum_{i=1}^{k} m_i = n$. *Also, let* $t_1 < t_1 + m_1 - 1 < t_2 < t_2 + m_2 - 1 < \cdots < t_{k-1} < t_{k-1} + m_{k-1} - 1 < \eta_{01} < \eta_{01} + 1 < \cdots < \eta_{m_k-1,m} < \eta_{m_k-1,m} + 1 < t_k$ *in* $\mathbb{Z}$ *and* $\alpha_{01}, \ldots, \alpha_{m_k-1,m} \in \mathbb{R}$ *be given. Then, there exists* $\epsilon > 0$ *such that, if* $\delta_{01}, \ldots, \delta_{m_k-1,k}, \beta_{01}, \ldots, \beta_{m_k-1,m} \in \mathbb{R}$ *with* $|\delta_{ij}| < \epsilon,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k$ *and* $|\alpha_{ip} - \beta_{ip}| < \epsilon,\ 0 \leq i \leq m_k - 1$ *and* $1 \leq p \leq m$, *the nonlocal boundary value problem* (4.1) *satisfying*

$$w(t_j + i) = y(t_j + i) + \delta_{ij},\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k - 1,$$
$$w(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}) = y(t_k + i) - \sum_{p=1}^{m} \beta_{ip} y(\eta_{ip}) + \delta_{ik},\ 0 \leq i \leq m_k - 1,$$

has a unique solution:

$$\begin{aligned} w\Big(&t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \beta_{01}, \ldots, \beta_{m_k-1,m}, y(t_1) + \delta_{01}, \ldots, \\ &y(t_1 + m_1 - 1) + \delta_{m_1-1,1}, \ldots, y(t_k) - \sum_{p=1}^{m} \beta_{0p} y(\eta_{0p}) + \delta_{0k}, \ldots, \\ &y(t_k + m_k - 1) - \sum_{p=1}^{m} \beta_{m_k-1,p} y(\eta_{m_k-1,p}) + \delta_{m_k-1,k}\Big). \end{aligned}$$

Moreover, as $\epsilon \to 0$, *this solution converges to* $y(t)$ *on* $\mathbb{Z}$.

4.2 Second Order Problem

The following section is dedicated to the results of Hopkins, Kim, Lyons, and Speer found within [30]. These results establish a characterization of partial derivatives and partial differences of the second order discrete boundary value problem

$$w(t + 2) = f(t, w(t), w(t + 1)),\ t \in \mathbb{Z}, \tag{4.7}$$

satisfying

$$w(t_1) = w_1, \quad w(t_2) - \sum_{i=1}^{r} \alpha_i w(\eta_i) = w_2, \tag{4.8}$$

where $r \in \mathbb{N}$, $t_1 < t_1 + 1 < \eta_1 < \eta_1 + 1 < \cdots < \eta_r < \eta_r + 1 < t_2$ in $\mathbb{Z}$, and $w_1, w_2, \alpha_1, \ldots, \alpha_r \in \mathbb{R}$. Given a solution $w(t)$ of (4.7) the related variational equation along $w(t)$ is

$$z(t+2) = \frac{\partial f}{\partial d_1}(t, w(t), w(t+1))z(t) + \frac{\partial f}{\partial d_2}(t, w(t), w(t+1))z(t+1). \tag{4.9}$$

We now present, under certain conditions, the recent work of Hopkins, Kim, Lyons, and Speer in [30]. The first theorem deals with various partial derivatives and the second deals with partial differences. For completeness, we include the proofs.

Theorem 4.5. *Assume conditions* (i), (ii)*, and* (iii) *are satisfied, that* (4.7) *satisfies Property (U) on* $\mathbb{Z}$*, and that the variational equation* (4.9) *satisfies Property (U) along solutions of* (4.7)*. Suppose* $r \in \mathbb{N}$ *and*

$$w(t) = w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$$

is the solution of (4.7) *on* $\mathbb{Z}$ *satisfying*

$$w(t_1) = w_1, \qquad w(t_2) - \sum_{i=1}^{r} \alpha_i w(\eta_i) = w_2,$$

where $t_1 < t_1 + 1 < \eta_1 < \eta_1 + 1 < \cdots < \eta_r < \eta_r + 1 < t_2$ *in* $\mathbb{Z}$ *and* $\alpha_1, \ldots, \alpha_r, w_1, w_2 \in \mathbb{R}$*.*

Then:

(a) for $1 \leq j \leq r$, $\dfrac{\partial w}{\partial \alpha_j}(t)$ *exists on* $\mathbb{Z}$, *and* $p_j(t) := \dfrac{\partial w}{\partial \alpha_j}(t)$ *is the solution of* (4.9) *along* $w(t)$ *satisfying*

$$p_j(t_1) = 0, \quad p_j(t_2) - \sum_{i=1}^{r} \alpha_i p_j(\eta_i) = w(\eta_j).$$

(b) for $j = 1, 2$, $\dfrac{\partial w}{\partial w_j}(t)$ *exists on* $\mathbb{Z}$, *and* $z_j(t) := \dfrac{\partial w}{\partial w_j}(t)$ *is the solution of* (4.9) *along* $w(t)$ *satisfying*

$$z_j(t_1) = \delta_{1j}, \;\; z_j(t_2) - \sum_{i=1}^{r} \alpha_i z_j(\eta_i) = \delta_{2j}.$$

Proof. We first deal with part (a). Let $\epsilon > 0$ be as in Theorem 4.4 and let $0 < |h| < \epsilon$ be given. Fix $1 \leq j \leq r$. For brevity, we denote $w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$ by $w(t, \alpha_j)$. Consider the quotient,

$$p_{jh}(t) = \frac{1}{h}[w(t, \alpha_j + h) - w(t, \alpha_j)].$$

Notice that for $h \neq 0$,

$$\begin{aligned} p_{jh}(t_1) &= \frac{1}{h}[w(t_1, \alpha_j + h) - w(t_1, \alpha_j)] \\ &= \frac{1}{h}[w_1 - w_1] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} p_{jh}(t_2) - \sum_{i=1}^{r} \alpha_i p_{jh}(\eta_i) &= \frac{1}{h}\left[\Big(w(t_2, \alpha_j + h) - \sum_{i=1}^{r} \alpha_i w(\eta_i, \alpha_j + h)\Big)\right. \\ &\qquad \left. - \Big(w(t_2, \alpha_j) - \sum_{i=1}^{r} \alpha_i w(\eta_i, \alpha_j)\Big)\right] \\ &= \frac{1}{h}\Big[w(t_2, \alpha_j + h) - hw(\eta_j, \alpha_j + h) + hw(\eta_j, \alpha_j + h) \\ &\qquad - \sum_{i=1}^{r} \alpha_i w(\eta_i, \alpha_j + h) - w_2\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}[w_2 + hw(\eta_j, \alpha_j + h) - w_2] \\
&= w(\eta_j, \alpha_j + h).
\end{aligned}
$$

Also note, as $h \to 0$, $p_{jh}(t_2) - \sum_{i=1}^{r} \alpha_i p_{jh}(\eta_i) \to w(\eta_j)$.

Now view $w(t)$ in terms of the solution of an initial value problem of (4.7). Let

$$\sigma_2 = w(t_1 + 1, \alpha_j)$$

and

$$\epsilon_2 = \epsilon_2(h) = w(t_1 + 1, \alpha_j + h) - \sigma_2.$$

Then $\epsilon_2 \to 0$ as $h \to 0$ by Theorem 4.4.

Thus, using Theorem 4.2 and the Mean Value Theorem,

$$
\begin{aligned}
p_{jh}(t) =& \frac{1}{h}[u(t, t_1, w_1, \sigma_2 + \epsilon_2) - u(t, t_1, w_1, \sigma_2)] \\
=& \frac{1}{h}\beta_2(t, u(t, t_1, w_1, \sigma_2 + \bar{\epsilon}_2))(\sigma_2 + \epsilon_2 - \sigma_2) \\
=& \frac{\epsilon_2}{h}\beta_2(t, u(t, t_1, w_1, \sigma_2 + \bar{\epsilon}_2)),
\end{aligned}
$$

where $\sigma_2 + \bar{\epsilon}_2$ is between σ_2 and $\sigma_2 + \epsilon_2$. Hence, for $\lim_{h\to 0} p_{jh}(t)$ to exist, we need $\lim_{h \to 0} \frac{\epsilon_2}{h}$ to exist. Note that $\beta_2(t_1) = 0$ and $\beta_2(t_1 + 1) = 1$. Thus, $\beta_2(t, u(\cdot))$ is a nontrivial solution of (4.9). By Property (U) for the variational equation,

$$\beta_2(t_2, u(\cdot)) - \sum_{i=1}^{r} \alpha_i \beta_2(\eta_i, u(\cdot)) \neq 0.$$

Recall,

$$p_{jh}(t_2, \alpha_j) - \sum_{i=1}^{r} \alpha_i p_{jh}(\eta_i, \alpha_j) = w(\eta_j, \alpha_j + h),$$

but $p_{jh}(t)$ gives us

$$\frac{\epsilon_2}{h}\left[\beta(t_2, u(t, t_1, w_1, \sigma_2 + \bar{\epsilon}_2)) - \sum_{i=1}^{r} \alpha_i \beta_2(\eta_i, u(t, t_1, w_1, \sigma_2 + \bar{\epsilon}_2))\right] = w(\eta_j, \alpha_j + h).$$

Consequently,

$$\frac{\epsilon_2}{h} = \frac{w(\eta_j, \alpha_j + h)}{\beta(t_2, u(t, t_1, w_1, \sigma_2 + \bar{\epsilon}_2)) - \sum_{i=1}^{r} \alpha_i \beta_2(\eta_i, u(t, t_1, w_1, \sigma_2 + \bar{\epsilon}_2))}$$

$$\to \frac{w(\eta_j, \alpha_j)}{\beta(t_2, u(\cdot)) - \sum_{i=1}^{r} \alpha_i \beta_2(\eta_i, u(\cdot))} := A$$

as $h \to 0$ using Theorems 4.1 and 4.4, and so,

$$p_{jh}(t) = \frac{\epsilon_2}{h}\beta_2(t, u(\cdot)) \to A\beta_2(t, u(t, t_1, w_1, \sigma_2)) = A\beta_2(t, w(t))$$

as $h \to 0$. Thus,

$$p_j(t) = \lim_{h\to 0} p_{jh}(t) = \frac{\partial w}{\partial \alpha_j}(t)$$

exists and solves (4.9) along $w(t)$. Also,

$$p_j(t_1) = \lim_{h\to 0} p_{jh}(t_1) = 0$$

and

$$p_j(t_2) - \sum_{i=1}^{r} \alpha_i p_j(\eta_i) = \lim_{h\to 0}\left[p_{jh}(t_2) - \sum_{i=1}^{r} \alpha_i p_{jh}(\eta_i)\right] = w(\eta_j).$$

Whence, we conclude part (a).

Next, we consider part (b). We will only show the proof of $\frac{\partial w}{\partial w_1}$, as the proof of $\frac{\partial w}{\partial w_2}$ is similar. Let $\epsilon > 0$ be as in Theorem 4.4, $0 < |h| < \epsilon$ be given, and denote $w(t, t_1, t_2, \eta_1, \ldots, \eta_m, \alpha_1, \ldots, \alpha_m, w_1, w_2)$ by $w(t, w_1)$. Consider the quotient,

$$z_{1h}(t) = \frac{1}{h}[w(t, w_1 + h) - w(t, w_1)].$$

Our goal is to show that $\lim_{h\to 0} z_{1h}(t)$ exists on $\mathbb{Z}$.

Note that, for $h \neq 0$,

$$\begin{aligned} z_{1h}(t_1) &= \frac{1}{h}[w(t_1, w_1 + h) - w(t_1, w_1)] \\ &= \frac{1}{h}[w_1 + h - w_1] \\ &= 1, \end{aligned}$$

and

$$z_{1h}(t_2) - \sum_{i=1}^{r} \alpha_i z_{1h}(\eta_i) = \frac{1}{h}[w(t_2, w_1 + h) - \sum_{i=1}^{r} \alpha_i w(\eta_i, w_1 + h)$$

$$-w(t_2,w_1) - \sum_{i=1}^{r} \alpha_i w(\eta_i, w_1)$$
$$= \frac{1}{h}[w_2 - w_2]$$
$$= 0.$$

Now view $z_{1h}(t)$ in terms of solutions of initial value problems of (4.7) at t_1. Let

$$\sigma_2 = w(t_1 + 1, w_1)$$

and

$$\epsilon_2 = \epsilon_2(h) = w(t_1 + 1, w_1 + h) - \sigma_2.$$

By Theorem 4.4, $\epsilon_2 \to 0$ as $h \to 0$. Now we rewrite $w(t)$ in terms of the solution of an inital value problem and use the Mean Value Theorem as well as Theorem 4.3 to see

$$\begin{aligned} z_{1h}(t) =& \frac{1}{h}[u(t,t_1,w_1+h,\sigma_2+\epsilon_2) - u(t,t_1,w_1,\sigma_2)] \\ =& \frac{1}{h}[u(t,t_1,w_1+h,\sigma_2+\epsilon_2) - u(t,t_1,w_1,\sigma+\epsilon_2) \\ & + u(t,t_1,w_1,\sigma+\epsilon_2) - u(t,t_1,w_1,\sigma_2)] \\ =& \frac{1}{h}[\beta_1(t,u(t,t_1,w_1+\bar{h},\sigma_2+\epsilon_2))h + \beta_2(t,u(t,t_1,w_1,\sigma_2+\bar{\epsilon}_2))\epsilon_2], \end{aligned}$$

where $\beta_i(t,u(\cdot)),\ i = 1,2$, denotes the solution of (4.9) along $u(\cdot)$ satisfying

$$\beta_1(t_1,u(\cdot)) = 1,\quad \beta_1(t_1+1,u(\cdot)) = 0,\quad \beta_2(t_1,u(\cdot)) = 0,\quad \beta_2(t_1+1,u(\cdot)) = 1.$$

In addition, $w_1 + \bar{h}$ is between w_1 and $w_1 + h$, and $\sigma_2 + \bar{\epsilon}_2$ is between σ_2 and $\sigma_2 + \epsilon_2$. Thus

$$z_{1h}(t) = \beta_1(t,u(t,t_1.w_1+\bar{h},\sigma_2+\epsilon_2)) + \frac{\epsilon_2}{h}\beta_2(t,u(t,t_1,w_1,\sigma_2+\bar{\epsilon}_2)).$$

For $\lim_{h\to 0} z_{1h}(t)$ to exist, it suffices to show $\lim_{h\to 0} \frac{\epsilon_2}{h}$ exists. Recall,

$$z_{1h}(t_2) - \sum_{i=1}^{r} \alpha_i z_{1h}(\eta_i) = 0,$$

and hence,

$$\frac{\epsilon_2}{h} = \frac{-[\beta_1(t_2, u(t, t_1, w_1 + \bar{h}, \sigma_2 + \epsilon_2)) - \sum_{i=1}^{r} \alpha_i \beta_1(\eta_i, u(t, t_1, w_1 + \bar{h}, \sigma_2 + \epsilon_2))]}{\beta_2(t_2, u(t, t_1, w_1, \sigma_2 + \bar{\epsilon}_2)) - \sum_{i=1}^{r} \alpha_i \beta_2(\eta_i, u(t, t_1, w_1, \sigma_2 + \bar{\epsilon}_2))},$$

which has nonzero denominator by Property (U) for the variational equation. Applying Theorem 4.1 and Theorem 4.4, let $\lim_{h \to 0} \frac{\epsilon_2}{h} := B$. Then,

$$\begin{aligned} z_1(t) &= \lim_{h \to 0} z_{1h}(t) \\ &= \beta_1(t, u(t, t_1, w_1, \sigma_2)) + B\beta_2(t, u(t, t_1, w_1, \sigma_2)) \\ &= \beta_1(t, u(\cdot)) + B\beta_2(t, u(\cdot)) \end{aligned}$$

is the solution of (4.7) along $u(\cdot)$. Moreover,

$$z_1(t_1) = \lim_{h \to 0} z_{1h}(t_1) = 1,$$

and

$$z_1(t_2) - \sum_{i=1}^{r} \alpha_i z_1(\eta_i) = \lim_{h \to 0} \left[z_{1h}(t_2) - \sum_{i=1}^{r} \alpha_i z_{1h}(\eta_i) \right] = 0.$$

Therefore, $z_1(t) = \frac{\partial w}{\partial w_1}(t)$. □

Now we establish an analogue for Theorem 4.3 involving boundary value probelms.

