ABSTRACT

The Isoperimetric Inequality on Natural Subsets

Harrison Jansma

Director: Brian Simanek Ph.D.

The isoperimetric problem is an exercise of classical geometry posing the following question. If a closed Jordan region on the plane has area A, what is the smallest perimeter that the gure can attain? This question was solved, yet recently an interesting reformulation of the question was posed. By viewing sets of natural numbers as objects, volume was defined as the sum of a sets elements, while perimeter was defined as the sum of all elements in a set with adjacent numbers not contained in a set. This new isoperimetric problem over the naturals then posed the question, If a subset of 0,1,2,... has volume n, what is the smallest possible value of its perimeter. In this thesis we seek to create tight bounds on this perimeter function, as well as construct an explicit set of minimal perimeter for all natural numbers.

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Brian Simanek Ph.D, Department of Mathematics

APPROVED BY THE HONORS PROGRAM

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Dr. Elizabeth Corey, Director.

DATE:________________________

The Isoperimetric Inequality on Natural Subsets

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Harrison Jansma

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Contents

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I would like to thank my parents for supporting me through college, and all of the professors of mathematics at Baylor who showed me that math can be so much more than derivatives and limits.

CHAPTER 1

PRELIMINARIES

Introduction

The isoperimetric problem [3] is an exercise of classical geometry posing the following question:

If a closed Jordan region on the plane has area A, what is the smallest perimeter that the figure can attain?

In Euclidean space, with given area, the figure of least perimeter is the disk. The formula for its perimeter as a function of area is $2\sqrt{A\pi}$, where A is area.

In 2011, Miller et al. [2] created a fresh take on the isoperimetric problem. By viewing sets of integers as geometric objects, Miller et. al established a definition for the volume and perimeter of an integer set. Given any subset A of the nonnegative integers, the volume of A is the sum of its elements, whereas the perimeter of A is the sum of $x \in A$ such that $\{x-1, x+1\} \not\subseteq A$.

The purpose of Miller's paper was to find a relationship between a set's volume and its perimeter, by answering the following:

If a subset of $\{0, 1, \ldots\}$ *has volume n, what is the smallest possible value of its perimeter?* (1.1)

Let $P(n)$ be the smallest attainable perimeter among sets of volume n. The paper [2] established bounds for this function, which were later refined by Patrick Devlin in 2012 [1]. In this thesis we will tighten Devlin's bounds on $P(n)$, and prove the optimality of these bounds. With a tighter and more operable bound we will be able to estimate $P(n)$ with significantly more ease and accuracy.

However, Devlin's paper did more than just create tighter bounds for $P(n)$. The paper [1] established a simple iterative formula for $P(n)$ for all $n \geq 0$. For $n \geq 0$ Devlin established that $P(n) = F(n) + P(G(n))$ for explicit functions F and G where $0 \leq G(n) < n$. In this way $P(n)$ could be determined via an iterative composition of the formula until, at some i^{th} step, $G^i(n) = 0$. After this step you would have a sum of explicit functions which could ideally be used to calculate $P(n)$. Unfortunately Devlin also found 177 numbers, which we will call counterexamples, where $P(n) \neq F(n) + P(G(n))$.

Given this discovery, the process breaks down for other values of $n \geq 0$. If we tried to iterate the function $P(n) = F(n) + P(G(n))$ for some $n \geq 0$ and on some i^{th} step found that $G^i(n)$ was a counterexample, then we could continue no further, given that we would have,

$$P(n) = \sum_{j=0}^{i} F(G^j(n)) + P(G^i(n)) \neq \sum_{j=0}^{i+1} F(G^j(n)) + P(G^{i+1}(n))$$

Luckily, [1] was able to compute values of $P(n)$ for all of the 177 numbers. Allowing $P(n)$ to be calculated for all $n \geq 0$, given access to the 177 values of $P(n)$ found in [1].

A related problem is to find for each n, an explicit set with volume n and perimeter $P(n)$. The construction of these *perimeter minimizing sets*, will be our second main goal. Using Devlin's iterative algorithm and calculations of $P(n)$ for the 177 counterexamples, in Section 5 we will construct an explicit perimeter minimizing set for all $n \geq 0$.

The contents of this thesis are as follows. The second section will consist of definitions, and a more detailed description of the function $P(n)$. The third section summarizes our main conclusions, namely a tighter bound on $P(n)$ as well as an explicit perimeter minimizing set of volume n, for all $n \geq 0$. In the fourth section we construct bounds on $P(n)$, culminating in Theorem 2.0.1, which proves a tighter bound than that of [1] with optimal coefficients. In Section 5 we construct perimeter minimizing sets for the counterexamples, by dividing the 177 numbers into 6 distinct groups. Lastly, Section 6 establishes an explicit perimeter minimizing set of volume n for all $n \geq 0$.

Definitions

Let A be a finite subset of $\{0,\ 1,\ 2,\ \ldots\}$, and $\partial A = \{x \in A : \{x-1, x+1\} \not\subseteq A\}$. Define

$$Vol(A) := \sum_{x \in A} x, \qquad Per(A) := \sum_{z \in \partial A} z.$$

Let A^c denote the compliment of A, with $A^c \subseteq \{0, 1, 2, \ldots\}$. For all $n \geq 0$, P(n) and Q(n) are defined as follows.

$$P(n) = \min_{A \subseteq \{0,1,2,\ldots\}} \{Per(A) | Vol(A) = n\}$$

$$Q(n) = \min_{A \subseteq \{0,1,2\ldots\}} \{Per(A^c) | Vol(A) = n\}$$

For $n \geq 0$, define $f(n)$ and $g(n)$ to be the unique integers that satisfy,

$$n = 1 + 2 + \cdots + f(n) - g(n) \qquad \text{and} \qquad 0 \leq g(n) < f(n)$$

It was shown in [1] that $f(n)$ and $g(n)$ are given by,

$$f(n) = \left\lceil -1 + \sqrt{\frac{(1+8n)}{2}} \right\rceil \qquad\qquad g(n) = \frac{f(n)(f(n)+1)}{2} - n$$

where $\lceil x \rceil$ is the ceiling function.

Discussion and Examples

To further understand the problem, it will be helpful to discuss a situation in which the question (1.1) has a simple answer. This discussion will show the inherent utility of the functions $f(n)$ and $g(n)$.

A triangle number is a member of the set $\{1, 3, 6, \ldots\}$ attained through continued summation of the natural numbers. Since $f(n)$ and $g(n)$ satisfy $[1 + 2 + 3 + \cdots + f(n)] - g(n) = n$ and $0 \leq g(n) < f(n)$, notice that n is a triangle number if and only if $g(n) = 0$. When $n \geq 0$ is a triangle number, $P(n)$ is trivial to calculate given the existence of the set of volume n, $\{0, 1, \ldots, f(n)\}$. This set is a perimeter minimizer among sets of volume n.

Lets try an example where $n \geq 0$ is a triangle number:

Example. Let $n = 10$, then $f(10) = 4$ and $g(10) = 0$. The following are examples of sets with volume 10:

$\{0, 1, 2, 3, 4\}, \{1, 2, 3, 4\}, \{2, 3, 5\}, \{4, 6\}, \{3, 7\}, \{1, 2, 7\}, \{2, 8\}, \{1, 9\}, \{10\}$

Notice that $\{0, 1, 2, 3, 4\}$ has the smallest perimeter, implying $P(10) = 4 = f(10)$.

The more challenging cases emerge when n is not a triangle number. In these cases $g(n) > 0$, making the choice of a perimeter minimizing set non-obvious. Let's try an example:

Example. Let $n = 7$, then $f(7) = 4$ and $g(10) = 3$.
The sets of volume 7 are;
$\{0, 1, 2, 4\}, \{1, 2, 4\}, \{0, 2, 5\}, \{2, 5\}, \{0, 1, 6\}, \{1, 6\}, \{0, 7\}, \{7\}$
After examining these sets, we find that the set {0,1,2,4} has the smallest perimeter, implying $P(7) = 6 > f(7)$.

When $n \geq 0$ is large and $g(n) > 0$, the number of sets of volume n grows very quickly. The Q-Pochammer function explicitly states the number of sets of volume n as the coefficient of an infinite series. [4] In this case, the process of listing sets of volume n becomes impractical. Herein lies the advantage of Devlin's formula, which can be used to find $P(n)$ for all $n \geq 0$ given knowledge of $P(n)$ for the counterexamples. Before we can continue we must state the main results of [1], which we seek to build on in this thesis.

The following Theorems are taken from [1]. Theorem 1.0.1, is Devlin's bound on $P(n)$. It is also the most recent bound put forth in the literature. Theorem 1.0.2 establishes the iterative formula for $P(n)$. Notice that one can find $P(n)$ through successive compositions of the two stated functions.

Theorem 1.0.1. *For all $n > 2$,*

$$\sqrt{2n} - 1/2 < P(n) \leq \sqrt{2n} + (2^{3/4}n^{1/4} + 1)[\log_2(\log_2(n/2)) - 1] + 7$$

Theorem 1.0.2. *For all $n \geq 0$ if n is not one of the 177 known counterexamples tabulated in Table 1 of the appendix [1] (in particular, for all $n > 149,894$), we have*

$$P(n) = f(n) + Q(g(n)) \qquad Q(n) = 1 + f(n) + P(g(n))$$

One last thing needs to be said about the properties of $f(n)$ and $g(n)$. Recall that $f(n)$ and $g(n)$ satisfy $[1+2+3+\cdots+f(n)] - g(n) = n$ and $0 \leq g(n) < f(n)$. Given that for all $n > 0$, we have $f(n) \leq n$, it can be inferred that the two functions satisfy,

$$\lim_{i\to\infty} f^i(n) = 0 \qquad \lim_{i\to\infty} g^i(n) = 0$$

where $g(n)$ and $f(n)$ both reach 0 in a finite number of compositions. This property will be key to our understanding of Theorem 1.0.2. Given that for any $n \geq 0$, $P(n)$ depends on values of $P(g^i(n))$ or for some smaller number $g^i(n)$ with $i \geq 1$. In section 6 we will be able to use this understanding to construct a perimeter minimizing set for all $n \geq 0$.

