ABSTRACT

On a Ring Associated to F[x]

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For a field F and the polynomial ring $F[x]$ in a single indeterminate, we define $\widehat{F[x]} = \{\alpha \in \mathrm{End}_F(F[x]) : \alpha(f) \in fF[x] \text{ for all } f \in F[x]\}$. Then $\widehat{F[x]}$ is naturally isomorphic to $F[x]$ if and only if F is infinite. If F is finite, then $\widehat{F[x]}$ has cardinality continuum. We study the ring $\widehat{F[x]}$ for finite fields F. For the case that F is finite, we discuss many properties and the structure of $\widehat{F[x]}$.

On a Ring Associated to F[x]

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CHAPTER ONE

History and Introduction

In this dissertation, we focus on the ring $\widehat{F[x]}$ when our field F is finite. The ring $\widehat{F[x]}$ was first studied by Dr. Joshua Buckner and Dr. Manfred Dugas. In the *Israel Journal of Mathematics, vol 160* from 2007 [3], Buckner and Dugas presented their findings on this ring when F was infinite. They found that in this case, $\widehat{F[x]} = F[x]\cdot$. In other words, they found that the α's in $\widehat{F[x]}$ were multiplications with some polynomial within the given polynomial ring.

The question of the structure of $\widehat{F[x]}$ when F is finite stayed open until Dr. Dugas and I started working on it in Spring 2012. This dissertation will contain all of our findings, most of which come from using the Chinese Remainder Theorem.

The inclusion of the Chinese Remainder Theorem into our research is due to a suggestion by Dr. Lindsay Childs [6]. After hearing a presentation given by Dr. Dugas during an Algebra conference in 2010, Dr. Childs sent an email to Dr. Dugas mentioning that the problem might be more approachable using the Chinese Remainder Theorem. Thanks to this suggestion, we now know much more about the ring $\widehat{F[x]}$.

The dissertation starts with an introduction to the Chinese Remainder Theorem and the various forms of it that we used throughout our research. From there we discuss how to construct the ring $\widehat{F[x]}$ as well as some of our preliminary findings. We end with our main theorem which is part of the sister theorem of Buckner and Dugas's result and a few corollaries we mention as results of our findings.

CHAPTER TWO

Chinese Remainder Theorem

2.1 Introduction

There have been many different versions of the Chinese Remainder Theorem. Although it originated with the work of the 3rd-century-AD Chinese mathematician Sun Zi, the first statement of the whole theorem was in 1247 by Qin Jiushao [7]. In our research we use two specific variations. Before discussing the two types, we state the general version.

Theorem 2.1. *[4] Let $n_1, n_2, \ldots, n_r$ be positive integers such that $\gcd(n_i, n_j) = 1$, for $i \neq j$. Then the system of linear congruences*

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_r \pmod{n_r} \end{aligned}$$

has a simultaneous solution, which is unique modulo the integer $n_1 n_2 \cdots n_r$.

2.2 The Chinese Remainder Theorem with Respect to Abelian Groups

We next need to approach the Chinese Remainder Theorem (CRT) with respect to abelian groups. The following proposition is equivalent to stating the CRT. However, to use this proposition, we need the following definition:

Definition 2.1. Let A be an abelian group and $\{N_1, N_2, \ldots, N_k\}$ be a list of subgroups of A. We call this list distributive if for all $1 \leq \ell \leq k$ we have

$$\bigcap_{1 \leq j \leq k, j \neq \ell} (N_j + N_\ell) = \Big(\bigcap_{1 \leq j \neq \ell \leq k, j \neq \ell} N_j\Big) + N_\ell.$$

Proposition 2.1. *Let A be an abelian group, $k \in \mathbb{N}$, $a_i \in A$ for each $i = 1, \ldots, k$, and $\{N_1, N_2, \ldots, N_k\}$ a distributive list of subgroups of A. The following are equivalent:*

(1) $\bigcap_{1\leq j\leq k} (a_j + N_j) \neq \emptyset$

(2) $a_i - a_j \in N_i + N_j$ *for all* $1 \leq i < j \leq k$.

Proof. Suppose there exists some $x \in \bigcap_{1\leq j\leq k} (a_j + N_j)$. Then $x = a_j + n_j$ for some $n_j \in N_j$, and it follows that $0 = a_i + n_i - (a_j + n_j)$ for all $1 \leq i < j \leq k$. We infer that $a_i - a_j = -n_i + n_j \in N_i + N_j$.