Theorem 4.6. *Assume conditions* (i),(ii), *and* (iii) *hold and that* (4.7) *satisfies Property (U) on* $\mathbb{Z}$. *Let* $r \in \mathbb{N}$ *and*

$$w(t) = w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$$

be the solution of (4.7), (4.8) *on* $\mathbb{Z}$, *where* $t_1 < t_1 + 1 < \eta_1 < \eta_1 + 1 < \cdots < \eta_r < \eta_r + 1 < t_2$ *in* $\mathbb{Z}$ *and* $\alpha_1, \ldots, \alpha_r, w_1, w_2 \in \mathbb{R}$. *Then:*

(a) $$\begin{aligned} \nu_1(t) &:= \Delta_{t_1} w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2) \\ &= w(t, t_1 + 1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2) \\ &\quad - w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2) \end{aligned}$$

(b) $\nu_2(t) := \Delta_{t_2} w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$

$$= w(t, t_1, t_2 + 1, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$$

$$-w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$$

are solutions of the respective linear difference equations

$$\nu_i(t+2) = A_{i1}(t)\nu_i(t) + A_{i2}(t)\nu_i(t+1), \ i = 1, 2,$$

where, for $i = 1, 2$*,*

$$A_{i1}(t) = \int_0^1 \frac{\partial f}{\partial d_1}(t, sw(t, t_i + 1) + (1 - s)w(t, t_i), w(t + 1, t_i + 1))ds,$$

$$A_{i2}(t) = \int_0^1 \frac{\partial f}{\partial d_2}(t, w(t, t_i), sw(t + 1, t_i + 1) + (1 - s)w(t + 1, t_i))ds,$$

with boundary conditions

$$\nu_1(t_1) = -\Delta_t w(t, t_1 + 1)|_{t=t_1}, \quad \nu_1(t_2) - \sum_{i=1}^{r} \alpha_i \nu_1(\eta_i) = 0,$$

$$\nu_2(t_1) = 0, \quad \nu_2(t_2) - \sum_{i=1}^{r} \alpha_i \nu_2(\eta_i) = -\Delta_t w(t, t_2 + 1)|_{t=t_2}.$$

Furthermore,

(c) for $1 \le j \le r$*,*

$$\xi_j(t) := \Delta_{\eta_j} w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$$

$$= w(t, t_1, t_2, \eta_1, \ldots, \eta_j + 1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$$

$$- w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$$

is a solution of the linear difference equation

$$\xi_j(t+2) = A_{j1}(t)\xi_i(t) + A_{j2}(t)\xi_i(t+1),$$

where

$$A_{j1}(t) = \int_0^1 \frac{\partial f}{\partial d_1}(t, sw(t, \eta_j + 1) + (1 - s)w(t, \eta_j), w(t + 1, \eta_j + 1))ds,$$

$$A_{j2}(t) = \int_0^1 \frac{\partial f}{\partial d_2}(t, w(t,\eta_j), sw(t+1,\eta_j+1) + (1-s)w(t+1,\eta_j))ds,$$

with boundary conditions

$$\xi_j(t_1) = 0, \quad \xi_j(t_2) - \sum_{i=1}^{r} \alpha_i \xi_j(\eta_i) = \alpha_j \Delta_t(t, \eta_j+1)|_{t=\eta_j}.$$

Proof. First note that parts (a) and (b) are similar, and thus we only verify the boundary conditions for part (b). Again for brevity, we denote $w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$ by $w(t, t_1)$.

Using a telescoping sum and the Mean Value Theorem, we have

$$\begin{aligned}
\nu_1(t+2) =& w(t+2, t_1+1) - w(t+2, t_1) \\
=& f(t, w(t, t_1+1), w(t+1, t_1+1)) \\
&- f(t, w(t, t_1), w(t+1, t_1)) \\
=& f(t, w(t, t_1+1), w(t+1, t_1+1)) - f(t, w(t, t_1), w(t+1, t_1+1)) \\
&+ f(t, w(t, t_1), w(t+1, t_1+1)) - f(t, w(t, t_1), w(t+1, t_1)) \\
=& \int_0^1 \frac{\partial f}{\partial d_1}(t, sw(t, t_1+1) + (1-s)w(t, t_1), w(t+1, t_1+1))ds \\
&\times (w(t, t_1+1) - w(t, t_1)) \\
&+ \int_0^1 \frac{\partial f}{\partial d_2}(t, w(t, t_1), sw(t+1, t_1+1) + (1-s)w(t+1, t_1))ds \\
&\times (w(t+1, t_1+1) - w(t+1, t_1)) \\
=& A_{11}\nu_1(t) + A_{12}\nu_1(t+1).
\end{aligned}$$

In addition, for part (a),

$$\begin{aligned}
\nu_1(t_1) &= w(t_1, t_1+1) - w(t_1, t_1) \\
&= w(t_1, t_1+1) - w(t_1+1, t_1+1) + w(t_1+1, t_1+1) - w(t_1, t_1) \\
&= -\Delta_t w(t, t_1+1)|_{t=t_1} + w_1 - w_1 \\
&= -\Delta_t w(t, t_1+1)|_{t=t_1},
\end{aligned}$$

and

$$\nu_1(t_2) - \sum_{i=1}^{r} \alpha_i \nu_1(\eta_i) = w(t_2, t_1+1) - w(t_2, t_1)$$
$$- \sum_{i=1}^{r} \alpha_i [w(\eta_i, t_1+1) - w(\eta_i, t_1)]$$
$$= [w_2 - w_2]$$
$$= 0.$$

For part (b), we have

$$\begin{aligned} \nu_2(t_1) &= w(t_1, t_2+1) - w(t_1, t_2) \\ &= [w_1 - w_1] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \nu_2(t_2) - \sum_{i=1}^{r} \alpha_i \nu_2(\eta_i) \quad &= \quad w(t_2, t_2+1) - w(t_2, t_2) \\ & \quad - \sum_{i=1}^{r} \alpha_i [w_2(\eta_i, t_2+1) + w_2(\eta_i, t_2)] \\ & \quad = w(t_2, t_2+1) - \sum_{i=1}^{r} \alpha_i w_2(\eta_i, t_2+1) \\ & \quad - w_2 - w(t_2+1, t_2+1) + w(t_2+1, t_2+1) \\ & \quad = w(t_2, t_2+1) - w(t_2+1, t_2+1) - w_2 + w_2 \\ &= \quad -\Delta_t w(t, t_2+1)|_{t=t_2}. \end{aligned}$$

Finally, we consider part (c). Fix $1 \le j \le r$ and denote $w(t, t_1, t_2, \eta_1, \ldots, \eta_r, \alpha_1, \ldots, \alpha_r, w_1, w_2)$ by $w(t, \eta_j)$. Then $\xi_j(t) = w(t, \eta_j+1) - w(t, \eta_j)$.

Again, using a telescoping sum and the Mean Value Theorem, we have

$$\xi_j(t+2) = w(t+2, \eta_j+1) - w(t+2, \eta_j)$$

$$
\begin{aligned}
&= f(t, w(t, \eta_j + 1), w(t+1, \eta_j + 1)) - f(t, w(t, \eta_j), w(t+1, \eta_j)) \\
&= f(t, w(t, \eta_j + 1), w(t+1, \eta_j + 1)) - f(t, w(t, \eta_j), w(t+1, \eta_j + 1)) \\
&\quad + f(t, w(t, \eta_j), w(t+1, \eta_j + 1)) - f(t, w(t, \eta_j), w(t+1, \eta_j)) \\
&= \int_0^1 \frac{\partial f}{\partial d_1}(t, sw(t, \eta_j + 1) + (1-s)w(t, \eta_j), w(t+1, \eta_j))ds \\
&\quad \times (w(t, \eta_j + 1) - w(t, \eta_j)) \\
&\quad + \int_0^1 \frac{\partial f}{\partial d_2}(t, w(t, \eta_j + 1), sw(t+1, \eta_j + 1) + (1-s)w(t+1, \eta_j))ds \\
&\quad \times (w(t+1, \eta_j + 1) - w(t_1, \eta_j)) \\
&= A_{j1}\xi_j(t) + A_{j2}\xi_j(t+1).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
\xi_j(t_1) &= w(t_1, \eta_j + 1) - w(t_1, \eta_j) \\
&= [w_1 - w_1] \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_j(t_2) - \sum_{i=1}^{r} \alpha_i \xi_j(\eta_i) &= w(t_2, \eta_j + 1) - w(t_2, \eta_j) - \sum_{i=1}^{r} \alpha_i [w(\eta_i, \eta_j + 1) - w(\eta_i, \eta_j)] \\
&= w(t_2, \eta_j + 1) - \alpha_j w(\eta_j + 1, \eta_j + 1) + \alpha_j w(\eta_j + 1, \eta_j + 1) \\
&\quad - \sum_{i=1}^{r} \alpha_i w(\eta_i, \eta_j + 1) - w_2 \\
&= w_2 + \alpha_j w(\eta_j + 1, \eta_j + 1) - \alpha_j w(\eta_j, \eta_j + 1) - w_2 \\
&= \alpha_j \Delta_t w(t, \eta_j + 1)|_{t=\eta_j}.
\end{aligned}
$$

□

4.3 General nth Order Problem

With the ideas and strategies of the previous section, we are now able to turn to the general nth order difference equation

$$w(t+n) = f(t, w(t), w(t+1), \ldots, w(t+n-1)),\ t \in \mathbb{Z},\ n \geq 2, \tag{4.10}$$

satisfying

$$\begin{aligned} &w(t_j + i) = w_{ij},\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1, \\ &w(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}) = w_{ik},\ 0 \leq i \leq m_k - 1 \end{aligned} \tag{4.11}$$

where $2 \leq k \leq n,\ m \in \mathbb{N},\ m_1, \ldots, m_k$ are positive integers such that $\sum_{i=1}^{k} m_i = n,\ t_1 < t_1+m_1-1 < t_2 < t_2+m_2-1 < \cdots < t_{k-1} < t_{k-1}+m_{k-1}-1 < \eta_{01} < \eta_{01}+1 < \cdots < \eta_{m_k-1,m} < \eta_{m_k-1,m}+1 < t_k$ in $\mathbb{Z}$, and $\alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k} \in \mathbb{R}$.

Remark 4.3. Note we need not always be given m values for η and α as is done in this section. This is done simply to ease the burdensome notation.

The variational equation for (4.10) given a solution $w(t)$ of (4.10) is defined by

$$z(m+n) = \sum_{i=1}^{n} \frac{\partial f}{\partial d_i}(t, w(t), w(t+1), \ldots, w(t+n-1)) z(t+i-1). \tag{4.12}$$

As in the previous section, we now present two results; one pertaining to partial derivatives and the other pertaining to partial differences.

Theorem 4.7. *Let $n \geq 2,\ 2 \leq k \leq n$, and $m \in \mathbb{N}$ be given, and let $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$. Assume conditions* (i), (ii), *and* (iii) *are satisfied, that* (4.10) *satisfies Property (U) on $\mathbb{Z}$, and that the variational equation* (4.12) *satisfies Property (U) along solutions of* (4.10). *Suppose*

$$w(t) = w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})$$

is the solution of (4.10) *on $\mathbb{Z}$ where*

$$w(t_j + i) = w_{ij},\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

$$w(t_k+i)-\sum_{p=1}^{m}\alpha_{ip}w(\eta_{ip})=w_{ik},\ 0\le i\le m_k-1,$$

$t_1<t_1+m_1-1<t_2<t_2+m_2-1<\cdots<t_{k-1}+m_{k-1}-1<\eta_{01}<\eta_{01}+1<\cdots<\eta_{0m}<\eta_{0m}+1<\cdots<\eta_{m_k-1,1}<\eta_{m_k-1,1}+1<\cdots<\eta_{m_k-1,m}<\eta_{m_k-1,m}+1<t_k$ *in* $\mathbb{Z}$, *and* $\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}\in\mathbb{R}$. *Then:*

(a) for $0\le q\le m_k-1$ *and* $1\le l\le m$, $\dfrac{\partial w}{\partial\alpha_{ql}}(t)$ *exists on* $\mathbb{Z}$, *and* $p_{ql}(t):=\dfrac{\partial w}{\partial\alpha_{ql}}(t)$ *is the solution of* (4.12) *along* $w(t)$ *satisfying*

$$\begin{aligned}
p_{ql}(t_j+i) &= 0, && 0\le i\le m_j-1,\ 1\le j\le k-1,\\
p_{ql}(t_k+i) &- \sum_{p=1}^{m}\alpha_{ip}p_{ql}(\eta_{ip})=0, && 0\le i\le m_k-1,\ i\ne q,\\
p_{ql}(t_k+q) &- \sum_{p=1}^{m}\alpha_{qp}p_{ql}(\eta_{qp})=w(\eta_{ql}).
\end{aligned}$$

(b) for $1\le l\le k-1$ *and* $0\le q\le m_l-1$, $\dfrac{\partial w}{\partial w_{ql}}(t)$ *exists on* $\mathbb{Z}$, *and* $z_{ql}(t):=\dfrac{\partial w}{\partial w_{ql}}(t)$ *is the solution of* (4.12) *along* $w(t)$ *satisfying*

$$\begin{aligned}
z_{ql}(t_j+i) &= 0, && 0\le i\le m_j-1,\ 1\le j\le k-1,\ j\ne l,\\
z_{ql}(t_l+i) &= 0, && 0\le i\le m_l-1,\ i\ne q,\\
z_{ql}(t_l+q) &= 1,\\
z_{ql}(t_k+i) &- \sum_{p=1}^{m}\alpha_{ip}z_{ql}(\eta_{ip})=0, && 0\le i\le m_k-1,
\end{aligned}$$

and for $0\le q\le m_k-1$, $\dfrac{\partial w}{\partial w_{qk}}(t)$ *exists on* $\mathbb{Z}$, *and* $z_{qk}(t):=\dfrac{\partial w}{\partial w_{rk}}(t)$ *is the solution of* (4.12) *along* $w(t)$ *satisfying*

$$\begin{aligned}
z_{qk}(t_j+i) &= 0, && 0\le i\le m_j-1,\ 1\le j\le k-1,\\
z_{qk}(t_k+i) &- \sum_{p=1}^{m}\alpha_{ip}z_{qk}(\eta_{ip})=0, && 0\le m_k-1,\ i\ne q,\\
z_{qk}(t_k+q) &- \sum_{p=1}^{m}\alpha_{qp}z_{qk}(\eta_{qp})=1.
\end{aligned}$$

Proof. First we will deal with part (a). Let $\epsilon > 0$ be as in Theorem 4.4, and $0 < |h| < \epsilon$ be given. Fix $1 \leq q \leq m_k - 1$ and $1 \leq l \leq m$, and consider the quotient,

$$\begin{aligned} p_{qlh}(t) &= \frac{1}{h}[w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{ql} + h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\ &\quad -w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{ql}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})]. \end{aligned}$$

Notice that for $0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1$, and $h \neq 0$,

$$\begin{aligned} p_{qlh}(t_j + i) &= \frac{1}{h}[w(t_j + i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{ql} + h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\ &\quad -w(t_j + i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{ql}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\ &= [w_{ij} - w_{ij}] \\ &= 0, \end{aligned}$$

and for $0 \leq i \leq m_k - 1,\ i \neq q$, and $h \neq 0$,

$$\begin{aligned} p_{qlh}(t_k + i) &- \sum_{p=1}^{m} \alpha_{ip} p_{qlh}(\eta_{ip}) \\ &= \frac{1}{h}\Big[w(t_k + i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{ql} + h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\ &\quad -\sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad\qquad \alpha_{01}, \ldots, \alpha_{ql} + h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\ &\quad -w(t_k + i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{ql}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\ &\quad +\sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \end{aligned}$$

$$
\begin{aligned}
& \qquad\qquad \alpha_{01}, \ldots, \alpha_{ql}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})\Big] \\
&= [w_{ik} - w_{ik}] \\
&= 0.
\end{aligned}
$$

Also, we have, for $h \neq 0$,

$$
\begin{aligned}
p_{qlh}(t_k+q) \; &- \sum_{p=1}^{m} \alpha_{qp} p_{qlh}(\eta_{qp}) \\
&= \frac{1}{h}\Big[(w(t_k+q, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}+h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&- \sum_{p=1}^{m} \alpha_{qp} w(\eta_{qp}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}+h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
&- w(t_k+q, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&+ \sum_{p=1}^{m} \alpha_{qp} w(\eta_{qp}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})\Big] \\
&= \frac{1}{h}\Big[w(t_k+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}+h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&- hw(\eta_{ql}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}+h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&+ hw(\eta_{ql}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}+h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&- \sum_{p=1}^{m} \alpha_{qp} w(\eta_{qp}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}+h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) - w_{qk}\Big] \\
&= \frac{1}{h}[w_{qk} - w_{qk} + hw(\eta_{ql}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{ql}+h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})]
\end{aligned}
$$

$$= w(\eta_{ql}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m},$$
$$\alpha_{01}, \ldots, \alpha_{ql} + h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}).$$

Also note, as $h \to 0$, $p_{qlh}(t_k + q) - \sum_{p=1}^{m} \alpha_{qp} p_{qlh}(\eta_{qp}) \to w(\eta_{ql})$.