CHAPTER 2

RESULTS

The first main goal of this thesis is to provide improved bounds for $P(n)$ by eliminating the logarithmic factors from the bounds in Theorem 1.0.1. Further results focus on the construction of an explicit perimeter minimizing set of volume n, for all $n \geq 0$. The key results are as follows,

Theorem 2.0.1. *For all* $n > 4$

$$\sqrt{2n} - 1/2 < P(n) < 7/2 + \sqrt{2n} + 2^{3/4}n^{1/4} + 2^{7/8}n^{1/8} + 2(2^{17/16}n^{1/16})$$

Furthermore, the coefficients $2^{1/2}$ *and* $2^{3/4}$ *are optimal.*

Theorem 2.0.2. *For all* $n \geq 5$

$$f(n) + f(g(n)) \leq P(n) \leq L + \sum_{i=0}^{2L-1} f(g^i(n)) + g^{2L}(n)$$

with $L \in \mathbb{N}$.

The following theorems describe perimeter minimizing sets for all $n \geq 0$. First, some preliminary remarks on notation. If $x, y \in \{0, 1, \ldots\}$ with $0 \leq x \leq y$, then the following notation,

$$[x, y]$$

represents the set $\{x, x+1, \ldots, y-1, y\}$.

Theorem 2.0.3. *For $n \geq 0$ satisfying $P(n) \neq f(n) + Q(g(n))$ or $Q(n) \neq 1 + f(n) + P(g(n))$, n falls into one of the following 6 categories. These categories are pairwise disjoint. Furthermore, for each category, a set σ_n of volume n is provided.*

- *Category 1: if $n \leq 29$ and $g(n) = f(n) - 1$*

$$\sigma_n = [0, f(n) - 2] \cup \{f(n)\}$$

- *Category 2: if $n \leq 29$ and $g(n) = f(n) - 2$*

$$\sigma_n = [0, f(n) - 2] \cup \{f(n) + 1\}$$

- *Category 3: if $n > 29$ and $g(f(n) + g(n) + 1) = 0$*

$$\sigma_n = [f(f(n) + g(n) + 1) + 1, f(n) + 1]$$

- *Category 4: if $n > 29$ and $g(f(n) + g(n) + 2) = 0$*

$$\sigma_n = \{0, 1\} \cup [f(f(n) + g(n) + 2) + 1, f(n) + 1]$$

- *Category 5: if $n > 29$, $g(f(n) + g(n) + 1) > 0$, $g^3(f(n) + g(n) + 1) = 0$, and $f(g^2(f(n) + g(n) + 1) + 1 = 1$*

$$\sigma_n = [f(g^2(f(n)+g(n)+1))+1, f(g(f(n)+g(n)+1))] \cup [f(f(n)+g(n)+1)+1, f(n)+1]$$

- *Category 6: if $n > 29$, $g(f(n) + g(n) + 1) > 0$, $g^3(f(n) + g(n) + 1) \neq 0$, and*

$f(g^2(f(n)+g(n)+1)+1=1$

$$\sigma_n = [0, f(g(f(n)+g(n)+1))] \cup [f(f(n)+g(n)+1)+1, f(n)+1]$$

Theorem 2.0.4. *If $n \geq 0$ is such that $P(n) \neq f(n) + Q(g(n))$, then a perimeter minimizing set is given by σ_n, where σ_n represents the set of volume n given in Theorem 2.0.3.*

Theorem 2.0.5. *If $n \geq 0$ is such that $P(n) = f(n) + Q(g(n))$ and $Q(g(n)) \neq 1 + f(g(n)) + P(g^2(n))$, then a perimeter minimizing set for n is given by,*

$$\{0, 1, 2, \ldots, f(n)\} \setminus \sigma_{g(n)}$$

where $\sigma_{g(n)}$ is the set of volume $g(n)$ given in Theorem 2.0.3.

Theorem 2.0.6. *If $n \geq 0$ is such that $g^2(n) = 0$ and $P(n) = f(n) + Q(g(n))$ then a perimeter minimizing set is given by one of the following,*

if $g(n) = 0$;

$$\{0, 1, 2, \ldots, f(n)\}$$

if $g(n) > 0$;

$$[f(g(n)) + 1, f(n)]$$

Theorem 2.0.7. *For all $n \geq 0$, with $g^2(n) > 0$, $P(n) = f(n) + Q(g(n))$, and $Q(g(n)) = 1 + f(g(n)) + P(g^2(n))$, let*

$$\begin{aligned}\rho = \min\{\ell \in \mathbb{N} | P(g^{2\ell}(n)) \neq f(g^{2\ell+1}(n)) + Q(g^{2\ell+1}(n)),\\ Q(g^{2\ell+1}(n)) \neq 1 + f(g^{2\ell+2}(n)) + P(g^{2\ell+2}(n)), \text{ or}\\ g^{2\ell}(n) = 0\}.\end{aligned}$$

Then a perimeter minimizing set for n is given by,

$$\bigcup_{i=0}^{\rho-1}[f(g^{2i+1}(n))+1, f(g^{2i}(n))] \cup B_{g^{2\rho}(n)},$$

where $B_{g^{2\rho}(n)}$ is the perimeter minimizing set for $g^{2\rho}(n)$ given by one of Theorems 2.0.6, 2.0.5, or 2.0.4. Furthermore this theorem, along with Theorems 2.0.6, 2.0.5, and 2.0.4, construct a perimeter minimizing sets for all $n \geq 0$.

CHAPTER 3

PROOF OF THEOREM 2.0.1

The main purpose of this chapter is to prove Theorem 2.0.1 of the Results Section. To do so, we will first need to establish a basic knowledge of $P(n)$, $f(n)$, and $g(n)$. After this, we will construct bounds on $P(n)$ in terms of $f(n)$ and $g(n)$. Lastly, with these new bounds and our knowledge of $f(n)$, we will be able to prove Theorem 2.0.1.

Intro to P(n)

We first seek to establish a basic knowledge of $P(n)$. The following lemma and corollary are reproduced from Devlin's paper [1], though we present original proofs.

Lemma 3.0.1. *[1] Let A be a finite subset of $\{0, 1, ...\}$, and let m be the maximum element contained in A.*

$$m \leq Per(A) \leq Vol(A) \leq \frac{m(m+1)}{2}$$

Proof. Let m be the maximum element of A, then $m + 1 \notin A$ and $m \in \partial A$, therefore $m \leq Per(A)$. Notice, $\partial A \subseteq A$, implying $Per(A) \leq Vol(A)$. Finally, if $m(m+1)/2 < Vol(A)$, then m is not the maximum element of A. □

Corollary 3.0.2. *[1] If A is a finite subset of $\{0, 1, \ldots\}$, then*

$$\sqrt{2Vol(A)} - 1/2 \leq \frac{-1 + \sqrt{1 + 8Vol(A)}}{2} \leq Per(A)$$

This implies, for any $n \in \{0, 1, \ldots\}$

$$\sqrt{2n} - 1/2 < P(n)$$

Intro to f(n) and g(n)

Recall our discussion of triangle numbers in Section 2, which stated that if $n > 0$ is a triangle number, then n can be attained through continued summation of the natural numbers. We also established that finding $P(n)$ was relatively easy if n is a triangle number, given the existence of the set of volume n, given by $\{0, 1, 2, \ldots, f(n)\}$.

Finding values of $P(n)$ became more difficult when we looked at non-triangle numbers. In this case, there are more potential sets of volume n, none of which are obvious perimeter minimizers. If we are tasked to find $P(n)$ for a non-triangle number, one way to offset the difficulty might be to modify the perimeter minimizing set of a nearby triangle number.

This viewpoint of considering the next greater triangle number is critical to our understanding of $P(n)$. Consider the following lemma reproduced from [1].

Lemma 3.0.3. *[1] For all $n \in \{0, 1, \ldots\}$ there exists an $f(n), g(n) \in \{0, 1, \ldots\}$ such that*

$$n = [0 + 1 + 2 + \cdots + f(n)] - g(n) \qquad \textit{with} \qquad 0 \leq g(n) < f(n). \tag{3.1}$$

with $f(n)$ and $g(n)$ given by,

$$f(n) = \lceil -1 + \sqrt{(1 + 8n)/2} \rceil \qquad\qquad g(n) = \frac{f(n)(f(n) + 1)}{2} - n$$

Proof. The statement is trivially true. □

In context to our previous discussion, for any $n \geq 0$, the sum $\sum_{i=0}^{f(n)} i$ is the smallest triangle number greater than n. In this sense, $g(n)$ becomes a measure of the distance from n to the set of triangle numbers. We will see that $P(n)$ generally reaches its highest values when n is as far from the next triangle number as it can possibly be. (e.g. when $g(n) = f(n) - 1$, or any triangle number plus one.)

For example consider the following sets of volume 6 and 7, where 6 is a triangle number, and 7 is far from a triangle number. Consider the sets of volume 6 with maximal element $f(6) = 3$:

$$\{0, 1, 2, 3\}, \{1, 2, 3\}$$

Then consider the sets of volume 7 with maximal element $f(7) = 4$:

$$\{0, 1, 2, 4\}, \{1, 2, 4\}, \{0, 3, 4\}, \{3, 4\}$$

Notice that the sets of volume 7 require more numbers to be in the perimeter given that $n = 1 + 2 + \cdots + f(n) - g(n)$ and $g(n)$ is very large.

The functions $f(n)$ and $g(n)$ will be used in almost every proof contained in this paper, so it is best to understand them before we move forward. With that goal in mind, the next two proofs are again reproduced from [1]. Proposition 3.0.1 will provide an ease of computation for $P(n)$. Whereas Corollary 3.0.4 will provide a nice bound for $f(n)$, which will be useful later when we improve on the bounds in [1].

Proposition 3.0.1. *[1] Let be $f(n)$ be as defined. For all $n \geq 0$*

$$\begin{aligned} f(n) &= \lceil -1 + \sqrt{(1+8n)/2} \rceil \\ &= \lceil \sqrt{2n} - 1/2 \rceil \\ &= [\sqrt{2n}] \end{aligned}$$

Where $[x]$ is the nearest integer function.

Proof. By way of contradiction, suppose the first two representations are not equal. Then this would imply that there exists integers $p \in \mathbb{Z}$ and $n \in \{0, 1, \ldots\}$ such that

$$\sqrt{2n} - 1/2 \leq p < \frac{-1 + \sqrt{1 + 8Vol(A)}}{2} \tag{3.2}$$

Which implies $8n \leq (2p+1)^2 < 8n+1$. Since n and p are integers, this forces $8n = (2p+1)^2$, which taken modulo 2 yields a contradiction. □

Corollary 3.0.4. *[1] For all $n \geq 0$*

$$\sqrt{2n} - 1/2 < f(n) < \sqrt{2n} + 1/2$$

Bounds for P(n) Using f(n) and g(n)

Before we can establish more practical bounds on $P(n)$ we first require some preliminary lemmas. The inequalities in Lemma 3.0.6 and Lemma 3.0.7 are particularly important. Each will be useful in the construction of bounds for $P(n)$ and minimizing sets for Devlin's 177 counterexamples.

Lemma 3.0.5. *For $n \geq 0$, the following inequality holds,*

$$f(n) \leq P(n).$$

Proof. Given Corollary 3.0.2, for $n \geq 0$, $\sqrt{2n} - 1/2 < P(n)$. Since $P(n)$ is an integer, we know that, $f(n) = \lceil \sqrt{2n} - 1/2 \rceil \leq P(n)$. □

Lemma 3.0.6. *Let $n \geq 0$ and $A \subseteq \{0, 1, \ldots\}$ with $Vol(A) = n$. Then the maximal element, $m \in A$ must be such that,*

$$f(n) \leq m.$$

Proof. Recall that for $n \geq 0$, we have $0 \leq g(n) < f(n)$. Suppose $f(n) > m$, then the following contradiction arises.

$$\begin{aligned} Vol(A) \leq \frac{m(m+1)}{2} &\leq \frac{f(n)(f(n)-1)}{2} \\ &= \frac{f(n)(f(n)+1)}{2} - f(n) < \frac{f(n)(f(n)+1)}{2} - g(n) = n \end{aligned}$$

□

Given any $n \geq 0$, Lemma 3.0.6 places a significant restriction on the composition of sets with volume n. Knowing that the maximal element m of any set of volume n must be such that $f(n) \leq m$ will allow the construction of perimeter minimizing sets in Section 5.

Lemma 3.0.7. *Let $n > 0$, such that $g(n) > 0$. Take $A \subseteq \{0, 1, \ldots\}$ with $Vol(A) = n$. Then there must exist a maximal $\ell \notin A$ with $\ell < m$. Furthermore, this ℓ satisfies the* *inequality,*

$$\ell \geq f(g(n)) - 1.$$

Proof. Take $A \subseteq \{0, 1, \ldots\}$ as the set with $Vol(A) = n$ and maximal element m. Since $g(n) > 0$, there also must exist a maximal $\ell \notin A$ with $\ell < m$. If the opposite were true, then the following contradiction arises,

$$Vol(A) = \frac{m(m+1)}{2} \geq \frac{f(n)(f(n)+1)}{2} > \frac{f(n)(f(n)+1)}{2} - g(n) = n$$

Suppose $\ell < f(g(n)) - 1$, then

$$\begin{aligned} Vol(A) &\geq \frac{m(m+1)}{2} - \frac{\ell(\ell+1)}{2} \\ &> \frac{f(n)(f(n)+1)}{2} - \frac{f(g(n))(f(g(n))-1)}{2} \\ &\geq \frac{f(n)(f(n)+1)}{2} - \frac{f(g(n))(f(g(n))+1)}{2} + g^2(n) \\ &= \frac{f(n)(f(n)+1)}{2} - g(n) \\ &= n \end{aligned}$$

This contradicts our assumption that $Vol(A) = n$, implying $\ell \geq f(g(n)) - 1$.

□

Now that we have Lemma 3.0.6 and Lemma 3.0.7, we can move to our first major result of this section. The following theorem will set a lower bound on $P(n)$, which will be crucial to our improvement of the bounds in Theorem 1.0.1.

Theorem 3.0.8. *For all $n \geq 5$,*

$$f(n) + f(g(n)) \leq P(n)$$

Proof. Given Lemma 3.0.5, the theorem holds when $g(n) = 0$. Suppose $g(n) > 0$. Take $A \subseteq \{0, 1, \ldots\}$ as the set with $Vol(A) = n$, maximal element m, and $Per(A) = P(n)$. Let ℓ be the maximal $\ell \notin A$ with $\ell < m$.

If $\ell < m - 1$ then $m, \ell + 1 \in \partial A$. Since $m \geq f(n)$ and $\ell \geq f(g(n)) - 1$, we have the following.

$$P(n) = Per(A) \geq m + \ell + 1 \geq f(n) + f(g(n))$$

If $\ell = m - 1$ then either $m \geq f(n) + f(g(n))$, in which case the inequality holds, or $m < f(n) + f(g(n))$.

Suppose $m < f(n) + f(g(n))$. Since $g(n) \leq f(n) - 1$ we have

$$m < f(n) + f(g(n)) \leq f(n) + f(f(n) - 1)$$

Our strategy from here will be to find some $N \geq 0$ where the Theorem holds for all $n \geq N$. After which, we can calculate the the finitely many values of $P(n)$ with $0 \leq n < N$.

See that for $n > 6$, the sum, $\sum_{i=1}^{f(n)} i$, in the equality, $n = \sum_{i=1}^{f(n)} i - g(n)$ has at least four terms. Since $g(n) < f(n)$ and $f(n) \leq n$, we have the following for $n > 6$,

$$\begin{aligned} n &= \sum_{i=1}^{f(n)} i - g(n) \\ &\geq \sum_{i=1}^{f(n)-2} i + f(n) \\ &= \sum_{i=2}^{f(n)-3} i + f(n) - 1 + f(n) \\ &\geq f(n) + f(f(n) - 1) \\ &\geq f(n) + f(g(n)) \end{aligned}$$

Thus for $n > 6$, we have $m < f(n) + f(g(n)) \leq n$. By assumption A had volume n, implying there must exists some maximal $h \in A \cap \{0, 1, \ldots, m-2\}$. Since $h \in \partial A$, it suffices to show that $h \geq f(g(n))$. Consider the following,

$$\sum_{x \in A \cap \{0,1,\ldots,m-2\}} x = n - m$$

This implies that $h \geq f(n - m)$. Given our assumption that $m < f(n) + f(g(n))$ and our knowledge that f is an increasing function, we have the following.

$$h \geq f(n - m) \geq f(n - f(n) - f(g(n))$$

Thus we must prove for some $n > 6$, $f(n - f(n) - f(g(n))) \geq f(g(n))$. Since f is an increasing function, this can be shown by proving that $n - f(n) - f(g(n)) \geq g(n)$ for $n > 15$.

For $n \geq 0$, we have $g(n) \leq f(n) - 1$. Consider the following inequality,

$$f(n) + f(g(n)) + g(n) \leq 2f(n) + f(f(n) - 1) - 1$$

For $n > 15$ the sum, $\sum_{i=1}^{f(n)} i$, in $n = \sum_{i=1}^{f(n)} i - g(n)$ has at least 6 terms. Since $g(n) < f(n)$ and $f(n) \leq n$, then for all $n > 15$ the following inequalities hold.

$$\begin{aligned} n &= \sum_{i=1}^{f(n)} i - g(n) \\ &\geq \sum_{i=1}^{f(n)-2} i + f(n) \\ &= \sum_{i=3}^{f(n)-4} i + 2f(n) + f(n) - 2 \\ &\geq 2f(n) + f(f(n) - 1) - 1 \\ &\geq 2f(n) + f(g(n)) - 1 \\ &\geq f(n) + f(g(n)) + g(n) \end{aligned}$$

Thus for $n > 15$ we have $h \geq f(n - f(n) - f(g(n)) \geq f(g(n))$ as desired. All that is left is to check that the inequality holds for $n \leq 15$ with $g(n) > 0$. This can be done by hand.

$$\begin{aligned} P(14) &= 7 & f(14) + f(g(14)) &= 6 \\ P(13) &= 9 & f(13) + f(g(13)) &= 7 \\ P(12) &= 8 & f(12) + f(g(12)) &= 7 \\ P(11) &= 8 & f(11) + f(g(11)) &= 8 \\ P(9) &= 6 & f(9) + f(g(9)) &= 5 \\ P(8) &= 7 & f(8) + f(g(8)) &= 6 \\ P(7) &= 6 & f(7) + f(g(7)) &= 6 \\ P(4) &= 4 & f(4) + f(g(4)) &= 5 \end{aligned}$$

Note $n = 4$ is the first incidence where the theorem does not hold, hence the restriction that $n \geq 5$. □

Now that we have constructed a lower bound for $P(n)$, lets move on to the construction of an upper bound. Notice that $P(n)$ was defined as the minimum perimeter among sets of volume n. By constructing a set of volume $n \geq 0$ with known perimeter we will also set an upper bound on $P(n)$.