Let $1 \leq l \leq k-1$, and for $m_l \leq i \leq n-1$, let

$$\sigma_i = w(t_l + i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m},$$
$$\alpha_{01}, \ldots, \alpha_{ql}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}),$$

and

$$\epsilon_i = \epsilon_i(h) = w(t_l + i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m},$$
$$\alpha_{01}, \ldots, \alpha_{ql} + h, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) - \sigma_i.$$

Then, by Theorem 4.4, $\epsilon_i \to 0$ as $h \to 0$ for $m_l \leq i \leq n-1$. Now view $w(t)$ in terms of the solution of an initial value problem of (4.10), and denote $w(t) = u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1}, \ldots, \sigma_{n-1})$. Thus, we have

$$p_{qlh}(t) = \frac{1}{h}[u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l} + \epsilon_{m_l}, \sigma_{m_l+1} + \epsilon_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1})$$
$$-u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1}, \ldots, \sigma_{n-1})].$$

Now we implement a telescoping sum to yield

$$p_{qlh}(t) = \frac{1}{h}[u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l} + \epsilon_{m_l}, \sigma_{m_l+1} + \epsilon_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1})$$
$$-u(t, t_1, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1} + \epsilon_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1})]$$
$$+[u(t, t_1, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1} + \epsilon_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1})$$
$$-u(t, t_1, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1})]$$
$$+ - \cdots$$
$$+[u(t, t_1, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1})$$
$$-u(t, t_1, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1}, \ldots, \sigma_{n-1})]$$

Thus, by using Theorem 4.2 and the Mean Value Theorem,

$$\begin{aligned} p_{qlh}(t) &= \frac{\epsilon_{m_l}}{h}\beta_{m_l}(t, u(t, t_l, w_{0l}, \dots, w_{m_l-1,l}, \sigma_{m_l} + \bar{\epsilon}_{m_l}, \\ &\qquad \sigma_{m_l+1} + \epsilon_{m_l+1}, \dots, \sigma_{n-1} + \epsilon_{n-1}) \\ &\quad + \frac{\epsilon_{m_l+1}}{h}\beta_{m_l+1}(t, u(t, t_l, w_{0l}, \dots, w_{m_l-1,l}, \sigma_{m_l}, \\ &\qquad \sigma_{m_l+1} + \bar{\epsilon}_{m_l+1}, \dots, \sigma_{n-1} + \epsilon_{n-1}) \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_{n-1}}{h}\beta_{n-1}(t, u(t, t_l, w_{0l}, \dots, w_{m_l-1,l}, \sigma_{m_l}, \\ &\qquad \sigma_{m_l+1}, \dots, \sigma_{n-1} + \bar{\epsilon}_{n-1}), \end{aligned}$$

where $\sigma_i + \bar{\epsilon}_i$ is between σ_i and $\sigma_i + \epsilon_i$, for $m_l \leq i \leq n-1$, and $\beta_j(t_l + i) = \delta_{ij}$, for $0 \leq i, j \leq n-1$, and solves (4.12). Hence, for $\lim_{h \to 0} p_{qlh}(t)$ to exist, we need $\lim_{h \to 0} \frac{\epsilon_i}{h}$ to exist for $m_l \leq i \leq n-1$. Now, from the construction of $p_{qlh}(t)$,

$$p_{qlh}(t_j + i) = 0, \quad 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,\ j \neq l,$$

$$p_{qlh}(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} p_{qlh}(\eta_{ip}) = 0, \quad 0 \leq i \leq m_k - 1,\ i \neq q,$$

$$p_{qlh}(t_k + q) - \sum_{p=1}^{m} \alpha_{qp} p_{qlh}(\eta_{qp}) = u(\eta_{ql}, \cdot, \alpha_{ql} + h, \cdot).$$

Hence, we have a system of $n - m_l$ linear equations with $n - m_l$ unknowns:

$$\begin{aligned} 0 &= \frac{\epsilon_{m_l}}{h}\beta_{m_l}(t_j + i, u(t, t_l, w_{0l}, \dots, w_{m_l-1,l}, \sigma_{m_l} + \bar{\epsilon}_{m_l}, \\ &\qquad \sigma_{m_l+1} + \epsilon_{m_l+1}, \dots, \sigma_{n-1} + \epsilon_{n-1}) \\ &\quad + \frac{\epsilon_{m_l+1}}{h}\beta_{m_l+1}(t_j + i, u(t, t_l, w_{0l}, \dots, w_{m_l-1,l}, \sigma_{m_l}, \\ &\qquad \sigma_{m_l+1} + \bar{\epsilon}_{m_l+1}, \dots, \sigma_{n-1} + \epsilon_{n-1}) + \cdots \\ &\quad + \frac{\epsilon_{n-1}}{h}\beta_{n-1}(t_j + i, u(t, t_l, w_{0l}, \dots, w_{m_l-1,l}, \sigma_{m_l}, \\ &\qquad \sigma_{m_l+1}, \dots, \sigma_{n-1} + \bar{\epsilon}_{n-1})), \end{aligned}$$

$$0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l,$$

$$\begin{aligned} 0 &= \frac{\epsilon_{m_l}}{h}[\beta_{m_l}(t_k + i, u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l} + \bar{\epsilon}_{m_l}, \\ &\qquad \sigma_{m_l+1} + \epsilon_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1})) \\ &\quad - \sum_{p=1}^{m} \alpha_{ip}\beta_{m_l}(\eta_{ip}, u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l} + \bar{\epsilon}_{m_l}, \\ &\qquad \sigma_{m_l+1} + \epsilon_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1}))] + \cdots \\ &\quad + \frac{\epsilon_{n-1}}{h}[\beta_{n-1}(t_k + i, u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \\ &\qquad \sigma_{m_l+1}, \ldots, \sigma_{n-1} + \bar{\epsilon}_{n-1})) \\ &\quad - \sum_{p=1}^{m} \alpha_{ip}\beta_{n-1}(\eta_{ip}, u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \\ &\qquad \sigma_{m_l+1}, \ldots, \sigma_{n-1} + \bar{\epsilon}_{n-1}))], \end{aligned}$$

$$0 \le i \le m_k - 1,\ i \ne q,$$

and

$$\begin{aligned} u(\eta_{ql}, \cdot, \alpha_{ql} + h, \cdot) &= \frac{\epsilon_{m_l}}{h}\Big[\beta_{m_l}(t_k + q, u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l} + \bar{\epsilon}_{m_l}, \\ &\qquad \sigma_{m_l+1} + \epsilon_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1})) \\ &\quad - \sum_{p=1}^{m} \alpha_{qp}\beta_{m_l}(\eta_{qp}, u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l} + \bar{\epsilon}_{m_l}, \\ &\qquad \sigma_{m_l+1} + \epsilon_{m_l+1}, \ldots, \sigma_{n-1} + \epsilon_{n-1}))\Big] + \cdots \\ &\quad + \frac{\epsilon_{n-1}}{h}\Big[\beta_{n-1}(t_k + q, u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \\ &\qquad \sigma_{m_l+1}, \ldots, \sigma_{n-1} + \bar{\epsilon}_{n-1})) \\ &\quad - \sum_{p=1}^{m} \alpha_{qp}\beta_{n-1}(\eta_{qp}, u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \\ &\qquad \sigma_{m_l+1}, \ldots, \sigma_{n-1} + \bar{\epsilon}_{n-1}))\Big]. \end{aligned}$$

From this point forward, we will frequently suppress the arguments of β, subscripts of α and η, as well as the limits of summation.

As $u(\cdot)$ is not consistent throughout the preceeding system of equations, we must consider the matrix

$$M := \begin{pmatrix} \beta_{m_l}(t_1, u(x)) & \beta_{m_l+1}(t_1, u(x)) & \cdots & \beta_{n-1}(t_1, u(x)) \\ \beta_{m_l}(t_1+1, u(x)) & \beta_{m_l+1}(t_1+1, u(x)) & \cdots & \beta_{n-1}(t_1+1, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_1+m_1-1, u(x)) & \beta_{m_l+1}(t_1+m_1-1, u(x)) & \cdots & \beta_{n-1}(t_1+m_1-1, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_{l-1}+m_{l-1}-1, u(x)) & \beta_{m_l+1}(t_{l-1}+m_{l-1}-1, u(x)) & \cdots & \beta_{n-1}(t_{l-1}+m_{l-1}-1, u(x)) \\ \beta_{m_l}(t_{l+1}, u(x)) & \beta_{m_l+1}(t_{l+1}, u(x)) & \cdots & \beta_{n-1}(t_{l+1}, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_k, u(x)) - & \beta_{m_l+1}(t_k, u(x)) - & & \beta_{n-1}(t_k, u(x)) - \\ \sum \alpha\beta_{m_l}(\eta, u(x)) & \sum \alpha\beta_{m_l+1}(\eta, u(x)) & \cdots & \sum \alpha\beta_{n-1}(\eta, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_k+m_k-1, u(x)) - & \beta_{m_l+1}(t_k+m_k-1, u(x)) - & & \beta_{n-1}(t_k+m_k-1, u(x)) - \\ \sum \alpha\beta_{m_l}(\eta, u(x)) & \sum \alpha\beta_{m_l+1}(\eta, u(x)) & \cdots & \sum \alpha\beta_{n-1}(\eta, u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R},\ m_l \leq i \leq n-1$, not all zero such that

$$p_{m_l} \begin{pmatrix} \beta_{m_l}(t_1, u(t)) \\ \beta_{m_l}(t_1+1, u(t)) \\ \vdots \\ \beta_{m_l}(t_{l-1}+m_{l-1}-1, u(t)) \\ \beta_{m_l}(t_{l+1}, u(t)) \\ \vdots \\ \beta_{m_l}(t_k, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \\ \vdots \\ \beta_{m_l}(t_k+m_k-1, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \beta_{n-1}(t_1, u(t)) \\ \beta_{n-1}(t_1+m_1-1, u(t)) \\ \vdots \\ \beta_{n-1}(t_{l-1}+m_{l-1}-1, u(t)) \\ \beta_{n-1}(t_{l+1}, u(t)) \\ \vdots \\ \beta_{n-1}(t_k, u(t)) - \\ \sum \alpha\beta_{n-1}(\eta, u(t)) \\ \vdots \\ \beta_{n-1}(t_k+m_k-1, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$\omega(t, u(t)) := p_{m_l}\beta_{m_l}(t, u(t)) + \cdots + p_{n-1}\beta_{n-1}(t, u(t)).$$

Then, $\omega(t, u(t))$ is a nontrivial solution of (4.12), but

$$\omega(t_j + i, u(t)) = 0,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

and

$$\omega(t_k+i,u(t))-\sum_{p=1}^{m}\alpha_{ip}\omega(\eta_{ip},u(t))=0,\ 0\le i\le m_k-1,$$

which when coupled with Property (U) implies $\omega(t,u(t))\equiv 0$ in turn implying $p_i=0,\ m_l\le i\le n-1$. This is a contradiction. Hence $\det(M)\ne 0$. Thus, as a result of continuous dependence, for $h\ne 0$ and sufficiently small, $\det(M(h))\ne 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, for each $m_l\le i\le n-1$, we can solve $\epsilon_i(h)/h$, by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h}=\frac{1}{|M(h)|}\times$$

$$\begin{vmatrix}
\beta_{m_l}(t_1) & \cdots & \beta_{i-2}(t_1) & 0 & \beta_i(t_1) & \cdots & \beta_{n-1}(t_1)\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\begin{matrix}\beta_{m_l}(t_k)-\\ \sum\alpha\beta_{m_l}(\eta)\end{matrix} & \cdots & \begin{matrix}\beta_{i-2}(t_k)-\\ \sum\alpha\beta_{i-2}(\eta)\end{matrix} & 0 & \begin{matrix}\beta_i(t_k)-\\ \sum\alpha\beta_i(\eta)\end{matrix} & \cdots & \begin{matrix}\beta_{n-1}(t_k)-\\ \sum\alpha\beta_{n-l}(\eta)\end{matrix}\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\begin{matrix}\beta_{m_l}(t_k+q)-\\ \sum\alpha\beta_{m_l}(\eta)\end{matrix} & \cdots & \begin{matrix}\beta_{i-2}(t_k+q)\\ \sum\alpha\beta_{i-2}(\eta)\end{matrix} & \begin{matrix}u(\eta_{ql},\\ \alpha_{ql}+h)\end{matrix} & \begin{matrix}\beta_i(t_k+q)-\\ \sum\alpha\beta_i(\eta)\end{matrix} & \cdots & \begin{matrix}\beta_{n-1}(t_k+q)-\\ \sum\alpha\beta_{n-l}(\eta)\end{matrix}\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\begin{matrix}\beta_{m_l}(t_k+m_k-1)\\ -\sum\alpha\beta_{m_l}(\eta)\end{matrix} & \cdots & \begin{matrix}\beta_{i-2}(t_k+m_k-1)\\ -\sum\alpha\beta_{i-2}(\eta)\end{matrix} & 0 & \begin{matrix}\beta_i(t_k+m_k-1)\\ -\sum\alpha\beta_i(\eta)\end{matrix} & \cdots & \begin{matrix}\beta_{n-1}(t_k+m_k-1)\\ -\sum\alpha\beta_{n-l}(\eta)\end{matrix}
\end{vmatrix}.$$

Note as $h\to 0$, $\det(M(h))\to\det(M)$, and so for $m_l\le i\le n-1$, $\epsilon_i(h)/h\to\det(M_i)/\det(M):=A_i$ as $h\to 0$, where M_i is the $n-m_l\times n-m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\mathrm{col}[0,\ldots,0,\mathrm{u}(\eta_{\mathrm{ql}}),0,\ldots,0].$$

Now let $p_{ql}(t)=\lim_{h\to 0}p_{qlh}(t)$, and note by construction of $p_{qlh}(t)$,

$$p_{ql}(t)=\frac{\partial w}{\partial\alpha_{ql}}(t).$$

Furthermore,

$$p_{ql}(t)=\lim_{h\to 0}p_{qlh}(t)=\sum_{i=m_l}^{n-1}A_i\beta_i(t,u(t)),$$

which is the solution of the variational equation (4.12) along $w(t)$. In addition,

$$p_{ql}(t_j+i) = \lim_{h\to 0} p_{qlh}(t_j+i) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

$$\begin{aligned} p_{ql}(t_k+i) - \sum_{p=1}^{m} \alpha_{ip} p_{ql}(\eta_{il}) &= \lim_{h\to 0}\left[p_{qlh}(t_k+i) - \sum_{p=1}^{m} \alpha_{ip} p_{qlh}(\eta_{il})\right] \\ &= 0,\ 0 \le m_k - 1,\ i \ne q, \end{aligned}$$

and

$$p_{ql}(t_k+q) - \sum_{p=1}^{m} \alpha_{qp} p_{ql}(\eta_{ql}) = \lim_{h\to 0}\left[p_{qlh}(t_k+q) - \sum_{p=1}^{m} \alpha_{qp} p_{qlh}(\eta_{ql})\right] = u(\eta_{ql}).$$

This completes the argument for $\frac{\partial w}{\partial \alpha_{ql}}$.