This set of volume n will be constructed with the functions $f(n)$ and $g(n)$. However, before we can continue, we must recall that if $x, y \in \{0, 1, \ldots\}$ with $0 \leq x \leq y$, then the following notation,

$$[x, y]$$

represents the set $\{x, x+1, \ldots, y-1, y\}$.

Lemma 3.0.9. *Let $n \geq 0$. Take $L \in \mathbb{N}$ such that $L \leq \min\{\ell \in \mathbb{N} | g^{2\ell}(n) = 0\}$. Then the intervals,*

$$\{[f(g^{2i+1}(n)) + 1, f(g^{2i}(n))]\}_{i=0}^{L-1}$$

are pairwise disjoint, and $g^{2L}(n)$ is in none of them.

Proof. Knowing $0 \leq g(n) < f(n) \leq n$, we can infer that for all $i \geq 0$, if $g^i(n) > 0$ then

$$f(g^{i+1}(n)) \leq g(g^i(n)) < f(g^i(n)) \tag{3.3}$$

This shows that the stated intervals are well defined. Let $L' = \min\{\ell \in \mathbb{N} | g^{2\ell}(n) = 0\}$. For all $i \leq 2(L'-1)$, we have $g^i(n) > 0$. Implying, $f(g^{2i+2}(n)) < f(g^{2i+1}(n))$, given (3.3), showing that the intervals are pairwise disjoint.

Lastly, $g^{2L}(n)$ must be such that $g^{2L}(n) \notin \bigcup_{i=0}^{L-1}[f(g^{2i+1}(n)) + 1, f(g^{2i}(n))]$ given the following,

$$g^{2L}(n) < f(g^{2L-1}(n))$$

We can have equality when $g^{2L-1}(n) = 0$, however in this case the lemma still holds.

□

The following theorem, along with Theorem 3.0.8, will prove the upper bound stated in Theorem 2.0.2 of the Results Section.

Theorem 3.0.10. *Given $n \in \mathbb{N}$, for all $L \in \mathbb{N}$ the following inequality holds,*

$$P(n) \leq L + \sum_{i=0}^{2L-1} f(g^i(n)) + g^{2L}(n)$$

Proof. Given Lemma 3.0.3, for all $n \geq 0$ there exists a $f(n)$ and $g(n)$ such that

$$n = [0 + 1 + 2 + \cdots + f(n)] - g(n) \qquad 0 \leq g(n) < f(n)$$

Since $g(n) \geq 0$

$$n = [0 + 1 + 2 + \cdots + f(n)] - (1 + 2 + \cdots + f(g(n)) - g^2(n))$$

$$= \frac{f(n)(f(n)+1)}{2} - \frac{f(g(n))(f(g(n))+1)}{2} + g^2(n)$$

where $g^2(n) \geq 0$.

Iterating this procedure we can construct a sum with volume n. After L iterations of this method we have,

$$n = \sum_{i=0}^{2L-1} \frac{f(g^i(n))(f(g^i(n))+1)(-1)^i}{2} + g^{2L}(n) \tag{3.4}$$

This equality holds for all $L \in \mathbb{N}$, and will be useful later on. Define $A_L \subseteq \{0, 1, \ldots\}$ by

$$A_L = \{g^{2L}(n)\} \cup \bigcup_{i=0}^{L-1} [f(g^{2i+1}(n)) + 1, f(g^{2i}(n))]$$

where $L \in \mathbb{N}$ satisfies $L \leq \min\{\ell \in \mathbb{N} | g^{2\ell}(n) = 0\}$.

Given Lemma 3.0.9, the volume of A_L is the volume of each interval contained in the union. Knowing this, see that for all $L \in \mathbb{N}$ with $L \leq \min\{\ell \in \mathbb{N} | g^{2\ell}(n) = 0\}$,

$$\begin{aligned} Vol(A_L) &= g^{2L}(n) + \sum_{i=0}^{L-1} \left[\frac{f(g^{2i}(n))(f(g^{2i}(n)) + 1)}{2} - \frac{f(g^{2i+1}(n))(f(g^{2i+1}(n)) + 1)}{2} \right] \\ &= \sum_{i=0}^{2L-1} \frac{f(g^i(n))(f(g^i(n)) + 1)(-1)^i}{2} + g^{2L}(n) \\ &= n \end{aligned}$$

Therefore the perimeter of each A_L creates a bound for $P(n)$.

$$\begin{aligned} P(n) \leq Per(A_L) &\leq g^{2L}(n) + \sum_{i=0}^{L-1} \left[f(g^{2i+1}(n)) + 1 + f(g^{2i}(n)) \right] \\ &= L + \sum_{i=0}^{2L-1} f(g^i(n)) + g^{2L}(n). \end{aligned}$$

Let $L' = \min\{\ell \in \mathbb{N} | g^{2\ell}(n) = 0\}$. Notice that for $L \in \mathbb{N}$ with $L > L'$, the inequality holds given, for all $i \geq 2L'$, we have $g^i(n) = 0$.

$$\begin{aligned} P(n) \leq L' + \sum_{i=0}^{2L'-1} f(g^i(n)) + g^{2L'}(n) &= L' + \sum_{i=0}^{2L-1} f(g^i(n)) + g^{2L}(n) \\ &< L + \sum_{i=0}^{2L-1} f(g^i(n)) + g^{2L}(n) \end{aligned}$$

□

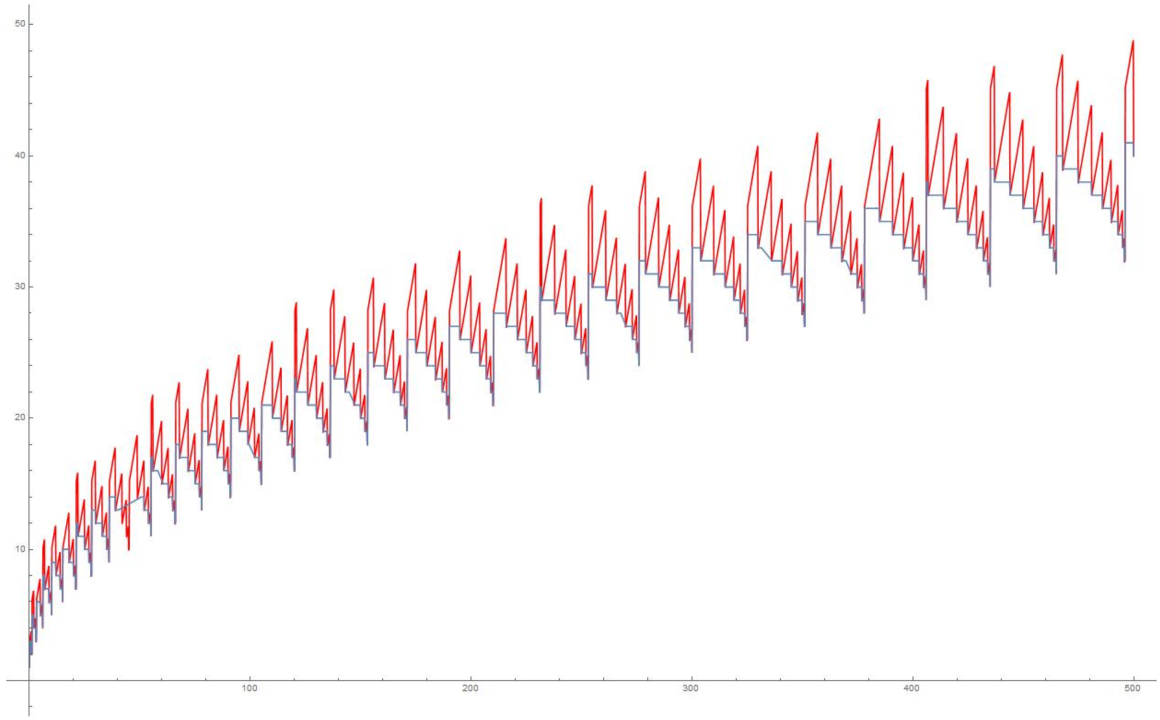

Figure 3.1: The bounds pictured are that of Theorem 2.0.2, where $L = 1$.

Notice that Theorem 2.0.2 of the Results Section, is a combination of Theorems 3.0.8 and 3.0.10. These new bounds on $P(n)$ will allow us to improve on the bounds established in Theorem 1.0.1 [1]. The bounds of Theorem 2.0.2 are pictured above.

It is worth mentioning that these bounds are significantly sharper than those of Theorem 1.0.1, however the difficulty of computing $g(n)$ by hand limits the usefulness of these new bounds. In the following section we construct sharp bounds that can be attained using a elementary calculation.

Improving Analytic Bounds

Given the new bounds established by Theorem 2.0.2, we can now improve on the bounds put forth in [1]. With a tighter and more easily computable bound on $P(n)$, the ease and accuracy of estimating $P(n)$ will be greatly improved. Below is the proof of Theorem 2.0.1, which was stated in the Results section.

Proof. Given Lemma 3.0.10, for all $n \geq 0$, we have $P(n) \leq 2 + \sum_{i=0}^{3} f(g^i(n)) + g^4(n)$. Recall that for $n \geq 0$, we have $g(n) \leq f(n) - 1$ and $f(n) < \sqrt{2n} + 1/2$. With this in mind we have the following inequalities,

$$
\begin{aligned}
P(n) &\leq 2 + f(n) + f(g(n)) + f(g^2(n)) + f(g^3(n)) + g^4(n) \\
&\leq 2 + f(n) + f(f(n) - 1) + f(f(g(n)) - 1)) + f(f(g^2(n)) - 1) + g^4(n) \\
&\leq 1 + f(n) + f(f(n) - 1) + f(f(f(n) - 1) - 1)) + 2f(f(f(f(n) - 1) - 1)) - 1) \\
&< 7/2 + \sqrt{2n} + \sqrt{2(\sqrt{2n} - 1/2)} + \sqrt{2\sqrt{2(\sqrt{2n} - 1/2)} - 1/2} \\
&+ 2\sqrt{2\sqrt{2(\sqrt{2\sqrt{2n} - 1/2} - 1/2)} - 1/2} \\
&< 7/2 + \sqrt{2n} + 2^{3/4}n^{1/4} + 2^{7/8}n^{1/8} + 2(2^{17/16}n^{1/16})
\end{aligned}
$$

To prove that the coefficient $2^{1/2}$ is optimal, suppose that there existed a coefficient $\chi < 2^{1/2}$ such that $P(n) < 7/2 + \chi\sqrt{n} + 2^{3/4}n^{1/4} + 2^{7/8}n^{1/8} + 2(2^{17/16}n^{1/16})$.

Consider the following,

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{4 + 2^{3/4}n^{1/4} + 2^{7/8}n^{1/8} + 2(2^{17/16}n^{1/16})}{n^{1/2}} \\
&= \lim_{n\to\infty} (4)n^{-1/2} + 2^{3/4}n^{-3/4} + 2^{7/8}n^{-5/8} + 2(2^{17/16}n^{-9/16}) \\
&= 0
\end{aligned}
$$

Then since $\sqrt{2n} - 1/2 < P(n)$, the following contradiction arises,

$$\sqrt{2n} - 1/2 < 7/2 + \chi\sqrt{n} + 2^{3/4}n^{1/4} + 2^{7/8}n^{1/8} + 2(2^{17/16}n^{1/16})$$

implies,

$$
\begin{aligned}
&\sqrt{2n} - 2^{3/4}n^{1/4} - 2^{7/8}n^{1/8} - 2(2^{17/16}n^{1/16}) - 4 < \chi\sqrt{n} \\
&1 + \frac{-2^{3/4}n^{1/4} - 2^{7/8}n^{1/8} - 2(2^{17/16}n^{1/16}) - 4}{\sqrt{2n}} < \frac{\chi\sqrt{n}}{\sqrt{2n}} \\
&lim_{n\to\infty}1 + \frac{-2^{3/4}n^{1/4} - 2^{7/8}n^{1/8} - 2(2^{17/16}n^{1/16}) - 4}{\sqrt{2n}} < \frac{\chi}{\sqrt{2}} \\
&1 < \frac{\chi}{\sqrt{2}}
\end{aligned}
$$

This violates our assumption that $\chi < 2^{1/2}$. Thus the coefficient $2^{1/2}$ in our upper bound for $P(n)$ is optimal.

For a proof that the coefficient $2^{3/4}$ is optimal, suppose that there existed a coefficient $\chi < 2^{3/4}$ such that $P(n) < 7/2 + \sqrt{2n} + \chi n^{1/4} + 2^{7/8}n^{1/8} + 2(2^{17/16}n^{1/16})$.

Yet we also know that for $n \geq 5$

$$f(n) + f(g(n)) \leq P(n)$$

Suppose that $n \geq 5$ is such that $g(n) = f(n) - 1$, then

$$\sqrt{2n} + \sqrt{2(\sqrt{2n} - 3/2)} - 1 < \sqrt{2n} + \sqrt{2(f(n) - 1)} - 1 < P(n)$$

Consider the following,

$$
\begin{aligned}
\sqrt{2(\sqrt{2n} - 3/2)} &= \sqrt{\sqrt{8n} - 3} \\
&= \sqrt{\sqrt{8n}(1 - \frac{3}{\sqrt{8n}})}
\end{aligned}
$$

By Taylor expansion, we have

$$\sqrt{1-\frac{3}{\sqrt{8n}}} = \sum_{i=0}^{\infty}\binom{1/2}{i}(\frac{3}{\sqrt{8n}})^i(-1)^i$$
$$= 1-\frac{3}{\sqrt{8n}}\sum_{i=1}^{\infty}\binom{1}{i}(\frac{3}{\sqrt{8n}})^{i-1}(-1)^{i-1}.$$

If we let $\frac{3}{\sqrt{8n}}\sum_{i=1}^{\infty}\binom{1}{i}(\frac{3}{\sqrt{8n}})^{i-1}(-1)^{i-1} = \epsilon$, then we have the following,

$$\sqrt{2n}+\sqrt{2(\sqrt{2n}-3/2)}-1 = \sqrt{2n}+\sqrt{\sqrt{8n}(1-\frac{3}{\sqrt{8n}})}-1$$
$$= \sqrt{2n}+2^{3/4}n^{1/4}-\epsilon$$

Notice the following,

$$\lim_{n\to\infty}\frac{7/2+2^{7/8}n^{1/8}+2(2^{17/16}n^{1/16})}{n^{1/4}} = \lim_{n\to\infty}(7/2)n^{-1/4}+2^{7/8}n^{-7/8}+2(2^{17/16}n^{-13/16}) = 0$$

$$\lim_{n\to\infty}\epsilon/n^{1/4} = 0$$

Since $\chi < 2^{3/4}$ and $P(n) < 7/2+\sqrt{2n}+\chi n^{1/4}+2^{7/8}n^{1/8}+2(2^{17/16}n^{1/16})$, we have the following contradiction for $n \geq 5$.

$$\sqrt{2n}+2^{3/4}n^{1/4}-\epsilon < 7/2+\sqrt{2n}+\chi n^{1/4}+2^{7/8}n^{1/8}+2(2^{17/16}n^{1/16})$$

which implies,

$$2^{3/4}n^{1/4}-\epsilon-7/2-2^{7/8}n^{1/8}-2(2^{17/16}n^{1/16}) < \chi n^{1/4}$$
$$1-\frac{7/2+2^{7/8}n^{1/8}+2(2^{17/16}n^{1/16})}{2^{3/4}n^{1/4}} < \frac{\chi}{2^{3/4}}$$
$$1-\lim_{n\to\infty}\frac{7/2+2^{7/8}n^{1/8}+2(2^{17/16}n^{1/16})}{2^{3/4}n^{1/4}} < \frac{\chi}{2^{3/4}}$$
$$1 < \frac{\chi}{2^{3/4}}$$

This violates the assumption that $\chi < 2^{3/4}$, implying the coefficient $2^{3/4}$ in our upper bound is optimal.

□

With this theorem proven we have completed our first goal of sharpening the bounds in [1]. These new bounds are much easier to calculate, and are significantly tighter for large n. Our second major goal, constructing perimeter minimizing sets for all $n \geq 0$, will be pursued in the following chapter.

CHAPTER 4

PERIMETER MINIMIZING SETS

Intro to Q(n)

Before we begin a discussion on perimeter minimizing sets, we first need to prove some introductory lemmas.

Let us begin by proving equality for $P(n)$ and $Q(n)$ when $n \geq 0$ is a triangle number. Though the Lemma stated below is relatively obvious, this will be a good exercise in proving that a given set is a perimeter minimizer. Recall the inequality from Lemma 3.0.6. Namely, for any set A of volume n, the maximal element $m \in A$ must satisfy $f(n) \leq m$.

Lemma 4.0.1. *If $n \geq 0$ is such that $g(n) = 0$, then*

$$P(n) = f(n) \qquad\qquad Q(n) = f(n) + 1$$

Proof. Let $n \geq 0$ be such that $g(n) = 0$. Take $A \subseteq \{0, 1 \ldots\}$ with maximal element $m \in A$ and volume n. By our argument in Lemma 3.0.5, we know that $m \in \partial A$ and $m + 1 \in \partial A^c$. This implies $f(n) \leq P(n)$ and $f(n) + 1 \leq Q(n)$.

By assumption $n = 1 + \cdots + f(n)$ implying the set, $\{0, 1, \ldots, f(n)\}$, has volume n. The perimeter of this set and its compliment establish upper bounds for $P(n)$ and $Q(n)$. Squeezing the two functions to the above equalities. $\square$

Strategy for Constructing a Perimeter Minimizing Set

Devlin's recursive analysis of the P(n) function found 177 counterexamples that contradicted the statement,

$$P(n) = f(n) + Q(g(n)) \qquad \text{and} \qquad Q(n) = 1 + f(n) + P(g(n))$$

Knowing these two formulas, if we wanted to find a perimeter minimizing set for some $n \geq 0$, as long as n satisfies $P(n) = f(n) + Q(g(n))$ and $Q(g(n)) = 1 + f(g(n)) + P(g^2(n))$, we have

$$P(n) = 1 + f(n) + f(g(n)) + P(g^2(n))$$

Define $B_{g^2(n)}$ as a perimeter minimizing set for $g^2(n) > 0$. Then since $g^2(n) < f(g(n))$, we know that $B_{g^2(n)}$ and $[f(g(n)) + 1, f(n)]$ are pairwise disjoint sets. Knowing this, we can infer that the union given by,

$$B_{g^2(n)} \cup [f(g(n)) + 1, f(n)]$$

Has volume n given the following,

$$\begin{aligned}
Vol =& \frac{f(n)(f(n)+1)}{2} - \frac{f(g(n))(f(g(n))+1)}{2} + g^2(n) \\
=& \frac{f(n)(f(n)+1)}{2} - \left(\frac{f(g(n))(f(g(n))+1)}{2} - g^2(n)\right) \\
=& \frac{f(n)(f(n)+1)}{2} - g(n) \\
=& n
\end{aligned}$$

With perimeter given by, $1+f(n)+f(g(n))+P(g^2(n)) = P(n)$. This implies that the above set is a perimeter minimizing set for $n \geq 0$.

Thus the only issues arise when n is such that $P(n) \neq f(n)+Q(g(n))$ or $Q(g(n)) \neq 1+f(g(n))+P(g^2(n))$. The following section will seek to provide explicit perimeter minimizing sets for $n \geq 0$ satisfying one of the two above conditions.