Now we move to the proof of part (b). First, let $\epsilon > 0$ be as in Theorem 4.4, and let $0 < |h| < \epsilon$ be given. Fix $1 \le l \le k-1,\ 0 \le q \le m_l - 1$, and consider the quotient,

$$\begin{aligned} z_{qlh}(t) &= \frac{1}{h}[w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}+h, \ldots, w_{m_k-1,k}) \\ &\quad - w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}, \ldots, w_{m_k-1,k})]. \end{aligned}$$

Notice that for $0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l$, and $h \ne 0$,

$$\begin{aligned} z_{qlh}(t_j+i) &= \frac{1}{h}[w(t_j+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}+h, \ldots, w_{m_k-1,k}) \\ &\quad - w(t_j+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}, \ldots, w_{m_k-1,k})] \\ &= [w_{ij} - w_{ij}] \\ &= 0, \end{aligned}$$

for $0 \leq i \leq m_l - 1,\ i \neq q$, and $h \neq 0$,

$$\begin{aligned} z_{qlh}(t_l+i) &= \frac{1}{h}[w(t_l+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}+h, \ldots, w_{m_k-1,k}) \\ &\quad -w(t_l+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}, \ldots, w_{m_k-1,k})] \\ &= [w_{il} - w_{il}] = 0, \end{aligned}$$

and for $0 \leq i \leq m_k - 1$ and $h \neq 0$,

$$\begin{aligned} z_{qlh}(t_k+i) &- \sum_{p=1}^{m} \alpha_{ip} z_{qlh}(\eta_{ip}) \\ &= \frac{1}{h}\Big[(w(t_k+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}+h, \ldots, w_{m_k-1,k}) \\ &- \sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}+h, \ldots, w_{m_k-1,k})) \\ &- w(t_k+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}, \ldots, w_{m_k-1,k}) \\ &+ \sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}, \ldots, w_{m_k-1,k})\Big] \\ &= [w_{ik} - w_{ik}] \\ &= 0. \end{aligned}$$

Also, we have, for $h \neq 0$,

$$\begin{aligned} z_{qlh}(t_l+q) &= \frac{1}{h}[w(t_l+q, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{ql}+h, \ldots, w_{m_k-1,k}) \\ &\quad -w(t_l+q, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \end{aligned}$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{ql},\ldots,w_{m_k-1,k})]$$
$$= \frac{1}{h}[w_{ql}+h-w_{ql}]$$
$$= 1.$$

Now view $w(t)$ in terms of the solution of an initial value problem of (4.10).

For $m_l \le i \le n-1$, let

$$\sigma_i = w(t_l+i,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},$$
$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{ql},\ldots,w_{m_k-1,k}),$$

and

$$\epsilon_i=\epsilon_i(h) = w(t_l+i,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},$$
$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{ql}+h,\ldots,w_{m_k-1,k})-\sigma_i.$$

Then, by Theorem 4.4, $\epsilon_i \to 0$ as $h \to 0$ for $m_l \le i \le n-1$. Thus, we have $w(t)=u(t,t_l,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\ldots,\sigma_{n-1})$, and

$$z_{qlh}(t) = \frac{1}{h}[u(t,t_l,w_{0l},\ldots,w_{ql}+h,\ldots,w_{m_l-1,l},\sigma_{m_l}+\epsilon_{m_l},$$
$$\sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1})$$
$$-u(t,t_l,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\ldots,\sigma_{n-1})].$$

Now we implement a telescoping sum to yield

$$z_{qlh}(t) = \frac{1}{h}[u(t,t_l,w_{0l},\ldots,w_{ql}+h,\ldots,w_{m_l-1,l},\sigma_{m_l}+\epsilon_{m_l},$$
$$\sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1})$$
$$-u(t,t_1,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l}+\epsilon_{m_l},$$
$$\sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1})]$$

$$+[u(t,t_1,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l}+\epsilon_{m_l},$$

$$\sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1})$$

$$-u(t,t_1,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1})]$$

$$+-\cdots$$

$$+[u(t,t_1,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1})$$

$$-u(t,t_1,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\ldots,\sigma_{n-1})].$$

Thus by using Theorem 4.2 and the Mean Value Theorem,

$$\begin{aligned} z_{qlh}(t) \quad = \quad & \beta_q(t,u(t,t_l,w_{0l},\ldots,w_{ql}+\bar{h},\ldots,w_{m_l-1,l},\sigma_{m_l}+\epsilon_{m_l}, \\ & \qquad \sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1}) \\ & +\frac{\epsilon_{m_l}}{h}\beta_{m_l}(t,u(t,t_l,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l}+\bar{\epsilon}_{m_l}, \\ & \qquad \sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1}) \\ & +\frac{\epsilon_{m_l+1}}{h}\beta_{m_l+1}(t,u(t,t_l,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l}, \\ & \qquad \sigma_{m_l+1}+\bar{\epsilon}_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1}) \\ & +\cdots \\ & +\frac{\epsilon_{n-1}}{h}\beta_{n-1}(t,u(t,t_l,w_{0l},\ldots,w_{ql},\ldots,w_{m_l-1,l},\sigma_{m_l}, \\ & \qquad \sigma_{m_l+1},\ldots,\sigma_{n-1}+\bar{\epsilon}_{n-1}), \end{aligned}$$

where $w_{ql}+\bar{h}$ is between w_{ql} and $w_{ql}+h$, $\sigma_i+\bar{\epsilon}_i$ is between σ_i and $\sigma_i+\epsilon_i$ for $m_l \le i \le n-1$, and $\beta_j(t_l+i)=\delta_{ij}$, for $0 \le i,j \le n-1$, and solves (4.12).

Hence, for $\lim_{h\to 0} p_{qlh}(t)$ to exist, we need $\lim_{h\to 0}\frac{\epsilon_i}{h}$ to exist for each $m_l \le i \le n-1$. Now, from the construction of $z_{qlh}(t)$,

$$z_{qlh}(t_j+i)=0, \qquad 0 \le i \le m_j-1,\ 1 \le j \le k-1,\ j \ne l,$$

and

$$z_{qlh}(t_k+i)-\sum_{p=1}^{m}\alpha_{ip}z_{qlh}(\eta_{ip})=0, \quad 0 \le i \le m_k-1.$$

Hence, we have a system of $n-m_l$ linear equations with $n-m_l$ unknowns:

$$
\begin{aligned}
&-\beta_q(t_j+i, u(t, t_l, w_{0l}, \ldots, w_{ql}+\bar{h}, \ldots, w_{m_l-1,l}, \sigma_{m_l}+\epsilon_{m_l},\\
&\qquad \sigma_{m_l+1}+\epsilon_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1})\\
&\quad =\frac{\epsilon_{m_l}}{h}\beta_{m_l}(t_j+i, u(t, t_l, w_{0l}, \ldots, w_{ql}, \ldots, w_{m_l-1,l}, \sigma_{m_l}+\bar{\epsilon}_{m_l},\\
&\qquad\qquad \sigma_{m_l+1}+\epsilon_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1})\\
&\quad +\frac{\epsilon_{m_l+1}}{h}\beta_{m_l+1}(t_j+i, u(t, t_l, w_{0l}, \ldots, w_{ql}, \ldots, w_{m_l-1,l}, \sigma_{m_l},\\
&\qquad\qquad \sigma_{m_l+1}+\bar{\epsilon}_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1})\\
&\quad +\cdots\\
&\quad +\frac{\epsilon_{n-1}}{h}\beta_{n-1}(t_j+i, u(t, t_l, w_{0l}, \ldots, w_{ql}, \ldots, w_{m_l-1,l}, \sigma_{m_l},\\
&\qquad\qquad \sigma_{m_l+1}, \ldots, \sigma_{n-1}+\bar{\epsilon}_{n-1})),\\
&\quad 0 \leq i \leq m_j-1,\ 1 \leq j \leq k-1,\ j \neq l,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\beta_q(t_k+i, u(t, t_l, w_{0l}, \ldots, w_{ql}+\bar{h}, \ldots, w_{m_l-1,l}, \sigma_{m_l}+\epsilon_{m_l},\\
&\qquad \sigma_{m_l+1}+\epsilon_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1})\\
&\quad +\sum_{p=1}^{m}\alpha_{ip}\beta_q(\eta_{ip}, u(t, t_l, w_{0l}, \ldots, w_{ql}+\bar{h}, \ldots, w_{m_l-1,l}, \sigma_{m_l}+\epsilon_{m_l},\\
&\qquad\qquad \sigma_{m_l+1}+\epsilon_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1})\\
&\quad =\frac{\epsilon_{m_l}}{h}\Big[\beta_{m_l}(t_k+i, u(t, t_l, w_{0l}, \ldots, w_{ql}, \ldots, w_{m_l-1,l}, \sigma_{m_l}+\bar{\epsilon}_{m_l},\\
&\qquad\qquad \sigma_{m_l+1}+\epsilon_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1}))\\
&\quad -\sum_{p=1}^{m}\alpha_{ip}\beta_{m_l}(\eta_{ip}, u(t, t_l, w_{0l}, \ldots, w_{ql}, \ldots, w_{m_l-1,l}, \sigma_{m_l}+\bar{\epsilon}_{m_l},\\
&\qquad\qquad \sigma_{m_l+1}+\epsilon_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1}))\Big]\\
&\quad +\cdots\\
&\quad +\frac{\epsilon_{n-1}}{h}\Big[\beta_{n-1}(t_k+i, u(t, t_l, w_{0l}, \ldots, w_{ql}, \ldots, w_{m_l-1,l}, \sigma_{m_l},\\
&\qquad\qquad \sigma_{m_l+1}, \ldots, \sigma_{n-1}+\bar{\epsilon}_{n-1}))
\end{aligned}
$$

$$- \sum_{p=1}^{m} \alpha_{ip} \beta_{n-1}(\eta_{ip}, u(t, t_l, w_{0l}, \ldots, w_{ql}, \ldots, w_{m_l-1,l}, \sigma_{m_l},$$

$$\sigma_{m_l+1}, \ldots, \sigma_{n-1} + \bar{\epsilon}_{n-1}))\Big],$$

$$0 \le i \le m_k - 1.$$

From now on, we will, at times, suppress the arguments of β, the subscripts of η and α, and the limits of summation. Note that $u(\cdot)$ is not necessarily the same within the system of equations so we consider the matrix

$$M := \begin{pmatrix} \beta_{m_l}(t_1, u(x)) & \beta_{m_l+1}(t_1, u(x)) & \cdots & \beta_{n-1}(t_1, u(x)) \\ \beta_{m_l}(t_1+1, u(x)) & \beta_{m_l+1}(t_1+1, u(x)) & \cdots & \beta_{n-1}(t_1+1, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_1+m_1-1, u(x)) & \beta_{m_l+1}(t_1+m_1-1, u(x)) & \cdots & \beta_{n-1}(t_1+m_1-1, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_{l-1}+m_{l-1}-1, u(x)) & \beta_{m_l+1}(t_{l-1}+m_{l-1}-1, u(x)) & \cdots & \beta_{n-1}(t_{l-1}+m_{l-1}-1, u(x)) \\ \beta_{m_l}(t_{l+1}, u(x)) & \beta_{m_l+1}(t_{l+1}, u(x)) & \cdots & \beta_{n-1}(t_{l+1}, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix}\beta_{m_l}(t_k, u(x)) - \\ \sum \alpha\beta_{m_l}(\eta, u(x))\end{matrix} & \begin{matrix}\beta_{m_l+1}(t_k, u(x)) - \\ \sum \alpha\beta_{m_l+1}(\eta, u(x))\end{matrix} & \cdots & \begin{matrix}\beta_{n-1}(t_k, u(x)) - \\ \sum \alpha\beta_{n-1}(\eta, u(x))\end{matrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix}\beta_{m_l}(t_k+m_k-1, u(x)) - \\ \sum \alpha\beta_{m_l}(\eta, u(x))\end{matrix} & \begin{matrix}\beta_{m_l+1}(t_k+m_k-1, u(x)) - \\ \sum \alpha\beta_{m_l+1}(\eta, u(x))\end{matrix} & \cdots & \begin{matrix}\beta_{n-1}(t_k+m_k-1, u(x)) - \\ \sum \alpha\beta_{n-1}(\eta, u(x))\end{matrix} \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R}$, $m_l \le i \le n-1$, not all zero such that

$$p_{m_l} \begin{pmatrix} \beta_{m_l}(t_1, u(t)) \\ \beta_{m_l}(t_1+1, u(t)) \\ \vdots \\ \beta_{m_l}(t_{l-1}+m_{l-1}-1, u(t)) \\ \beta_{m_l}(t_{l+1}, u(t)) \\ \vdots \\ \beta_{m_l}(t_k, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \\ \vdots \\ \beta_{m_l}(t_k+m_k-1, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \beta_{n-1}(t_1, u(t)) \\ \beta_{n-1}(t_1+m_1-1, u(t)) \\ \vdots \\ \beta_{n-1}(t_{l-1}+m_{l-1}-1, u(t)) \\ \beta_{n-1}(t_{l+1}, u(t)) \\ \vdots \\ \beta_{n-1}(t_k, u(t)) - \\ \sum \alpha\beta_{n-1}(\eta, u(t)) \\ \vdots \\ \beta_{n-1}(t_k+m_k-1, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$\omega(t,u(t)) := p_{m_l}\beta_{m_l}(t,u(t)) + \cdots + p_{n-1}\beta_{n-1}(t,u(t)).$$

Then, $\omega(t,u(t))$ is a nontrivial solution of (4.12), but

$$\omega(t_j+i,u(t)) = 0,\ 0 \le i \le m_j-1,\ 1 \le j \le k-1,$$

and

$$\omega(t_k+i,u(t)) - \sum_{p=1}^{m} \alpha_{ip}\omega(\eta_{ip},u(t)) = 0,\ 0 \le i \le m_k-1,$$

which when coupled with Property (U) implies $\omega(t,u(t)) \equiv 0$. Thus, $p_i = 0,\ m_l \le i \le n-1$, which is a contradiction to the choice of the $p_i's$. Hence $\det(M) \neq 0$. Thus, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, for each $m_l \le i \le n-1$, we can solve $\epsilon_i(h)/h$, by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|}\times$$

$$\begin{vmatrix} \beta_{m_l}(t_1) & \cdots & \beta_{i-2}(t_1) & -\beta_q(t_1) & \beta_i(t_1) & \cdots & \beta_{n-1}(t_1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_1+m_1-1) & \cdots & \beta_{i-2}(t_1+m_1-1) & -\beta_q(t_1+m_1-1) & \beta_i(t_1+m_1-1) & \cdots & \beta_{n-1}(t_1+m_1-1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_k) & & \beta_{i-2}(t_k) & -\beta_q(t_k) & \beta_i(t_k) & & \beta_{n-1}(t_k) \\ -\sum\alpha\beta_{m_l}(\eta) & \cdots & -\sum\alpha\beta_{i-2}(\eta) & +\sum\alpha\beta_q(\eta) & -\sum\alpha\beta_i(\eta) & \cdots & -\sum\alpha\beta_{n-l}(\eta) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_k+m_k-1) & & \beta_{i-2}(t_k+m_k-1) & -\beta_q(t_k+m_k-1) & \beta_i(t_k+m_k-1) & & \beta_{n-1}(t_k+m_k-1) \\ -\sum\alpha\beta_{m_l}(\eta) & \cdots & -\sum\alpha\beta_{i-2}(\eta) & +\sum\alpha\beta_q(\eta) & -\sum\alpha\beta_i(\eta) & \cdots & -\sum\alpha\beta_{n-l}(\eta) \end{vmatrix}$$

Note as $h \to 0$, $\det(M(h)) \to \det(M)$, and so for $m_l \le i \le n-1$, $\epsilon_i(h)/h \to \det(M_i)/\det(M) := B_i$ as $h \to 0$, where M_i is the $n-m_l \times n-m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\text{col}\Big[\ -\beta_q(t_1,u(t)),\ldots,-\beta_q(t_1+m_1-1,u(t)),\ldots,$$

$-\beta_q(t_{l-1}, u(t)), \ldots, -\beta_q(t_1 + m_{l-1} - 1, u(t)), -\beta_q(t_{l+1}, u(t)), \ldots,$

$-\beta_q(t_1 + m_{l+1} - 1, u(t)), \ldots,$

$-\beta_q(t_k, u(t)) + \sum_{p=1}^{m} \alpha_{0p}\beta_q(\eta_{0p}, u(t)), \ldots,$

$-\beta_q(t_k + m_k - 1, u(t)) + \sum_{p=1}^{m} \alpha_{m_k-1,p}\beta_q(\eta_{m_k-1,p}, u(t))\Big].$

Now let $z_{ql}(t) = \lim_{h\to 0} z_{qlh}(t)$, and note by construction of $z_{qlh}(t)$,

$$z_{ql}(t) = \frac{\partial w}{\partial w_{ql}}(t).$$

Furthermore,

$$z_{ql}(t) = \lim_{h\to 0} z_{qlh}(t) = \sum_{i=m_l}^{n-1} B_i\beta_i(t, u(t)),$$

which is a solution of the variational equation (4.12) along $w(t)$. In addition,

$$\begin{aligned}
z_{ql}(t_j + i) &= \lim_{h\to 0} z_{qlh}(t_j + i) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l,\\
z_{ql}(t_l + i) &= \lim_{h\to 0} z_{qlh}(t_l + i)0,\ 0 \le i \le m_l - 1,\ i \ne q,\\
z_{ql}(t_l + q) &= \lim_{h\to 0} z_{qlh}(t_l + q) = 1,
\end{aligned}$$

and

$$\begin{aligned}
z_{ql}(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} z_{ql}(\eta_{ip}) &= \lim_{h\to} \left[z_{qlh}(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} z_{qlh}(\eta_{ip})\right]\\
&= 0,\ 0 \le i \le m_k - 1.
\end{aligned}$$

This completes the argument for $\frac{\partial w}{\partial w_{ql}}$.