Counterexamples

We will start this section by classifying all 177 counterexamples. By defining 6 distinct categories that subdivide the set of counterexamples, we will be able to construct perimeter minimizing sets for all $n \geq 0$ with $P(n) \neq f(n)+Q(g(n))$ or $Q(g(n)) \neq 1+f(g(n))+P(g^2(n))$. The 6 categories introduced in the following sections will be crucial to all of the results that follow, most notable is the construction of a perimeter minimizing set for all $n \geq 0$.

Counterexamples of Low Volume

The following subsection seeks to categorize and construct common perimeter minimizing sets for counterexamples with low values of n. Specifically $n \leq 29$. We start with Theorem 4.0.2 which establishes a classification system for all counterexamples with $n \leq 29$.

Theorem 4.0.2. *Every counterexamples with $n \leq 29$ falls into one of the two following categories,*

Category 1: $g(n) = f(n) - 1$

Category 2: $g(n) = f(n) - 2$

Furthermore, these categories are pairwise disjoint.

Proof. A computer search proves that all counterexamples with $n \leq 29$ satisfy either $g(n) = f(n) - 1$ or $g(n) = f(n) - 2$. Lastly, $g(n)$ was defined as a unique value for all $n \geq 0$, proving mutual exclusivity of the two sets. □

Before we can attribute perimeter minimizing sets to both of the above categories, we must first prove some things about the following two sets,

$$[0, f(n) - 2] \cup \{f(n)\} \qquad [0, f(n) - 2] \cup \{f(n) + 1\}$$

Namely, we wish to prove that when $n \geq 0$ satisfies $g(n) = f(n) - 1$ or $g(n) = f(n) - 2$, one of the above sets will have volume n. After proving this, we will be able to state that these sets are perimeter minimizers for Category 1 and Category 2 given our knowledge of the values of $P(n)$ for these counterexamples calculated in [1].

Before we can begin, notice that when $n \geq 2$, we have $f(n) - 2 \geq 0$, given $f(2) = 2$ and $f(n)$ is a stepwise increasing function. This knowledge will be useful in the following proof.

Lemma 4.0.3. *For all $n \geq 2$ with $g(n) = f(n) - 1$, there exists a set of volume n given by,*

$$[0, f(n) - 2] \cup \{f(n)\}$$

Proof. Let $n \geq 2$ be such that $g(n) = f(n) - 1$, then given our previous discussion we have the following, $0 \leq f(n) - 2 < f(n)$. This implies that the intervals are well defined. To prove that the set above has volume n, consider the following.

$$\begin{aligned} n &= 1 + 2 + \cdots + f(n) - g(n) \\ &= 1 + 2 + \cdots + f(n) - (f(n) - 1) \\ &= 1 + 2 + \cdots + (f(n) - 2) + f(n) \\ &= Vol([0, f(n) - 2] \cup \{f(n)\}) \end{aligned}$$

□

Lemma 4.0.4. *For all $n \geq 2$ with $g(n) = f(n) - 2$, there exists a set of volume n given by,*

$$[0, f(n) - 2] \cup \{f(n) + 1\}$$

Proof. Let $n \geq 2$, with $g(n) = f(n) - 2$. As in the previous lemma, we know that the intervals are well defined, To prove that the above set has volume n consider the following.

$$\begin{aligned} n &= 1 + 2 + \cdots + f(n) - g(n) \\ &= 1 + 2 + \cdots + f(n) - (f(n) - 2) \\ &= 1 + 2 + \cdots + f(n) - (f(n) - 1) + 1 \\ &= 1 + 2 + \cdots + (f(n) - 2) + (f(n) + 1) \\ &= Vol([0, f(n) - 2] \cup \{f(n) + 1\}) \end{aligned}$$

□

With Lemmas 4.0.3 and 4.0.4 in hand we can now move on to the construction of perimeter minimizing sets for all $n \leq 29$ with $P(n) \neq f(n) + Q(g(n))$.

Recall that if we can find perimeter minimizing sets for all $n \geq 0$ with $P(n) \neq f(n) + Q(g(n))$ or $Q(g(n)) \neq 1 + f(g(n)) + P(g^2(n))$ we will be able to construct a perimeter minimizing set for all $n \geq 0$. Once we prove Theorem 4.0.5 we will be halfway done with the case where $P(n) \neq f(n) + Q(g(n))$.

Theorem 4.0.5. *All $n \leq 29$ with $P(n) \neq f(n) + Q(g(n))$ fall into one of the two following categories. Perimeter minimizing sets for all n in each category are given,*

Category 1: $g(n) = f(n) - 1$

$$[0, f(n) - 2] \cup \{f(n)\}$$

Category 2: $g(n) = f(n) - 2$

$$[0, f(n) - 2] \cup \{f(n) + 1\}$$

Proof. Since every $n \geq 0$ with $P(n) \neq f(n) + Q(g(n))$ is a counterexample, by Theorem 4.0.2, we know that all counterexamples with $n \leq 29$ fall into one of the two categories.

Given that there are a finite number of counterexamples, it is enough to check that each of Devlin's calculated $P(n)$ for counterexamples coincides with the perimeter of the suggested sets. A brief computation affirms this theorem.

□

The arguments in the following section will mirror that of Section 4. By proving that all counterexamples with $n > 29$ fall into 4 distinct categories, we can establish common perimeter minimizing sets for all $n > 29$ with $P(n) \neq f(n) + Q(g(n))$.

Counterexamples of High Volume

As in the previous section, all counterexamples with high volume ($n > 29$) fall into one of several distinct categories. This subdivision of the counterexamples is established in Theorem 4.0.6.

Theorem 4.0.6. *Every counterexample with $n > 29$ falls into one of the four following categories,*

Category 3: $g(f(n) + g(n) + 1) = 0$

Category 4: $g(f(n) + g(n) + 2) = 0$

Category 5: $g(f(n)+g(n)+1) > 0$, $g^3(f(n)+g(n)+1) = 0$, *and* $f(g^2(f(n)+g(n)+1)+1 = 1$

Category 6: $g(f(n)+g(n)+1) > 0$, $g^3(f(n)+g(n)+1) = 0$, *and* $f(g^2(f(n)+g(n)+1)+1 \neq 1$

Furthermore, these categories are pairwise disjoint.

Proof. An exhaustive computer search affirms that all counterexamples with $n > 29$ fall into one of the four categories. Furthermore, this search affirms that these categories are also pairwise disjoint. □

Lemma 4.0.7. *For $n > 10$*

$$f(n) \geq f(f(n) + g(n) + 2)$$

Proof. Since $f(n)$ is an increasing function, we just need to prove that for $n > 10$ the following inequality holds, $n \geq f(n) + g(n) + 2$.

Let $n > 10$, then the sum, $\sum_{i=1}^{f(n)} i$, in $n = \sum_{i=1}^{f(n)} i - g(n)$ has at least 5 terms.

Since $0 \leq g(n) < f(n) \leq n$, for all $n > 10$ the following inequalities hold,

$$\begin{aligned} n &= \sum_{i=1}^{f(n)} i - g(n) \\ &\geq \sum_{i=1}^{f(n)-2} i + f(n) \\ &\geq \sum_{i=3}^{f(n)-3} i + f(n) + f(n) - 1 + 2 \\ &\geq f(n) + g(n) + 2 \end{aligned}$$

□

As in Section 4, we will now establish that when $n \geq 0$ falls into one of the categories established in Theorem 4.0.6, one of the following sets will attain a volume n.

$$[f(f(n) + g(n) + 1) + 1, f(n) + 1]$$

$$\{0, 1\} \cup [f(f(n) + g(n) + 2) + 1, f(n) + 1]$$

$$[f(g^2(f(n) + g(n) + 1)) + 1, f(g(f(n) + g(n) + 1))] \cup [f(f(n) + g(n) + 1) + 1, f(n) + 1]$$

Lemma 4.0.8. *For all $n > 10$ with $g(f(n) + g(n) + 1) = 0$, there exists a set of volume n given by,*

$$[f(f(n) + g(n) + 1) + 1, f(n) + 1]$$

Proof. Let $n > 10$ be such that $g(f(n) + g(n) + 1) = 0$. Lemma 4.0.7 implies the interval, $[f(f(n) + g(n) + 1) + 1, f(n) + 1]$, is well defined. For proof that this set has volume n, consider the following.

$$
\begin{aligned}
n &= \frac{(f(n))(f(n)+1)}{2} - g(n) \\
&= \frac{(f(n)+1)(f(n)+2)}{2} - \left(\frac{(f(n)+1)(f(n)+2)}{2} - \frac{(f(n))(f(n)+1)}{2} + g(n)\right) \\
&= \frac{(f(n)+1)(f(n)+2)}{2} - (f(n)+g(n)+1) \\
&= \frac{(f(n)+1)(f(n)+2)}{2} - \left(\frac{f(f(n)+g(n)+1)(f(f(n)+g(n)+1)+1)}{2} - g(f(n)+g(n)+1)\right) \\
&= \frac{(f(n)+1)(f(n)+2)}{2} - \frac{f(f(n)+g(n)+1)(f(f(n)+g(n)+1)+1)}{2} \\
&= Vol([f(f(n)+g(n)+1)+1, f(n)+1])
\end{aligned}
$$

□

Lemma 4.0.9. *For all $n > 10$ with $g(f(n)+g(n)+2) = 0$, there exists a set of volume n given by,*

$$\{0,1\} \cup [f(f(n)+g(n)+2)+1, f(n)+1]$$

Proof. Let $n > 10$ be such that $g(f(n)+g(n)+2) = 0$. Given Lemma 4.0.7, the interval, $[f(f(n)+g(n)+2)+1, f(n)+1]$ is well defined. Now we must establish that the two sets are pairwise disjoint. Recall that for all $n \geq 0$ $f(n), g(n) \geq 0$. With this in mind, consider the following,

$$f(g(n)+f(n)+2)+1 \geq f(2)+1 = 3 > 1$$

Thus for all $n \geq 0$, we have $0, 1 \notin [f(f(n)+g(n)+2)+1, f(n)+1]$. For proof that the above set has volume n, consider the following.

$$\begin{aligned}
n &= \frac{(f(n))(f(n)+1)}{2} - g(n) \\
&= \frac{(f(n))(f(n)+1)}{2} - g(n) + (f(n)+1) - (f(n)+1) \\
&= \frac{(f(n)+1)(f(n)+2)}{2} + 1 - (f(n)+g(n)+2) \\
&= \frac{(f(n)+1)(f(n)+2)}{2} + 1 - \frac{f(f(n)+g(n)+2)(f(f(n)+g(n)+2)+1)}{2} \\
&+ g(f(n)+g(n)+2) \\
&= \frac{(f(n)+1)(f(n)+2)}{2} + 1 - \frac{f(f(n)+g(n)+2)(f(f(n)+g(n)+2)+1)}{2} \\
&= Vol(\{0,1\} \cup [f(f(n)+g(n)+2)+1, f(n)+1])
\end{aligned}$$

□

The next Lemma refers back to our work in Theorem 3.0.10. Specifically (3.4) reproduced below,

$$n = \sum_{i=0}^{2L-1} \frac{f(g^i(n))(f(g^i(n))+1)(-1)^i}{2} + g^{2L}(n)$$

where $n \geq 0$ and $L \in \mathbb{N}$.

Lemma 4.0.10. *If $n > 10$ is such that $g(f(n)+g(n)+1) > 0$ and $g^3(f(n)+g(n)+1) = 0$ there exists a set of volume n given by,*

$$[f(g^2(f(n)+g(n)+1))+1, f(g(f(n)+g(n)+1))] \cup [f(f(n)+g(n)+1)+1, f(n)+1]$$

Proof. Consider the following set,

$$B = [f(g^2(f(n)+g(n)+1))+1, f(g(f(n)+g(n)+1))] \cup [f(f(n)+g(n)+1)+1, f(n)+1]$$

Given the argument in Lemma 4.0.7, we know that the interval, $[f(f(n)+g(n)+1)+$

$1, f(n)+1]$, is well defined. For all $n \geq 0$, we have $0 \leq g(n) < f(n) \leq n$. Thus we can infer the following two inequalities,

$$f(g^2(f(n)+g(n)+1))+1 \leq g(g(f(n)+g(n)+1))+1 \leq f(g(f(n)+g(n)+1))$$
$$f(g(f(n)+g(n)+1)) \leq g(f(n)+g(n)+1) < f(f(n)+g(n)+1)$$

These imply that the set, $[f(g^2(f(n)+g(n)+1))+1, f(g(f(n)+g(n)+1))]$ is well defined, and also pairwise disjoint from the set, $[f(f(n)+g(n)+1)+1, f(n)+1]$.

To show that B has volume n, we refer to (3.4).

$$n = \sum_{i=0}^{2L-1} \frac{f(g^i(n))(f(g^i(n))+1)(-1)^i}{2} + g^{2L}(n)$$

where $L \in \mathbb{N}$. Recall that $\frac{f(n)(f(n)+1)}{2} \geq n$, which implies $\frac{(f(n)+1)(f(n)+2)}{2} > n$. Knowing this, we can infer that $\frac{(f(n)+1)(f(n)+2)}{2} - n \in \mathbb{N}$. Thus $\frac{(f(n)+1)(f(n)+2)}{2} - n$ satisfies (3.4), giving the following equality,

$$\begin{aligned}
&\frac{(f(n)+1)(f(n)+2)}{2} - n \\
&= \sum_{i=0}^{2L-1} \frac{f(g^i(\frac{(f(n)+1)(f(n)+2)}{2} - n))(f(g^i(\frac{(f(n)+1)(f(n)+2)}{2} - n))+1)(-1)^i}{2} \\
&+ g^{2L}(\frac{(f(n)+1)(f(n)+2)}{2} - n)
\end{aligned}$$

where $L \in \mathbb{N}$. Recall that for all $n \geq 0$, $n = \frac{f(n)(f(n)+1)}{2} - g(n)$. Implying

$$\begin{aligned}
\frac{(f(n)+1)(f(n)+2)}{2} - n &= \frac{(f(n)+1)(f(n)+2)}{2} - \frac{f(n)(f(n)+1)}{2} + g(n) \\
&= f(n)+1+g(n)
\end{aligned}$$

Lets apply this knowledge to our constructed sum,

$$\frac{(f(n)+1)(f(n)+2)}{2} - n$$
$$= \sum_{i=0}^{2L-1} \frac{f(g^i(f(n)+g(n)+1))(f(g^i(f(n)+g(n)+1))+1)(-1)^i}{2} + g^{2L}(f(n)+g(n)+1)$$

Solving for n on the left, and setting $L \in \mathbb{N}$ to $L = 2$, we have the following.

$$n = \frac{(f(n)+1)(f(n)+2)}{2} + \sum_{i=0}^{3} \frac{f(g^i(f(n)+g(n)+1))(f(g^i(f(n)+g(n)+1))+1)(-1)^{i+1}}{2}$$
$$+g^4(f(n)+g(n)+1)$$

Given our assumption that n is such that that $g(f(n)+g(n)+1) > 0$ and $g^3(f(n)+g(n)+1) = 0$, we can eliminate a few terms from this expression.

$$n = \frac{(f(n)+1)(f(n)+2)}{2} + \sum_{i=0}^{2} \frac{f(g^i(f(n)+g(n)+1))(f(g^i(f(n)+g(n)+1))+1)(-1)^{i+1}}{2}$$

Notice that the volume of $[f(f(n)+g(n)+1)+1, f(n)+1]$ is given by,

$$\begin{aligned} Vol =& \frac{(f(n)+1)(f(n)+2)}{2} \\ &- \frac{f(f(n)+g(n)+1)(f(f(n)+g(n)+1)+1)}{2} \end{aligned}$$

While the volume of $[f(g^2(f(n)+g(n)+1))+1, f(g(f(n)+g(n)+1))]$ is given by,

$$\begin{aligned} Vol =& \frac{f(g(f(n)+g(n)+1))(f(g(f(n)+g(n)+1))+1)}{2} \\ &- \frac{f(g^2(f(n)+g(n)+1))(f(g^2(f(n)+g(n)+1))+1)}{2} \end{aligned}$$

Since the two intervals are disjoint, the volume of their union is the sum of elements of both intervals. Implying the volume of the set B is given by,

$$\frac{(f(n)+1)(f(n)+2)}{2}+\sum_{i=0}^{2}\frac{f(g^i(f(n)+g(n)+1))(f(g^i(f(n)+g(n)+1))+1)(-1)^{i+1}}{2}$$

which we have established is equal to n.

□

With Theorem 4.0.11, given below, we will complete our construction of perimeter minimizing sets for all $n \geq 0$ with $P(n) \neq f(n) + Q(g(n))$. With this accomplished, our goal of constructing a perimeter minimizing set for all $n \geq 0$ is nearly complete.

Theorem 4.0.11. *All $n > 29$ with $P(n) \neq f(n) + Q(g(n))$ fall into one of the four following categories. Perimeter minimizing sets for all n in each category are given,*

Category 3: $g(f(n) + g(n) + 1) = 0$

$$[f(f(n) + g(n) + 1) + 1, f(n) + 1]$$

Category 4: $g(f(n) + g(n) + 2) = 0$

$$\{0, 1\} \cup [f(f(n) + g(n) + 2) + 1, f(n) + 1]$$

Category 5: $g(f(n)+g(n)+1) > 0$, $g^3(f(n)+g(n)+1) = 0$, *and* $f(g^2(f(n)+g(n)+1)+1 = 1$

$$[f(g^2(f(n)+g(n)+1))+1, f(g(f(n)+g(n)+1))]\cup[f(f(n)+g(n)+1)+1, f(n)+1]$$

Category 6: $g(f(n)+g(n)+1) > 0$, $g^3(f(n)+g(n)+1) \neq 0$, *and* $f(g^2(f(n)+g(n)+1)+1 = 1$

$$[0, f(g(f(n) + g(n) + 1))] \cup [f(f(n) + g(n) + 1) + 1, f(n) + 1]$$

Proof. Since every $n > 29$ with $P(n) \neq f(n)+Q(g(n))$ is a counterexample, Theorem 4.0.6, established that all counterexamples with $n > 29$ fall into one of the above categories.

Since we are considering counterexamples with $n > 29$, if a counterexample satisfies one of the above four conditions, the three previous lemmas establish the stated set has a volume of n. All that is left is to check if the perimeter of the corresponding set is equal to $P(n)$. Using the calculated values of $P(n)$ for the counterexamples in [1], an exhaustive computer calculation affirms this theorem. □

Recall Theorem 1.0.2 reproduced from [1]. Given $n \geq 0$ where n is not one of the 177 counterexamples,

$$P(n) = f(n) + Q(g(n)) \qquad \text{and} \qquad Q(n) = 1 + f(n) + P(g(n))$$

With Theorem 4.0.5 and Theorem 4.0.11, we now have explicit perimeter minimizing sets for all $n \geq 0$ that satisfy $P(n) \neq f(n) + Q(g(n))$. With this knowledge, we almost have everything we need to construct a perimeter minimizing set for all $n \geq 0$.

Notice that Theorems 2.0.3 and 2.0.4 of the Results Section are combinations of the theorems in the last two sections. Notice, also, by Theorem 2.0.3, all $n \geq 0$ with $Q(n) \neq 1 + f(n) + P(g(n))$ has a unique set σ_n of volume n. These described sets will be crucial to our understanding of the case when $n \geq 0$ satisfies $Q(g(n)) \neq 1 + f(g(n)) + P(g^2(n))$.