Next, we look at t_k. Let $\epsilon > 0$ be as in Theorem 4.4, and let $0 < |h| < \epsilon$ be given. Fix $0 \le q \le m_k - 1$, and consider the quotient,

$$\begin{aligned}
z_{qkh}(t) = \frac{1}{h}[&w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m},\\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{qk} + h, \ldots, w_{m_k-1,k})\\
&-w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m},
\end{aligned}$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{qk},\ldots,w_{m_k-1,k})].$$

Notice that for $0 \le i \le m_j - 1,\ 1 \le j \le k-1$, and $h \neq 0$,

$$\begin{aligned}
z_{qkh}(t_j+i) &= \frac{1}{h}[w(t_j+i,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{qk}+h,\ldots,w_{m_k-1,k})\\
&\quad -w(t_j+i,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{qk},\ldots,w_{m_k-1,k})]\\
&= [w_{ij}-w_{ij}]\\
&= 0,
\end{aligned}$$

and for $0 \le i \le m_k - 1,\ i \neq q$, and $h \neq 0$,

$$\begin{aligned}
z_{qkh}(t_k+i) &- \sum_{p=1}^{m}\alpha_{ip}z_{qkh}(\eta_{ip})\\
&= \frac{1}{h}\Big[(w(t_k+i,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{qk}+h,\ldots,w_{m_k-1,k})\\
&- \sum_{p=1}^{m}\alpha_{ip}w(\eta_{ip},t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{qk}+h,\ldots,w_{m_k-1,k}))\\
&- w(t_k+i,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{qk},\ldots,w_{m_k-1,k})\\
&+ \sum_{p=1}^{m}\alpha_{ip}w(\eta_{ip},t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{qk},\ldots,w_{m_k-1,k})\Big]\\
&= [w_{ik}-w_{ik}]\\
&= 0.
\end{aligned}$$

Also, we have, for $h \neq 0$,

$$z_{qkh}(t_k+q) - \sum_{p=1}^{m}\alpha_{qp}z_{qkh}(\eta_{qp})$$

$$
\begin{aligned}
&= \frac{1}{h}\Big[(w(t_k+q, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{qk}+h, \ldots, w_{m_k-1,k}) \\
&- \sum_{p=1}^{m} \alpha_{qp} w(\eta_{qp}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{qk}+h, \ldots, w_{m_k-1,k})) \\
&- w(t_k+q, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{qk}, \ldots, w_{m_k-1,k}) \\
&+ \sum_{p=1}^{m} \alpha_{qp} w(\eta_{qp}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{qk}, \ldots, w_{m_k-1,k})\Big] \\
&= \frac{1}{h}[w_{qk}+h-w_{qk}] \\
&= 1.
\end{aligned}
$$

Now view $w(t)$ in terms of the solution of an initial value problem of (4.10). Let $1 \le l \le k-1$. For $m_l \le i \le n-1$, let

$$
\begin{aligned}
\sigma_i &= w(t_l+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{qk}, \ldots, w_{m_k-1,k}),
\end{aligned}
$$

and

$$
\begin{aligned}
\epsilon_i = \epsilon_i(h) &= w(t_l+i, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{qk}+h, \ldots, w_{m_k-1,k}) - \sigma_i.
\end{aligned}
$$

Then, by Theorem 4.4, $\epsilon_i \to 0$ as $h \to 0$ for $m_l \le i \le n-1$. Thus, we have $w(t) = u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1}, \ldots, \sigma_{n-1})$, and

$$
\begin{aligned}
z_{qkh}(t) &= \frac{1}{h}[u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}+\epsilon_{m_l}, \sigma_{m_l+1}+\epsilon_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1}) \\
&\quad -u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}, \sigma_{m_l+1}, \ldots, \sigma_{n-1})].
\end{aligned}
$$

Now we implement a telescoping sum to yield

$$
z_{qkh}(t) = \frac{1}{h}[u(t, t_l, w_{0l}, \ldots, w_{m_l-1,l}, \sigma_{m_l}+\epsilon_{m_l}, \sigma_{m_l+1}+\epsilon_{m_l+1}, \ldots, \sigma_{n-1}+\epsilon_{n-1})
$$

$$-u(t,t_1,w_{0l},\dots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1}+\epsilon_{m_l+1},\dots,\sigma_{n-1}+\epsilon_{n-1})]$$

$$+[u(t,t_1,w_{0l}\dots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1}+\epsilon_{m_l+1},\dots,\sigma_{n-1}+\epsilon_{n-1})$$

$$-u(t,t_1,w_{0l},\dots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\dots,\sigma_{n-1}+\epsilon_{n-1})]$$

$$+-\cdots$$

$$+[u(t,t_1,w_{0l},\dots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\dots,\sigma_{n-1}+\epsilon_{n-1})$$

$$-u(t,t_1,w_{0l},\dots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\dots,\sigma_{n-1})]$$

Thus by using Theorem 4.2 and the Mean Value Theorem,

$$\begin{aligned}
z_{qkh}(t) \;=\; & \frac{\epsilon_{m_l}}{h}\beta_{m_l}(t,u(t,t_l,w_{0l},\dots,w_{m_l-1,l},\sigma_{m_l}+\bar{\epsilon}_{m_l},\\
& \qquad \sigma_{m_l+1}+\epsilon_{m_l+1},\dots,\sigma_{n-1}+\epsilon_{n-1})\\
& +\frac{\epsilon_{m_l+1}}{h}\beta_{m_l+1}(t,u(t,t_l,w_{0l},\dots,w_{m_l-1,l},\sigma_{m_l},\\
& \qquad \sigma_{m_l+1}+\bar{\epsilon}_{m_l+1},\dots,\sigma_{n-1}+\epsilon_{n-1})+\cdots\\
& +\frac{\epsilon_{n-1}}{h}\beta_{n-1}(t,u(t,t_l,w_{0l},\dots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\dots,\sigma_{n-1}+\bar{\epsilon}_{n-1}),
\end{aligned}$$

where $\sigma_i+\bar{\epsilon}_i$ is between σ_i and $\sigma_i+\epsilon_i$ for $m_l\le i\le n-1$, and $\beta_j(t_l+i)=\delta_{ij}$, for $0\le i,j\le n-1$, and solves (4.12).

Hence, for $\lim_{h\to 0} z_{qkh}(t)$ to exist, we need $\lim_{h\to 0}\frac{\epsilon_i}{h}$ to exist for each $m_l\le i\le n-1$. Now, from the construction of $z_{qkh}(t)$,

$$z_{qkh}(t_j+i)=0, \qquad 0\le i\le m_j-1,\ 1\le j\le k-1,\ j\ne l,$$

and

$$z_{qkh}(t_k+i)-\sum_{p=1}^{m}\alpha_{ip}z_{qkh}(\eta_{ip})=\delta_{iq},\ 0\le i\le m_k-1.$$

Hence, we have a system of $n-m_l$ linear equations with $n-m_l$ unknowns:

$$\begin{aligned}
0 \;=\; & \frac{\epsilon_{m_l}}{h}\beta_{m_l}(t_j+i,u(t,t_l,w_{0l},\dots,w_{m_l-1,l},\sigma_{m_l}+\bar{\epsilon}_{m_l},\\
& \qquad \sigma_{m_l+1}+\epsilon_{m_l+1},\dots,\sigma_{n-1}+\epsilon_{n-1})
\end{aligned}$$

$$+\frac{\epsilon_{m_l+1}}{h}\beta_{m_l+1}(t_j+i,u(t,t_l,w_{0l},\ldots,w_{m_l-1,l},\sigma_{m_l},$$
$$\sigma_{m_l+1}+\bar{\epsilon}_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1})+\cdots$$
$$+\frac{\epsilon_{n-1}}{h}\beta_{n-1}(t_j+i,u(t,t_l,w_{0l},\ldots,w_{m_l-1,l},\sigma_{m_l},$$
$$\sigma_{m_l+1},\ldots,\sigma_{n-1}+\bar{\epsilon}_{n-1})),$$
$$0\le i\le m_j-1,\ 1\le j\le k-1,\ j\ne l,$$

and

$$\begin{aligned}\delta_{iq} \quad = \quad & \frac{\epsilon_{m_l}}{h}\Big[\beta_{m_l}(t_k+i,u(t,t_l,w_{0l},\ldots,w_{m_l-1,l},\sigma_{m_l}+\bar{\epsilon}_{m_l},\\ & \qquad \sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1}))\\ & -\sum_{p=1}^{m}\alpha_{ip}\beta_{m_l}(\eta_{ip},u(t,t_l,w_{0l},\ldots,w_{m_l-1,l},\sigma_{m_l}+\bar{\epsilon}_{m_l},\\ & \qquad \sigma_{m_l+1}+\epsilon_{m_l+1},\ldots,\sigma_{n-1}+\epsilon_{n-1}))\Big]+\cdots\\ & +\frac{\epsilon_{n-1}}{h}\Big[\beta_{n-1}(t_k+i,u(t,t_l,w_{0l},\ldots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\ldots,\sigma_{n-1}+\bar{\epsilon}_{n-1}))\\ & -\sum_{p=1}^{m}\alpha_{ip}\beta_{n-1}(\eta_{ip},u(t,t_l,w_{0l},\ldots,w_{m_l-1,l},\sigma_{m_l},\sigma_{m_l+1},\ldots,\sigma_{n-1}+\bar{\epsilon}_{n-1}))\Big],\\ & 0\le i\le m_k-1.\end{aligned}$$

Note that we will occasionally suppress the arguments of β, the subscripts of α and η, and the limits of summation. As the preceeding system of equations were not along identical solutions, we consider the matrix

$$M := \begin{pmatrix} \beta_{m_l}(t_1, u(x)) & \beta_{m_l+1}(t_1, u(x)) & \cdots & \beta_{n-1}(t_1, u(x)) \\ \beta_{m_l}(t_1+1, u(x)) & \beta_{m_l+1}(t_1+1, u(x)) & \cdots & \beta_{n-1}(t_1+1, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_1+m_1-1, u(x)) & \beta_{m_l+1}(t_1+m_1-1, u(x)) & \cdots & \beta_{n-1}(t_1+m_1-1, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_{l-1}+m_{l-1}-1, u(x)) & \beta_{m_l+1}(t_{l-1}+m_{l-1}-1, u(x)) & \cdots & \beta_{n-1}(t_{l-1}+m_{l-1}-1, u(x)) \\ \beta_{m_l}(t_{l+1}, u(x)) & \beta_{m_l+1}(t_{l+1}, u(x)) & \cdots & \beta_{n-1}(t_{l+1}, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_k, u(x)) - & \beta_{m_l+1}(t_k, u(x)) - & & \beta_{n-1}(t_k, u(x)) - \\ \sum \alpha\beta_{m_l}(\eta, u(x)) & \sum \alpha\beta_{m_l+1}(\eta, u(x)) & \cdots & \sum \alpha\beta_{n-1}(\eta, u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m_l}(t_k+m_k-1, u(x)) - & \beta_{m_l+1}(t_k+m_k-1, u(x)) - & & \beta_{n-1}(t_k+m_k-1, u(x)) - \\ \sum \alpha\beta_{m_l}(\eta, u(x)) & \sum \alpha\beta_{m_l+1}(\eta, u(x)) & \cdots & \sum \alpha\beta_{n-1}(\eta, u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R},\ m_l \leq i \leq n-1$, not all zero such that

$$p_{m_l} \begin{pmatrix} \beta_{m_l}(t_1, u(t)) \\ \vdots \\ \beta_{m_l}(t_{l-1}+m_{l-1}-1, u(t)) \\ \beta_{m_l}(t_{l+1}, u(t)) \\ \vdots \\ \beta_{m_l}(t_k, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \\ \vdots \\ \beta_{m_l}(t_k+m_k-1, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \beta_{n-1}(t_1, u(t)) \\ \beta_{n-1}(t_1+m_1-1, u(t)) \\ \vdots \\ \beta_{n-1}(t_{l-1}+m_{l-1}-1, u(t)) \\ \beta_{n-1}(t_{l+1}, u(t)) \\ \vdots \\ \beta_{n-1}(t_k, u(t)) - \\ \sum \alpha\beta_{n-1}(\eta, u(t)) \\ \vdots \\ \beta_{n-1}(t_k+m_k-1, u(t)) - \\ \sum \alpha\beta_{m_l}(\eta, u(t)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$\omega(t, u(t)) := p_{m_l}\beta_{m_l}(t, u(t)) + \cdots + p_{n-1}\beta_{n-1}(t, u(t)).$$

Then, $\omega(t, u(t))$ is a nontrivial solution of (4.12), but

$$\omega(t_j + i, u(t)) = 0,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

and

$$\omega(t_k+i,u(t))-\sum_{p=1}^{m}\alpha_{ip}\omega(\eta_{ip},u(t))=0,\ 0\le i\le m_k-1,$$

which when coupled with Property (U) implies $\omega(t,u(t))\equiv 0$ forcing each $p_i = 0,\ m_l \le i \le n-1$, a contradiction. Hence $\det(M)\neq 0$. Thus, as a result of continuous dependence, for $h\neq 0$ and sufficiently small, $\det(M(h))\neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, for each $m_l\le i\le n-1$, we can solve $\epsilon_i(h)/h$, by using Cramer's rule:

$$\frac{\epsilon_i(h)}{h}=\frac{1}{|M(h)|}\times$$

$$\begin{vmatrix}
\beta_{m_l}(t_1) & \cdots & \beta_{i-2}(t_1) & 0 & \beta_i(t_1) & \cdots & \beta_{n-1}(t_1)\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\beta_{m_l}(t_k)- & & \beta_{i-2}(t_k)- & & \beta_i(t_k)- & & \beta_{n-1}(t_k)-\\
\sum\alpha\beta_{m_l}(\eta) & \cdots & \sum\alpha\beta_{i-2}(\eta) & 0 & \sum\alpha\beta_i(\eta) & \cdots & \sum\alpha\beta_{n-l}(\eta)\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\beta_{m_l}(t_k+q)- & & \beta_{i-2}(t_k+q)- & & \beta_i(t_k+q)- & & \beta_{n-1}(t_k+q)-\\
\sum\alpha\beta_{m_l}(\eta) & \cdots & \sum\alpha\beta_{i-2}(\eta) & 1 & \sum\alpha\beta_i(\eta) & \cdots & \sum\alpha\beta_{n-l}(\eta)\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\beta_{m_l}(t_k+m_k-1)- & & \beta_{i-2}(t_k+m_k-1)- & & \beta_i(t_k+m_k-1)- & & \beta_{n-1}(t_k+m_k-1)-\\
\sum\alpha\beta_{m_l}(\eta) & \cdots & \sum\alpha\beta_{i-2}(\eta) & 0 & \sum\alpha\beta_i(\eta) & \cdots & \sum\alpha\beta_{n-l}(\eta)
\end{vmatrix}$$

Note as $h\to 0,\ \det(M(h))\to\det(M)$, and so for $m_l\le i\le n-1,\ \epsilon_i(h)/h\to \det(M_i)/\det(M):=B_i$ as $h\to 0$, where M_i is the $n-m_l\times n-m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\mathrm{col}[0,\ldots,0,1,0,\ldots,0].$$

Now let $z_{qk}(t)=\lim_{h\to 0} z_{qkh}(t)$, and note by construction of $z_{qkh}(t)$,

$$z_{qk}(t)=\frac{\partial w}{\partial w_{qk}}(t).$$

Furthermore,

$$z_{qk}(t) = \lim_{h\to 0} z_{qkh}(t) = \sum_{i=m_l}^{n-1} B_i\beta_i(t, u(t)),$$

which is a solution of the variational equation (4.12) along $w(t)$. In addition,

$$z_{qk}(t_j + i) = \lim_{h\to 0} z_{qkh}(t_j + i) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

$$\begin{aligned} z_{qk}(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} z_{qk}(\eta_{ip}) &= \lim_{h\to 0}\left[z_{qkh}(t_k + i) - \sum_{p=1}^{m} \alpha_{ip} z_{qkh}(\eta_{ip})\right] \\ &= \delta_{iq},\ 0 \le i \le m_k - 1. \end{aligned}$$

This completes the argument for $\frac{\partial w}{\partial w_{qk}}$. □

Now we establish an analogue of Theorem 4.3 for boundary value problems.

Theorem 4.8. *Let* $n \ge 2,\ 2 \le k \le n$, *and* $m \in \mathbb{N}$ *be given and let* $m_1, \ldots, m_k$ *be positive integers such that* $\sum_{i=1}^{k} m_i = n$. *Assume conditions* (i), (ii), *and* (iii) *hold and that* (4.10) *satisfies Property (U) on* $\mathbb{Z}$. *Suppose*

$$w(t) = w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})$$

is the solution of (4.10) *on* $\mathbb{Z}$ *where*

$$w(x_j + i) = w_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

$$w(x_k + i) - \sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}) = w_{ik},\ 0 \le i \le m_k - 1,$$

$t_1 < t_1 + m_1 - 1 < t_2 < t_2 + m_2 - 1 < \cdots < t_{k-1} + m_{k-1} - 1 < \eta_{01} < \eta_{01} + 1 < \cdots < \eta_{m_k-1,m} < \eta_{m_k-1,m} + 1 < t_k$ *in* $\mathbb{Z}$, *and* $\alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k} \in \mathbb{R}$. *Then:*

(a) for $1 \le l \le k-1$,

$$\begin{aligned} \nu_l(t) :&= \Delta_{t_l} w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \end{aligned}$$

$$
\begin{aligned}
&= w(t,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})\\
&\quad -w(t,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})
\end{aligned}
$$

is a solution of the linear difference equation

$$\nu_l(t+n)=\sum_{r=1}^{n}A_{lr}(t)\nu_l(t+r-1),$$

where for $1\le r\le n$,

$$
\begin{aligned}
A_{lr}(t) &= \int_0^1 \frac{\partial f}{\partial d_r}(t,w(t,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\\
&\quad w(t+1,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\ldots,\\
&\quad sw(t+r-1,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})\\
&\quad +(1-s)w(t+r-1,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\ldots,\\
&\quad w(t+n-1,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}))ds,
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
\nu_l(t_j+i) &= 0,\ 0\le i\le m_j-1,\ 1\le j\le k-1,\ j\ne l,\\
\nu_l(t_l) &= -\Delta_t w(t,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})|_{t=t_l},\\
\nu_l(t_l+i) &= -\Delta_t w(t,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})|_{t=t_l+i-1},\ 0\le i\le m_l-1,
\end{aligned}
$$

$$\nu_l(t_k+i) \quad - \quad \sum_{p=1}^{m} \alpha_{ip}\nu_l(\eta_{ip}) = 0,\ 0 \le i \le m_k - 1,$$

(b)

$$\begin{aligned}\nu_k(t) : \quad &= \quad \Delta_{t_k} w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\ &= \quad w(t, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\ &\quad -w(t, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ &\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})\end{aligned}$$

is a solution of the linear difference equation

$$\nu_k(t+n) = \sum_{r=1}^{n} A_{kr}(t)\nu_k(t+r-1),$$

where for $1 \le r \le n$*,*

$$\begin{aligned}A_{kr}(t) \quad = \quad & \int_0^1 \frac{\partial f}{\partial d_r}(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ & \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\ & w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ & \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\ & sw(t+r-1, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ & \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\ & +(1-s)w(t+r-1, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ & \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\ & w(t+n-1, t_1, \ldots, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\ & \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))ds,\end{aligned}$$

with boundary conditions

$$\nu_k(t_i+i) \quad = \quad 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

$$
\begin{aligned}
\nu_k(t_k) &- \sum_{p=1}^{m} \alpha_{0p}\nu_k(\eta_{0p}) \\
&= -\Delta_t w(t, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))|_{t=t_k}, \\
\nu_k(t_k+i) &- \sum_{p=1}^{m} \alpha_{ip}\nu_k(\eta_{ip}) \\
&= -\Delta_t w(t, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))|_{t=t_k+i-1}, \\
&\quad 0 \le i \le m_k - 1.
\end{aligned}
$$

Furthermore,

(c) for $0 \le q \le m_k - 1$ *and* $1 \le l \le m$,

$$
\begin{aligned}
\xi_{ql}(t) : &= \Delta_{\eta_{ql}} w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&= w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad - w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})
\end{aligned}
$$

is a solution of the linear difference equation

$$
\xi_{ql}(t+n) = \sum_{r=1}^{n} A_{qlr}(t)\xi_{ql}(t+r-1),
$$

where

$$
\begin{aligned}
A_{qlr}(t) &= \int_0^1 \frac{\partial f}{\partial d_r}(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
&\quad w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots,
\end{aligned}
$$

$$sw(t+r-1,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},$$
$$\alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})$$
$$+(1-s)w(t+r-1,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql},\dots,\eta_{m_k-1,m},$$
$$\alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k}),\dots,$$
$$w(t+n-1,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},$$
$$\alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k}))ds,$$

with boundary conditions

$$\begin{aligned}
\xi_{ql}(t_j+i) &= 0,\ 0\le i\le m_j-1,\ 1\le j\le k-1,\\
\xi_{ql}(t_k+i) &- \sum_{p=1}^{m}\alpha_{ip}\xi_{ql}(\eta_{ip})=0,\ 0\le i\le m_k-1,\ i\ne q,\\
\xi_{ql}(t_k+q) &- \sum_{p=1}^{m}\alpha_{ip}\xi_{ql}(\eta_{ip})\\
&= \alpha_{ql}\Delta_t(t,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})|_{t=\eta_{ql}}.
\end{aligned}$$

Proof. The proofs of part (a) and (b) are very similar. In fact, they are the same as far as verifying that the partial difference does indeed solve the difference equation which we deal with first. Hence, let $1\le l\le k$, and we use a telescoping sum, the Mean Value Theorem, and difference calculus to obtain,

$$\begin{aligned}
\nu_l(t+n) &= w(t+n,t_1,\dots,t_l+1,\dots,t_k,\eta_{01},\dots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&\quad -w(t+n,t_1,\dots,t_l,\dots,t_k,\eta_{01},\dots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&= f(t,w(t,t_1,\dots,t_l+1,\dots,t_k,\eta_{01},\dots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k}),\\
&\quad w(t+1,t_1,\dots,t_l+1,\dots,t_k,\eta_{01},\dots,\eta_{m_k-1,m},
\end{aligned}$$

$$
\begin{aligned}
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
& -f(t, w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
= \; & [f(t, w(t, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
& -f(t, w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))] \\
& +[f(t, w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m},
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
& -f(t, w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))] \\
& + - \cdots \\
& +[f(t, w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
& -f(t, w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))] \\
= \; & \int_0^1 \frac{\partial f}{\partial d_1}(t, sw(t, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})
\end{aligned}
$$

$$
\begin{aligned}
&+(1-s)w(t,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\\
&w(t+1,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\ldots,\\
&w(t+n-1,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}))ds\\
&\times(w(t,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})\\
&-w(t,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}))\\
&+\int_0^1 \frac{\partial f}{\partial d_2}(t,w(t,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\\
&sw(t+1,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})\\
&+(1-s)w(t+1,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\\
&w(t+2,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\ldots,\\
&w(t+n-1,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}))ds\\
&\times(w(t+1,t_1,\ldots,t_l+1,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})\\
&-w(t,t_1,\ldots,t_l,\ldots,t_k,\eta_{01},\ldots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}))
\end{aligned}
$$

$$
\begin{aligned}
&+\cdots \\
&+\int_0^1 \frac{\partial f}{\partial d_n}(t, w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
&w(t+1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
&sw(t+n-1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&+(1-s)w(t+n-1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))ds \\
&\times(w(t+n-1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&-w(t, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
= \;& A_{l1}\nu_l(t) + A_{l2}\nu_l(t+1) + \cdots + A_{ln}\nu_l(t+n-1).
\end{aligned}
$$

All that remains is to verify the boundary conditions. We deal with those for part (a) first,

$$
\begin{aligned}
\nu_l(t_l) \;=\; & w(t_l, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&-w(t_l, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
=\; & w(t_l, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&-w(t_l+1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})
\end{aligned}
$$

$$
\begin{aligned}
&+w(t_l+1, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&-w(t_l, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
= \; & w_{0l} - w_{0l} - \Delta_t w(t, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=t_l} \\
= \; & -\Delta_t w(t, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=t_l},
\end{aligned}
$$

and for $1 \le i \le m_l - 1, 1 \le l \le k-1$,

$$
\begin{aligned}
\nu_l(t_l+i) \;=\; & w(t_l+i, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&-w(t_l+i, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
= \; & w(t_l+i, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&-w(t_l+i-1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&+w(t_l+i-1, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&-w(t_l+i, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
= \; & w_{i-1,l} - w_{i-1,l} - \Delta_t w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=t_l+i-1} \\
= \; & -\Delta_t w(t, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m},
\end{aligned}
$$

$$\alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=t_l+i-1}.$$

Also, for $0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,\ j \neq l$,

$$\begin{aligned}
\nu_l(t_j+i) &= w(t_j+i, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad -w(t_j+i, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&= [w_{ij} - w_{ij}] \\
&= 0,
\end{aligned}$$

and for $0 \leq i \leq m_k - 1$,

$$\begin{aligned}
\nu_l(t_k+i) &- \sum_{p=1}^{m} \alpha_{ip}\nu_l(\eta_{ip}) \\
&= \Big[w(t_k+i, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad -\sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}, t_1, \ldots, t_l+1, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})\Big] \\
&\quad -\Big[w(t_k+i, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad -\sum_{p=1}^{m} \alpha_{ip} w(\eta_{ip}, t_1, \ldots, t_l, \ldots, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})\Big] \\
&= [w_{ik} - w_{ik}] \\
&= 0.
\end{aligned}$$

Lastly, we verify the boundary conditions for t_k. We have,

$$\begin{aligned}
\nu_k(t_k) &- \sum_{p=1}^{m} \alpha_{0p}\nu_k(\eta_{0p}) \\
&= [w(t_k, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad -w(t_k, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\
&\quad -\sum_{p=1}^{m} \alpha_{0p}[w(\eta_{0p}, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad -w(\eta_{0p}, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\
&= [w(t_k, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad -w(t_k, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\
&\quad -\sum_{p=1}^{m} \alpha_{0p}[w(\eta_{0p}, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad -w(\eta_{0p}, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\
&\quad -w(t_k+1, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad +w(t_k+1, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&= w_{0k} - w_{0k} - \Delta_t w(t, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=t_k}
\end{aligned}$$

$$\begin{aligned}
&= -\Delta_t w(t, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=t_k},
\end{aligned}$$

and for $1 \leq i \leq m_k - 1$,

$$\begin{aligned}
\nu_k(t_k+i) \;-\; & \sum_{p=1}^{m} \alpha_{ip}\nu_k(\eta_{ip}) \\
= \;& [w(t_k+i, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
& -w(t_k+i, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\
& -\sum_{p=1}^{m} \alpha_{ip}[w(\eta_{ip}, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
& -w(\eta_{ip}, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\
= \;& [w(t_k+i, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
& -w(t_k+i, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\
& -\sum_{p=1}^{m} \alpha_{ip}[w(\eta_{ip}, t_1, \ldots, t_{k-1}, t_k+1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
& -w(\eta_{ip}, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})] \\
& -w(t_k+i-1, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
& +w(t_k+i-1, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})
\end{aligned}$$

$$
\begin{aligned}
&= w_{i-1,k} - w_{i-1,k} - \Delta_t w(t, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=t_k+i-1} \\
&= -\Delta_t w(t, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=t_k+i-1}.
\end{aligned}
$$

We also have for $0 \leq i \leq m_j - 1, 1 \leq j \leq k-1$,

$$
\begin{aligned}
\nu_k(t_j + i) &= w(t_j + i, t_1, \ldots, t_{k-1}, t_k + 1, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad - w(t_j + i, t_1, \ldots, t_{k-1}, t_k, \eta_{01}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&= [w_{ij} - w_{ij}] \\
&= 0.
\end{aligned}
$$

Hence the proofs of parts (a) and (b) are complete.

Finally, we conclude with part (c). Fix $0 \leq q \leq m_k - 1$ and $1 \leq l \leq m$. Again, we use a telescoping sum and the Mean Value Theorem to yield

$$
\begin{aligned}
\xi_{ql}(t+n) &= w(t+n, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql} + 1, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&\quad - w(t+n, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
&= f(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql} + 1, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
&\quad w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql} + 1, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
&\quad w(t+n-1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql} + 1, \ldots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))
\end{aligned}
$$

$$
\begin{aligned}
&-f(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
&w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
&w(t+n-1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
= \; &[f(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
&w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
&w(t+n-1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
&-f(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
&w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
&w(t+n-1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))] \\
&+[f(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
&w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
&w(t+n-1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
&\quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))
\end{aligned}
$$

$$
\begin{aligned}
& -f(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}))] \\
& + - \cdots \\
& +[f(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
& -f(t, w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \ldots, \\
& w(t+n-1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})) \\
= \; & \int_0^1 \frac{\partial f}{\partial d_1}(t, sw(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
& \qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
& +(1-s)w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}), \\
& w(t+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m},
\end{aligned}
$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\ldots,$$

$$w(t+n-1,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql}+1,\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})ds$$

$$\times(w(t,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql}+1,\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})$$

$$-w(t,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql},\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}))$$

$$+\int_0^1 \frac{\partial f}{\partial d_2}(t,w(t,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql},\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),$$

$$sw(t+1,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql}+1,\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})$$

$$+(1-s)w(t+1,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql},\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),$$

$$w(t+2,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql}+1,\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}),\ldots,$$

$$w(t+n-1,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql}+1,\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})ds$$

$$\times(w(t+1,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql}+1,\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})$$

$$-w(t+1,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql},\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k}))$$

$$+\cdots$$

$$+\int_0^1 \frac{\partial f}{\partial d_n}(t,w(t,t_1,\ldots,t_k,\eta_{01},\ldots,\eta_{ql},\ldots,\eta_{m_k-1,m},$$

$$\alpha_{01},\ldots,\alpha_{m_k-1,m},w_{01},\ldots,w_{m_k-1,k})$$

$$
\begin{aligned}
& w(t+1, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}, \dots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k}), \dots, \\
& sw(t+n-1, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}+1, \dots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k}) \\
& +(1-s)w(t+n-1, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}, \dots, \eta_{m_k-1,m}, \\
& \qquad \alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k}))ds \\
& \times(w(t+n-1, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}+1, \dots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k}) \\
& -w(t+n-1, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}, \dots, \eta_{m_k-1,m}, \\
& \quad \alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k})) \\
= \; & A_{ql1}\xi_{ql}(t) + A_{ql2}\xi_{ql}(t+1) + \cdots + A_{qln}\xi_{ql}(t+n-1).
\end{aligned}
$$

In addition, for $0 \leq i \leq m_j - 1, 1 \leq j \leq k-1$,

$$
\begin{aligned}
\xi_{ql}(t_j+i) &= w(t_j+i, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}+1, \dots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k}) \\
&\quad -w(t_j+i, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}, \dots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k}) \\
&= [w_{ij} - w_{ij}] \\
&= 0,
\end{aligned}
$$

and for $0 \leq i \leq m_k - 1, \; i \neq q$,

$$
\begin{aligned}
\xi_{ql}(t_k+i) &- \sum_{p=1}^{m} \alpha_{ip}\xi_{ql}(\eta_{ip}) \\
&= w(t_k+i, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}+1, \dots, \eta_{m_k-1,m}, \\
&\qquad \alpha_{01}, \dots, \alpha_{m_k-1,m}, w_{01}, \dots, w_{m_k-1,k}) \\
&\quad -w(t_k+i, t_1, \dots, t_k, \eta_{01}, \dots, \eta_{ql}, \dots, \eta_{m_k-1,m},
\end{aligned}
$$

$$\begin{aligned}
&\quad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&-\sum_{p=1}^{m}\alpha_{ip}\Big[w(\eta_{ip},t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&-w(\eta_{ip},t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql},\dots,\eta_{m_k-1,m},\\
&\quad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\Big]\\
=\ & [w_{ik}-w_{ik}]\\
=\ & 0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\xi_{ql}(t_k+q) \ -\ & \sum_{p=1}^{m}\alpha_{ip}\xi_{ql}(\eta_{ip})\\
=\ & w(t_k+q,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},\\
&\quad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&-w(t_k+q,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql},\dots,\eta_{m_k-1,m},\\
&\quad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&-\sum_{p=1}^{m}\alpha_{qp}[w(\eta_{qp},t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&-w(\eta_{qp},t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql},\dots,\eta_{m_k-1,m},\\
&\quad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})]\\
=\ & w(t_k+q,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},\\
&\quad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&-\alpha_{ql}w(\eta_{ql}+1,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})\\
&+\alpha_{ql}w(\eta_{ql}+1,t_1,\dots,t_k,\eta_{01},\dots,\eta_{ql}+1,\dots,\eta_{m_k-1,m},\\
&\qquad \alpha_{01},\dots,\alpha_{m_k-1,m},w_{01},\dots,w_{m_k-1,k})
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{p=1}^{m} \alpha_{qp} w(\eta_{qp}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
& \qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) - w_{ik} \\
= \; & w_{qk} + \alpha_{ql} w(\eta_{ql}+1, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
& \qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) \\
& -\alpha_{ql} w(\eta_{ql}, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}, \ldots, \eta_{m_k-1,m}, \\
& \qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k}) - w_{qk} \\
= \; & \alpha_{ql} \Delta_t w(t, t_1, \ldots, t_k, \eta_{01}, \ldots, \eta_{ql}+1, \ldots, \eta_{m_k-1,m}, \\
& \qquad\qquad \alpha_{01}, \ldots, \alpha_{m_k-1,m}, w_{01}, \ldots, w_{m_k-1,k})|_{t=\eta_{ql}}.
\end{aligned}
$$

□

4.4 An Example

Now we provide an example illustrating the results of Chapter 4.