Theorem 4.0.12. *For all $n \geq 0$ with $Q(n) \neq 1 + f(n) + P(g(n))$*

$$Q(n) = Per(\sigma_n^c)$$

Proof. Given Theorem 2.0.3, every $n \geq 0$ with $Q(n) \neq 1+f(n)+P(g(n))$ has a set of volume n, σ_n. Furthermore σ_n^c has a calculable perimeter. An exhaustive computer calculation using the values of $Q(n)$ for the 177 counterexamples calculated in [1] finds this Theorem to be true. □

With the following lemma, we will be able to expand on Theorem 4.0.12.

Lemma 4.0.13. *For all $n > 21$ the following inequality holds.*

$$f(g(n)) + 1 \leq f(n) - 2$$

Proof. Note that if $n > 6$, then $n \geq f(n) + 2$. If $n > 6$, then the $\sum_{i=1}^{f(n)} i$, in $n = \sum_{i=1}^{f(n)} i - g(n)$ has at least four terms. Thus for all $n > 6$ the following inequalities hold.

$$\begin{aligned} n &= \sum_{i=1}^{f(n)} -g(n) \\ &\geq \sum_{i=1}^{f(n)-2} +f(n) \\ &\geq \sum_{i=3}^{f(n)-2} +f(n) + 3 \\ &> f(n) + 2 \end{aligned}$$

Take $n > 21$ then $f(n) \geq 7$. If $g(n) > 6$ then $f(n) \geq g(n) + 1 \geq f(g(n)) + 3$ as desired. If $6 \geq g(n)$, then $f(n) \geq 7 \geq f(6) + 3 \geq f(g(n)) + 3$. □

With the previous lemma established, we know seek to prove Theorem 2.0.5 of the Results Section. The following theorem will mitigate the problems caused by $n \geq 0$ with $Q(g(n)) \neq 1 + f(g(n)) + P(g^2(n))$.

The Proof of Theorem 2.0.5

Proof. Suppose $n \geq 0$ is such that $P(n) = f(n) + Q(g(n))$, and $g(n)$ satisfies, $Q(g(n)) \neq 1 + f(g(n)) + P(g^2(n))$.

The minimal $n \geq 0$ that satisfies $Q(n) \neq 1 + f(n) + P(g(n))$ is $n = 92$, implying that $g(n) \geq 92$. Recall that $g(n) \leq f(n) - 1$. With this we have, $92 \leq f(n) - 1 < \sqrt{2n} - 1/2$. Implying n satisfies $4278 < n$, Given Lemma 4.0.13, we have

$$f(g(n)) + 1 \leq f(n) - 2$$

Let m be the maximal element in $m \in \sigma_{g(n)}$, then m satisfies $m \leq f(g(n)) + 1 \leq f(n) - 2$.

Thus, given Theorem 4.0.12, we have

$$\begin{aligned} Per([0, f(n)] \setminus \sigma_{g(n)}) &= f(n) + Per(\sigma^c_{g(n)}) \\ &= f(n) + Q(g(n)) \\ &= P(n) \end{aligned}$$

□

Constructing Perimeter Minimizing Sets

The following theorem comes from [1] and will be useful in constructing a common perimeter minimizing set for all $n \geq 0$.

Theorem 4.0.14. *[1] If n and $g(n)$ are not one of the 177 counterexamples (and in particular, if $g(n) > 149,894$), then we have*

$$P(n) = 1 + f(n) + f(g(n)) + P(g^2(n)) \qquad Q(n) = 1 + f(n) + f(g(n)) + Q(g^2(n)). \tag{4.1}$$

We now have everything we need to construct a perimeter minimizing set for all $n \geq 0$. The following proofs will do exactly that by affirming the theorems given in the latter half of the Results Section.

The Proof of Theorem 2.0.6

Proof. Suppose that $n \geq 0$ is such that $g^2(n) = 0$ and $P(n) = f(n) + Q(g(n))$. If $g(n) = 0$ then by the same argument as Lemma 4.0.1, the theorem holds. If $g(n) > 0$ then, given Lemma 4.0.1, $Q(g(n)) = 1 + f(g(n))$. This implies the set of volume n,

$$[f(g(n)) + 1, f(n)]$$

has perimeter given by

$$\begin{aligned} 1 + f(n) + f(g(n)) &= f(n) + Q(g(n)) \\ &= P(n) \end{aligned}$$

□

The Proof of Theorem 2.0.4

Proof. This theorem follows as a combination of Theorems 4.0.2, 4.0.5, 4.0.6, and 4.0.11. □

The Proof of Theorem 2.0.7

Proof. Let $n \geq 0$ be such that with $g^2(n) > 0$, $P(n) = f(n) + Q(g(n))$, and $Q(g(n)) = 1 + f(g(n)) + P(g^2(n))$. Let $\rho \in \mathbb{N}$ be given by the following,

$$\begin{aligned}\rho = \min\{\ell \in \mathbb{N} | P(g^{2\ell}(n)) &\neq f(g^{2\ell+1}(n)) + Q(g^{2\ell+1}(n)),\\ &Q(g^{2\ell+1}(n)) \neq 1 + f(g^{2\ell+2}(n)) + P(g^{2\ell+2}(n)), \text{ or}\\ &g^{2\ell}(n) = 0\}.\end{aligned}$$

Given Theorem 4.0.14, we know,

$$P(n) = \rho + \sum_{i=0}^{2\rho-1} f(g^i(n)) + P(g^{2\rho}(n))$$

If we take $B_{g^{2\rho}(n)}$ as the perimeter minimizing set given by Theorems 2.0.6, 2.0.5, or 2.0.4, then the maximal element $m \in B_{g^{2\rho}(n)}$ must be less than $g^{2\rho}(n)$. Since for $n \geq 0$, we have $g^{2\rho}(n) < f(g^{2\rho-1})$ we know that the sets $\bigcup_{i=0}^{\rho-1}[f(g^{2i+1}(n)) + 1, f(g^{2i}(n))]$ and $B_{g^{2\rho}(n)}$ are pairwise disjoint. This implies that the union of both sets has volume n (3.4) and perimeter

$$\begin{aligned}\sum_{i=0}^{2\rho-1}[f(g^{2i+1}(n)) + 1 + f(g^{2i}(n))] + P(g^{2\rho}(n)) &= \rho + \sum_{i=0}^{2\rho-1} f(g^i(n)) + P(g^{2\rho}(n)\\ &= P(n).\end{aligned}$$

Thus the union given below is a perimeter minimizer for all $n \geq 0$ with $g^2(n) > 0$, $P(n) = f(n) + Q(g(n))$, and $Q(g(n)) = 1 + f(g(n)) + P(g^2(n))$.

$$\bigcup_{i=0}^{\rho-1}[f(g^{2i+1}(n)) + 1, f(g^{2i}(n))] \cup B_{g^{2\rho}(n)}$$

As for the proof that Theorems 2.0.6, 2.0.5, 2.0.4, and 2.0.7 construct perimeter minimizing sets for all $n \geq 0$, consider the following argument. If $P(n) \neq f(n) + Q(g(n))$ then Theorem 2.0.6 can be applied. Likewise if n is such that $Q(g(n)) \neq 1 + f(g(n)) + P(g^2(n))$, apply Theorem 2.0.5. Theorem 2.0.4 takes care of all $n \geq 0$ with $g^2(n) = 0$. Lastly, Theorem 2.0.7 assigns a perimeter minimizing to all remaining $n \geq 0$ with $g^2(n) > 0$, $P(n) = f(n) + Q(g(n))$, and $Q(g(n)) = 1 + f(g(n)) + P(g^2(n))$. □

Note that the theorems in the Results Section construct perimeter minimizing sets for all $n \geq 0$. With access to the list of counterexamples found in [1] and the sets given in Theorem 2.0.3, let's try to find a perimeter minimizing set for a very large number.

Example. Let $n = 56714578492315796$.

First we must check all the values of $g^i(n)$ (with $i \geq 1$) to see if any members of the counterexamples show up. with a brief computation, we get the following:

$$\begin{aligned} g(n) &= 211751489 \\ g^2(n) &= 6421 \\ g^3(n) &= 20 \\ g^4(n) &= 1 \\ g^5(n) &= 0 \\ g^6(n) &= 0 \end{aligned}$$

None of these are counterexamples, yet $g^5(n) = 0$ and $g^6(n) = 0$ implying a perimeter minimizing set is given by

$$[0, f(g^4(n)))] \cup [f(g^3(n)) + 1, f(g^2(n))] \cup [f(g(n)) + 1, f(n)]$$

With a brief calculation, a perimeter minimizing set for $n = 56714578492315796$ is,

$$[0, 1] \cup [6, 113] \cup [20579, 336792454]$$

Example.

Let $n = 623234433348744547554648$, then we have the following,

$$\begin{aligned} g(n) &= 641296199802 \\ g^2(n) &= 611584 \\ g^3(n) &= 587 \\ g^4(n) &= 8 \end{aligned}$$

Since $P(g^4(n)) \neq f(g^4(n)) + Q(g^5(n))$, $g^4(n) \leq 29$ and $g^5(n) = f(g^4(n)) - 2$, we know that a perimeter minimizing set for $n = 62323443334874454754648$ is given by,

$$[0, f(g^4(n)) - 2] \cup \{f(g^4(n)) + 1\} \cup [f(g^3(n)) + 1, f(g^2(n))] \cup [f(g(n)) + 1, f(n)]$$

with a brief calculation, we have,

$$[0, 2] \cup \{5\} \cup [35, 1106] \cup [1132517, 1116453701099]$$

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