Example 4.1. Consider the BVP

$$\Delta^2 u(t) = 0, \tag{4.13}$$

$$u(t_1) = u_1, \qquad u(t_2) - \alpha u(\eta) = u_2, \tag{4.14}$$

where $t_1 < t_1 + 1 < \eta < \eta + 1 < t_2$ in $\mathbb{Z}$ and $u_1, u_2, \alpha \in \mathbb{R}$.

First of all, the variational equation along a solution $u(t)$ of (4.13) and the linear difference equations dealt with in Theorem 4.6 are simply $z(t+2) = -z(t) + 2z(t+1)$ which implies $\Delta^2 z(t) = 0$. Hence we have the variational equation and the linear difference equations are (4.13).

Our goal is to show (4.13) satisfies hypotheses (i)-(iii) as well as Property (U). Then we may apply the main result of this chapter.

As $f(t, u(t), u(t+1)) = 0$, it is clear that hypotheses (i) and (ii) are satisfied. Also note, as solutions of (4.13) are linear, the solutions will exist over all of $\mathbb{Z}$. Hence, hypothesis (iii) is satisfied. All that remains is to show (4.13) satisfies Property (U). If we find conditions that force a solution of (4.13) with generalized zero boundary conditions to be the zero solution, then Property (U) will be satisfied for all solutions of (4.13).

Thus, we consider (4.13) with BC's $u(t)$ and $u(t) - \alpha u(\eta)$ with generalized zeros at t_1 and t_2 respectively. Recall the definition of a generalized zero:

Definition 4.5. Let $v : \mathbb{Z} \to \mathbb{R}$. We say v has a generalized zero at $n_0 \in \mathbb{Z}$ provided either $v(n_0) = 0$ or there exists $k \in \mathbb{N}$ such that $(-1)^k v(n_0 - k)v(n_0) > 0$ and if $k > 1,\ v(n_0 - k + 1) = \cdots = v(n_0 - 1) = 0$.

Thus, we have four cases to deal with:

(C1) $u(t_1) = 0$ and $u(t_2) - \alpha u(\eta) = 0$.

(C2) $u(t_1) = 0$ and for some $k_2 \in \mathbb{N},\ (-1)^{k_2}[u(t_2 - k_2) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] > 0$ and if $k_2 > 1,\ u(t_2 - k_2 + 1) - \alpha u(\eta) = \cdots = u(t_2 - 1) - \alpha u(\eta) = 0$.

(C3) for some $k_1 \in \mathbb{N},\ (-1)^{k_1} u(t_1 - k_1)u(t_1) > 0$ and if $k_1 > 1,\ u(t_1 - k_1 + 1) = \cdots = u(t_1 - 1) = 0$ and $u(t_2) - \alpha u(\eta) = 0$.

(C4) for some $k_1 \in \mathbb{N},\ (-1)^{k_1} u(t_1 - k_1)u(t_1) > 0$ and if $k_1 > 1,\ u(t_1 - k_1 + 1) = \cdots = u(t_1 - 1) = 0$ and for some $k_2 \in \mathbb{N},\ (-1)^{k_2}[u(t_2 - k_2) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] > 0$ and if $k_2 > 1\,;\ u(t_2 - k_2 + 1) - \alpha u(\eta) = \cdots = u(t_2 - 1) - \alpha u(\eta) = 0$.

First, we deal with (C1). Assume (4.13) with boundary conditions

$$u(t_1) = 0, \qquad u(t_2) - \alpha u(\eta) = 0. \tag{4.15}$$

A general solution of (4.13) subject to the first condition of (4.15) is $u(t) = A(t - t_1)$ where A is some constant. Thus we need to find a condition to force $A = 0$. Well,

$$0 \;=\; u(t_2) - \alpha u(\eta)$$

$$
\begin{aligned}
&= A(t_2 - t_1) - \alpha A(\eta - t_1) \\
&= A(t_2 - t_1 - \alpha\eta + \alpha t_1).
\end{aligned}
$$

Hence, if $t_2 - t_1 - \alpha\eta + \alpha t_1 \neq 0$, then under (C1), $A = 0$ which implies $u(t) \equiv 0$.

Next, we look at (C2) which has two subcases:

(C2-S1) $u(t_1) = 0$ and $k_2 = 1$.

(C2-S2) $u(t_1) = 0$ and $k_2 > 1$.

As $u(t_1) = 0$, a solution of (4.13) subject to $u(t_1) = 0$ is $u(t) = A(t - t_1)$ where A is a constant.

We tackle (C2-S2) first. If $k_2 > 1$, then $(-1)^{k_2}[u(t_2 - k_2) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] > 0$ and $u(t_2 - k_2 + 1) - \alpha u(\eta) = \cdots = u(t_2 - 1) - \alpha u(\eta) = 0$.

If $k_2 > 2$, we are dealing with, for $1 \leq i < j \leq k_2 - 1$,

$$
\begin{aligned}
0 &= [u(t_2 - i) - \alpha u(\eta)] - [u(t_2 - j) - \alpha u(\eta)] \\
&= A(t_2 - i - t_1) - A(t_2 - j - t_1) \\
&= A(t_2 - i - t_1 - t_2 + j + t_1) \\
&= A(j - i) \\
\Rightarrow &\quad A = 0.
\end{aligned}
$$

If $k_2 = 2$, we have $[u(t_2 - k_2) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] > 0$. Therefore, $u(t_2) - \alpha u(\eta)$ and $u(t_2 - k_2) - \alpha u(\eta)$ have the same sign and $u(t_2 - 1) - \alpha u(\eta) = 0$ which is not possible for a linear function.

Therefore, we have $u(t) \equiv 0$ under (C2-S2) without making any new assumptions.

Now lets look at (C2-S1). If $k_2 = 1$, then $[u(t_2 - 1) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] < 0$. So we have

$$
0 > [u(t_2 - 1) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)]
$$

$$
\begin{aligned}
&= A[t_2 - 1 - t_1 - \alpha(\eta - t_1)] \cdot A[t_2 - t_1 - \alpha(\eta - t_1)] \\
&= A^2[t_2 - t_1 - \alpha(\eta - t_1)]^2 \left[1 - \frac{1}{t_2 - t_1 - \alpha(\eta - t_1)}\right] \\
\Leftrightarrow \quad & \left[1 - \frac{1}{t_2 - t_1 - \alpha(\eta - t_1)}\right] < 0 \\
\Leftrightarrow \quad & 1 < \frac{1}{t_2 - t_1 - \alpha(\eta - t_1)} \\
\Leftrightarrow \quad & 0 < t_2 - t_1 - \alpha(\eta - t_1) < 1.
\end{aligned}
$$

Under (C2-S1), we must assume $t_2 - t_1 - \alpha(\eta - t_1) < 0$ or $1 < t_2 - t_1 - \alpha(\eta - t_1)$ to force $A = 0$ and have $u(t) \equiv 0$.

Now on to (C3) which again has two subcases:

(C3-S1) $k_1 = 1$ and $u(t_2) - \alpha u(\eta) = 0$.

(C3-S2) $k_1 > 1$ and $u(t_2) - \alpha u(\eta) = 0$.

As $u(t_2) - \alpha u(\eta) = 0$, a solution of (4.13) subject to $u(t_2) - \alpha u(\eta) = 0$ is $u(t) = A\left[t + \frac{\alpha\eta - t_2}{1 - \alpha}\right]$ where A is a constant.

We begin with (C3-S2). If $k_1 > 1$, then $(-1)^{k_1} u(t_1 - k_1)u(t_1) > 0$ and $u(t_1 - k_1 + 1) = \cdots = u(t_1 - 1) = 0$.

Hence, if $k_1 > 2$, we are dealing with, for $1 \leq i < j \leq k_1 - 1$,

$$
\begin{aligned}
0 &= u(t_1 - i) - u(t_1 - j) \\
&= A\left[t_1 - i + \frac{\alpha\eta - t_2}{1 - \alpha}\right] - A\left[t_1 - j + \frac{\alpha\eta - t_2}{1 - \alpha}\right] \\
&= A\left[t_1 - i + \frac{\alpha\eta - t_2}{1 - \alpha} - t_1 + j - \frac{\alpha\eta - t_2}{1 - \alpha}\right] \\
&= A[j - i] \\
\Rightarrow \quad & A = 0.
\end{aligned}
$$

If $k_1 = 2$, we have $u(t_1 - k_1)u(t_1) > 0$. Therefore, $u(t_1)$ and $u(t_1 - k_1)$ have the same sign and $u(t_1 - 1) = 0$ which is not possible for a linear function.

Therefore, we have $u(t) \equiv 0$ under (C3-S2) without making any new assumptions.

Now lets look at (C3-S1). If $k_1 = 1$, then $u(t_1 - 1)u(t_1) < 0$. So we have

$$\begin{aligned}
0 &> u(t_1-1)u(t_1) \\
&= A\left[t_1 - 1 + \frac{\alpha\eta - t_2}{1-\alpha}\right] \cdot A\left[t_1 + \frac{\alpha\eta - t_2}{1-\alpha}\right] \\
&= A^2\left[t_1 + \frac{\alpha\eta - t_2}{1-\alpha}\right]^2 \left[1 - \frac{\alpha - 1}{t_2 - t_1 - \alpha(\eta - t_1)}\right] \\
\Leftrightarrow \quad & \left[1 - \frac{\alpha - 1}{t_2 - t_1 - \alpha(\eta - t_1)}\right] < 0 \\
\Leftrightarrow \quad & 1 < \frac{\alpha - 1}{t_2 - t_1 - \alpha(\eta - t_1)}.
\end{aligned}$$

So under (C3-S1), if we assume $\dfrac{\alpha - 1}{t_2 - t_1 - \alpha(\eta - t_1)} < 1$, we have $u(t) \equiv 0$.

Now we move to (C4). Assume (4.13) with boundary conditions

$$\begin{aligned}
&for\ some\ k_1 \in \mathbb{N},\ (-1)^{k_1}u(t_1 - k_1)u(t_1) > 0, \\
&and\ if\ k_1 > 1,\ u(t_1 - k_1 + 1) = \cdots = u(t_1 - 1) = 0, \\
&and\ for\ some\ k_2 \in \mathbb{N},\ (-1)^{k_2}[u(t_2 - k_2) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] > 0, \\
&and\ if\ k_2 > 1\ u(t_2 - k_2 + 1) - \alpha u(\eta) = \cdots = u(t_2 - 1) - \alpha u(\eta) = 0.
\end{aligned} \tag{4.16}$$

We find (C4) has four subcases:

(C4-S1) $k_1 > 1$ and $k_2 > 1$.

(C4-S2) $k_1 = 1$ and $k_2 > 1$.

(C4-S3) $k_1 > 1$ and $k_2 = 1$.

(C4-S4) $k_1 = 1$ and $k_2 = 1$.

For (C4-S1), we pick $k_1, k_2 > 1$ in $\mathbb{N}$ and have

$$\begin{aligned}
&u(t_1 - k_1 + 1) = \cdots = u(t_1 - 1) = 0, \\
&(-1)^{k_1}u(t_1 - k_1)u(t_1) > 0, \\
&u(t_2 - k_2 + 1) - \alpha u(\eta) = \cdots = u(t_2 - 1) - \alpha u(\eta) = 0, \\
&(-1)^{k_2}[u(t_2 - k_2) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] > 0.
\end{aligned}$$

Hence, if $k_1 > 2$, for $1 \leq i < j \leq k_1 - 1$, and a solution $u(t) = At + b$ of (4.13) with A and b constant,

$$\begin{aligned} 0 &= u(t_1 - i) - u(t_1 - j) \\ &= A(t_1 - i) + b - [A(t_1 - j) + b] \\ &= A[t_1 - i - t_1 + j] \\ &= A[j - i] \\ \Rightarrow \quad & A = 0 \\ \Rightarrow \quad & b = 0. \end{aligned}$$

If $k_1 = 2$, we have $u(t_1 - k_1)u(t_1) > 0$. Therefore, $u(t_1)$ and $u(t_1 - k_1)$ have the same sign and $u(t_1 - 1) = 0$ which is not possible for a linear function.

Therefore under (C4-S1), we need not make any new assumptions to require $u(t) \equiv 0$.

On to (C4-S2), let $k_2 > 1$ in $\mathbb{N}$, and we have

$$\begin{aligned} & u(t_1 - 1)u(t_1) < 0, \\ & (-1)^{k_2}[u(t_2 - k_2) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] > 0, \\ & u(t_2 - k_2 + 1) - \alpha u(\eta) = \cdots = u(t_2 - 1) - \alpha u(\eta) = 0. \end{aligned}$$

Well, $u(t) = At + b$ is a general solution of (4.13) with A and b constant. So, if $k_2 > 2$, for $1 \leq i < j \leq k_2 - 1$,

$$\begin{aligned} 0 &= [u(t_2 - j) - \alpha u(\eta)] - [u(t_2 - i) - \alpha u(\eta)] \\ &= A(t_2 - j) + b - [A(t_2 - i) + b] \\ &= A[t_2 - j - t_2 + i] \\ &= A[i - j] = 0 \\ \Rightarrow \quad & A = 0 \\ \Rightarrow \quad & b = 0. \end{aligned}$$

If $k_2 = 2$, we have $[u(t_2 - k_2) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] > 0$. Therefore, $u(t_2) - \alpha u(\eta)$ and $u(t_2 - k_2) - \alpha u(\eta)$ have the same sign and $u(t_2 - 1) - \alpha u(\eta) = 0$ which is not possible for a linear function.

Therefore, no assumptions are needed to have $u(t) \equiv 0$ under (C4-S2).

Next, for (C4-S3), let $k_1 > 1$ in $\mathbb{N}$, and we have

$$u(t_1 - k_1 + 1) = \cdots = u(t_1 - 1) = 0,$$

$$(-1)^{k_1} u(t_1 - k_1) u(t_1) > 0,$$

$$[u(t_2 - 1) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] < 0,$$

Well, $u(t) = At + b$ is the general solution of (4.13) with A and b constant. So, if $k_1 > 2$, for $1 \leq i < j \leq k_1 - 1$,

$$\begin{aligned} 0 &= u(t_1 - i) - u(t_1 - j) \\ &= A(t_1 - i) + b - [A(t_1 - j) + b] \\ &= A[t_1 - i - t_1 + j] \\ &= A[j - i] \\ \Rightarrow \quad & A = 0 \\ \Rightarrow \quad & b = 0. \end{aligned}$$

If $k_1 = 2$, we have $u(t_1 - k_1)u(t_1) > 0$. Therefore, $u(t_1)$ and $u(t_1 - k_1)$ have the same sign and $u(t_1 - 1) = 0$ which is not possible for a linear function.

Therefore, no assumptions are needed to have $u(t) \equiv 0$ under (C4-S3).

Thus all that remains is to find a conditions that satisfy (C4-S4). We have

$$u(t_1 - 1)u(t_1) < 0,$$

$$[u(t_2 - 1) - \alpha u(\eta)][u(t_2) - \alpha u(\eta)] < 0.$$

So we have for a general solution $u(t) = At + b$ of (4.13) where both A and b are constant,

$$0 > u(t_1 - 1)u(t_1)$$

$$
\begin{aligned}
&= [A(t_1-1)+b][At_1+b] \\
&= A^2[t_1+b/A]^2\left[1-\frac{1}{t_1+b/A}\right] \\
\Leftrightarrow\quad & \left[1-\frac{1}{t_1+b/A}\right]<0 \\
\Leftrightarrow\quad & 1<\frac{1}{t_1+b/A} \\
\Leftrightarrow\quad & 0<t_1+b/A<1.
\end{aligned}
$$

and

$$
\begin{aligned}
0 &> [u(t_2-1)-\alpha u(\eta)][u(t_2)-\alpha u(\eta)] \\
&= [A(t_2-1)+b-\alpha(A\eta+b)][At_2+b-\alpha(A\eta+b)] \\
&= A^2[t_2+b/A-\alpha(\eta+b/A)]^2\left[1-\frac{1}{t_2+b/A-\alpha(\eta+b/A)}\right] \\
\Leftrightarrow\quad & \left[1-\frac{1}{t_2+b/A-\alpha(\eta+b/A)}\right]<0 \\
\Leftrightarrow\quad & \frac{1}{t_2+b/A-\alpha(\eta+b/A)}<1 \\
\Leftrightarrow\quad & 0<t_2+b/A-\alpha(\eta+b/A)<1.
\end{aligned}
$$

Note $A \neq 0$ otherwise the assumptions of (C4-S4) are not satisfied. We combine the two previous results to find

$$-1 < t_2 - t_1 - \alpha(\eta + t_1) < 1 - \alpha.$$

Therefore, if we assume $1-\alpha < t_2 - t_1 - \alpha(\eta+t_1)$ or $t_2 - t_1 - \alpha(\eta+t_1) < -1$, then under (C4-S4), $u(t) \equiv 0$.

Thus, (C1)-(C4) yield 4 conditions we must place upon (4.13), (4.14) in order to satisfy Property (U):

1. $t_2 - t_1 - \alpha(\eta - t_1) \neq 0$,
2. $t_2 - t_1 - \alpha(\eta - t_1) < 0$ or $1 < t_2 - t_1 - \alpha(\eta - t_1)$,

   and

3. $\dfrac{\alpha - 1}{t_2 - t_1 - \alpha(\eta - t_1)} < 1,$

and

4. $t_2 - t_1 - \alpha(\eta + t_1) < -1$ or $1 - \alpha < t_2 - t_1 - \alpha(\eta + t_1)$.

After some finagling, we can link these three conditions and make more useful assumptions:

(A1) $\dfrac{t_2 - t_1}{\eta - t_1} < \alpha,$

or

(A2) If $\eta \geq 0$ and $\eta + t_1 > 0$,

$$\alpha < \frac{t_2 - t_1 - 1}{\eta + t_1 - 1}$$

or

(A3) If $t_1 \leq 0$ and $\eta + t_1 < 0$,

$$\alpha < \frac{t_2 - t_1 + 1}{\eta + t_1}$$

or

$$\frac{t_2 - t_1 - 1}{\eta + t_1 - 1} < \alpha < \frac{t_2 - t_1 + 1}{\eta - t_1 + 1}$$

and $\dfrac{t_2 - t_1 - 1}{\eta + t_1 - 1} < \alpha < \dfrac{t_2 - t_1 - 1}{\eta - t_1}$,

or

(A4) If $\eta = -t_1$,

$$t_1 - t_2 + 1 < \alpha < \frac{t_2 - t_1 + 1}{\eta - t_1 + 1}$$

and $t_1 - t_2 + 1 < \alpha < \dfrac{t_2 - t_1 - 1}{\eta - t_1}$.

Therefore, if we assume one of (A1)-(A4), the solution of (4.13), (4.14) is unique

$$u(t) = \left[\frac{u_1 - \alpha u_1 - u_2}{t_1 - t_2 - \alpha t_1 + \alpha\eta}\right] t - \frac{u_1 t_1 + \alpha u_1 t_1 + u_2 t_1}{t_1 - t_2 - \alpha t_1 + \alpha\eta} + u_1. \tag{4.17}$$

We now show that the assertion of Theorem 4.5 is satisfied.

(a) First we consider the partial with respect to u_1,

$$y_1(t) := \frac{\partial u}{\partial u_1}(t) = \frac{t - \alpha t - t_1 + \alpha t_1}{t_1 - t_2 - \alpha t_1 + \alpha\eta} + 1.$$

Clearly, $y_1(t)$ solves the variational equation as it is a linear function,

$$\begin{aligned} y_1(t_1) &= \frac{t_1 - \alpha t_1 - t_1 + \alpha t_1}{t_1 - t_2 - \alpha t_1 + \alpha\eta} + 1 \\ &= 0 + 1 \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} y_1(t_2) - \alpha y_1(\eta) &= \left[\frac{t_2 - \alpha t_2 - t_1 + \alpha t_1}{t_1 - t_2 - \alpha t_1 + \alpha\eta} + 1\right] \\ &\quad -\alpha\left[\frac{\eta - \alpha\eta - t_1 + \alpha t_1}{t_1 - t_2 - \alpha t_1 + \alpha\eta} + 1\right] \\ &= \frac{t_2 - t_1 + \alpha t_1 - \alpha\eta}{t_1 - t_2 - \alpha t_1 + \alpha\eta} + \frac{-\alpha t_2 - \alpha\eta + \alpha t_1 - \alpha t_1}{t_1 - t_2 - \alpha t_1 + \alpha\eta} + 1 - \alpha \\ &= -1 + \alpha + 1 - \alpha \\ &= 0. \end{aligned}$$

Now on to the partial with respect to u_2,

$$y_2(t) := \frac{\partial u}{\partial u_2}(t) = \frac{t_1 - t}{t_1 - t_2 - \alpha t_1 + \alpha\eta}.$$

Again, as $y_2(t)$ is linear, it solves the variational equation,

$$y_2(t_1) = \frac{t_1 - t_1}{t_1 - t_2 - \alpha t_1 + \alpha\eta} = 0,$$

and

$$\begin{aligned} y_2(t_2) - \alpha u_2(\eta) &= \left[\frac{t_1 - t_2}{t_1 - t_2 - \alpha t_1 + \alpha\eta}\right] - \alpha\left[\frac{t_1 - \eta}{t_1 - t_2 - \alpha t_1 + \alpha\eta}\right] \\ &= \frac{t_1 - t_2 - \alpha t_1 + \alpha\eta}{t_1 - t_2 - \alpha t_1 + \alpha\eta} \\ &= 1. \end{aligned}$$

(b) Next we look at the partial with respect to α,

$$v(t) := \frac{\partial u}{\partial \alpha}(t) = \frac{u_1(t_1 - t)}{t_1 - \alpha t_1 - t_2 + \alpha\eta} + \frac{(u_1 - \alpha u_1 - u_2)(\eta - t_1)(t_1 - t)}{(t_1 - t_2 - \alpha t_1 + \alpha\eta)^2}.$$

Once again, we have $v(t)$ solves the variational equation,

$$\begin{aligned} v(t_1) &= \frac{u_1(t_1 - t_1)}{t_1 - t_2 - \alpha t_1 + \alpha\eta} + \frac{(u_1 - \alpha u_1 - u_2)(\eta - y_1)(t_1 - t_1)}{(t_1 - t_2 - \alpha t_1 + \alpha\eta)^2} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} v(t_2) - \alpha v(\eta) &= \Big[\frac{u_1(t_1 - t_2)}{t_1 - t_2 - \alpha t_1 + \alpha\eta} \\ &\quad + \frac{(u_1 - \alpha u_1 - u_2)(\eta - t_1)(t_1 - t_2)}{(t_1 - t_2 - \alpha t_1 + \alpha\eta)^2}\Big] \\ &\quad -\alpha\Big[\frac{u_1(t_1 - \eta)}{t_1 - t_2 - \alpha t_1 + \alpha\eta} \\ &\quad + \frac{(u_1 - \alpha u_1 - u_2)(\eta - t_1)(t_1 - \eta)}{(t_1 - \alpha t_1 - t_2 + \alpha\eta)^2}\Big] \\ &= \frac{u_1(t_1 - \alpha t_1 - t_2 + \alpha\eta)}{t_1 - t_2 - \alpha t_1 + \alpha\eta} \\ &\quad + \frac{(u_1 - \alpha u_1 - u_2)(\eta - t_1)(t_1 - t_2 - \alpha t_1 + \alpha\eta)}{(t_1 - t_2 - \alpha t_1 + \alpha\eta)^2} \\ &= u_1 + \frac{(u_1 - \alpha u_1 - u_2)(\eta - t_1)}{(t_1 - t_2 - \alpha t_1 + \alpha\eta)} \\ &= u(\eta). \end{aligned}$$

Assume one of (A1)-(A4). Now, we show the results of Theorem 4.8.

(a) First, we investigate $\Delta_{t_1} u(t)$,

$$\nu_1(t) := \Delta_{t_1} u(t) = u(t, t_1 + 1, t_2, \eta, \alpha, u_1, u_2) - u(t, t_1, t_2, \eta, \alpha, u_1, u_2).$$

Our partial difference solves the special linear difference equation,

$$\begin{aligned} \nu_1(t_1) &= u(t_1, t_1 + 1, t_2, \eta, \alpha, u_1, u_2) - u(t_1, t_1, t_2, \eta, \alpha, u_1, u_2) \\ &= u(t_1, t_1 + 1, t_2, \eta, \alpha, u_1, u_2) - u(t_1 + 1, t_1 + 1, t_2, \eta, \alpha, u_1, u_2) \end{aligned}$$

$$
\begin{aligned}
&-u(t_1+1,t_1+1,t_2,\eta,\alpha,u_1,u_2)-u_1 \\
= \quad &-\Delta_t(t,t_1+1,t_2,\eta,\alpha,u_1,u_2)|_{t=t_1},
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_1(t_2)-\alpha\nu_1(\eta) &= u(t_2,t_1+1,t_2,\eta,\alpha,u_1,u_2)-u(t_2,t_1,t_2,\eta,\alpha,u_1,u_2) \\
&\quad -\alpha[u(\eta,t_1+1,t_2,\eta,\alpha,u_1,u_2)-u(\eta,t_1,t_2,\eta,\alpha,u_1,u_2)] \\
&= [u_2-u_2] \\
&= 0.
\end{aligned}
$$

Now we move to $\Delta_{t_2}u(t)$,

$$
\nu_2(t) := \Delta_{t_2}u(t) = u(t,t_1,t_2+1,\eta,\alpha,u_1,u_2)-u(t,t_1,t_2,\eta,\alpha,u_1,u_2).
$$

Our partial difference solves the special linear difference equation,

$$
\begin{aligned}
\nu_2(t_1) &= u(t_1,t_1,t_2+1,\eta,\alpha,u_1,u_2)-u(t_1,t_1,t_2,\eta,\alpha,u_1,u_2) \\
&= [u_1-u_1] \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_2(t_2)-\alpha\nu_2(\eta) &= u(t_2,t_1,t_2+1,\eta,\alpha,u_1,u_2)-u(t_2,t_1,t_2,\eta,\alpha,u_1,u_2) \\
&\quad -\alpha[u(\eta,t_1,t_2+1,\eta,\alpha,u_1,u_2)-u(\eta,t_1,t_2,\eta,\alpha,u_1,u_2)] \\
&= u(t_2,t_1,t_2+1,\eta,\alpha,u_1,u_2) \\
&\quad -\alpha u(\eta,t_1,t_2+1,\eta,\alpha,u_1,u_2) \\
&\quad -u(t_2+1,t_1,t_2+1,\eta,\alpha,u_1,u_2) \\
&\quad +u(t_2+1,t_1,t_2+1,\eta,\alpha,u_1,u_2)-u_2 \\
&= -\Delta_t u(t,t_1,t_2+1,\eta,\alpha,u_1,u_2)+u_2-u_2 \\
&= -\Delta_t u(t,t_1,t_2+1,\eta,\alpha,u_1,u_2)|_{t=t_2}.
\end{aligned}
$$

(b) Finally, we consider $\Delta_\eta u(t)$,

$$\xi(t) := \Delta_\eta u(t) = u(t, t_1, t_2, \eta + 1, \alpha, u_1, u_2) - u(t, t_1, t_2, \eta, \alpha, u_1, u_2).$$

Once again, we have $\xi(t)$ solves the special linear difference equation,

$$\begin{aligned} \xi(t_1) &= u(t_1, t_1, t_2, \eta + 1, \alpha, u_1, u_2) - u(t_1, t_1, t_2, \eta, \alpha, u_1, u_2) \\ &= [u_1 - u_1] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \xi(t_2) - \alpha\xi(\eta) &= u(t_2, t_1, t_2, \eta + 1, \alpha, u_1, u_2) - u(t_2, t_1, t_2, \eta, \alpha, u_1, u_2) \\ &\quad -\alpha[u(\eta, t_1, t_2, \eta + 1, \alpha, u_1, u_2) - u(\eta, t_1, t_2, \eta, \alpha, u_1, u_2)] \\ &= u(t_2, t_1, t_2, \eta + 1, \alpha, u_1, u_2) \\ &\quad -\alpha u(\eta + 1, t_1, t_2, \eta + 1, \alpha, u_1, u_2) \\ &\quad +\alpha u(\eta + 1, t_1, t_2, \eta + 1, \alpha, u_1, u_2) \\ &\quad -\alpha u(\eta, t_1, t_2, \eta + 1, \alpha, u_1, u_2) - u_2 \\ &= u_2 - \Delta_t u(t, t_1, t_2, \eta + 1, \alpha, u_1, u_2)|_{t=\eta} - u_2 \\ &= -\Delta_t u(t, t_1, t_2, \eta + 1, \alpha, u_1, u_2)|_{t=\eta}. \end{aligned}$$

CHAPTER FIVE

Conclusion and Future Work

The preceding work will hopefully prove quite instrumental when it comes to potential real life applications. The type of differential and difference equations we have studied occur frequently within various fields, and knowing, under fairly tolerable conditions, that we are able to take partial derivatives or partial differences of these equations that not only exist and are unique but solve the highly valuable variational equation associated to our original nonlocal boundary value problem could prove priceless.

Now, as we look forward to what future work this dissertation may lead to, there are several questions we might ask.

First, the conditions we place upon (3.1) and (3.3) are disconjugate type conditions, but as we look at Henderson's work, [23], we notice that he also worked with right disfocality. So naturally one might ponder, could the same right disfocality technique be adapted to the nonlocal case? If we inspect the work in Chapter 3 and that of [23], it would seem very plausible that this type of generalization may indeed work.

Another idea could involve a generalization of the domain that we use for these nonlocal boundary value problems. In this work, we found that we were able to adapt techniques for characterizing partial derivatives and partial differences over discrete domains and continuous domains. The logical next question would be can we merge the two into a result over an arbitrary time scale. The idea seems reasonable, but we would have to iron out what the variational equation would look like on an arbitrary time scale. Thankfully research into how we might define the variational equation is already underway by several mathematicians.

One last direction, would be to consider different techniques of proving all the results in this work. In [13], [14], and [15], we find the authors using functional analysis to prove results not too dissimilar from ones we find here. Can we adapt those techniques to prove the results of this dissertation? If so, do those techniques provide more or different insight into the results, and what might the insight tell us?

As is always the case, we are only able to provide a small sample of questions and ideas that may arise from this dissertation, but these thoughts seem to lead to great directions for future research.

APPENDIX

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