We prove the converse by induction over k. If $k = 1$, there is nothing to show, and for $k = 2$, we have $a_1 - a_2 \in N_1 + N_2$ and so $a_1 - a_2 = n_1 + n_2$ with $n_i \in N_i$. Thus, $x = a_1 - n_1 = a_2 + n_2 \in (a_1 + N_1) \cap (a_2 + N_2)$, and the proposition holds for $k = 2$. Assume the proposition holds for k and consider $a_i + N_i$ for $1 \leq i \leq k + 1$ such that $a_i - a_j \in N_i + N_j$ for all $1 \leq i < j \leq k + 1$. By the induction hypothesis, there exists some $x_0 \in \bigcap_{1\leq j\leq k} (a_j + N_j)$. Let $N = \bigcap_{1\leq j\leq k} N_j$, and consider the cosets $x_0 + N$ and $a_{k+1} + N_{k+1}$.

Note that $x_0 - a_{k+1} \in a_j - a_{k+1} + N_j \subseteq N_j + N_{k+1}$ for all $1 \leq j \leq k$. Thus, $x_0 - a_{k+1} \in \bigcap_{1\leq j\leq k} (N_j + N_{k+1}) = N + N_{k+1}$ by distributivity. By the case $k = 2$, we have some $x \in (x_0 + N) \cap (a_{k+1} + N_{k+1})$. Note that $x_0 + N \subseteq \bigcap_{1\leq j\leq k} (a_j + N_j)$ and so $x \in \bigcap_{1\leq j\leq k+1} (a_j + N_j)$ as desired. □

2.3 The Chinese Remainder Theorem with Respect to Principle Ideal Domains

Now that we have a version that works with abelian groups, we want to take it a step farther and look at a version for principle ideal domains (PIDs). This takes us to Ore's version of the CRT. Although this result is from Ore [11], we will prove the theorem a little differently. Our version of the proof requires a few claims to be made in order to get our desired result.

Theorem 2.2. *[11] R is a PID, $a_i \in R,\ 0 \neq m_i \in R,\ 1 \leq i \leq k$. Then the system of congruences $x \equiv a_i \pmod{m_i},\ 1 \leq i \leq k$ with $a_i \equiv a_j \pmod{d_{ij}}$, where $d_{ij} = \gcd(m_i, m_j)$, has a unique solution modulo M, where $M = lcm\{m_i : 1 \leq i \leq k\}$.*

Proof. Let $B_i = \frac{M}{m_i} \in R,\ 1 \leq i \leq k$. Our first claim is that the $\gcd\{B_i : 1 \leq i \leq k\} = 1$. To prove this, assume there is some irreducible $g \in R$ that divides all $B_i,\ 1 \leq i \leq k$, which implies that $g|M$. This shows that there exists an m_i with $g|m_i$. We may assume $i = 1$ and $g^e|m_1$ where e is the maximum such exponent. This implies that g^e is the highest power of g dividing M, and this in turn implies that $g \nmid \frac{M}{m_1} = B_1$. This is a contradiction; therefore, $\gcd\{B_i : 1 \leq i \leq k\} = 1$.

By the Extended Euclidian Algorithm we know that there exists some $c_i \in R$ with $\sum_{i=1}^{k} c_i B_i = 1$. Now let $x = \sum_{i=1}^{k} a_i c_i B_i$. We need to show that it solves the system of congruences. Fix j. By the hypothesis, $a_i \equiv a_j \pmod{d_{ij}}$ implies that $B_i a_i \equiv B_i a_j \pmod{B_i d_{ij}}$ and $B_i d_{ij} = \frac{M}{m_i} \gcd(m_i, m_j)$.

Our second claim is that $m_j|(\frac{M}{m_i} \gcd(m_i, m_j))$. Recall that $B_i = \frac{M}{m_i}$. So, we have $m_i B_i = M = m_j B_j$. Let $m_i = m_i' \gcd(m_i, m_j)$. This implies $m_i' \gcd(m_i, m_j) B_i = m_j' \gcd(m_i, m_j) B_j$ which in turn gives us $m_i' B_i = m_j' B_j$. As a result, $m_j'|m_i' B_i$, and since $\gcd(m_i', m_j') = 1$, we have $m_j'|B_i$. Hence, $m_j = m_j' \gcd(m_i, m_j)|B_i \gcd(m_i, m_j)$.

Now, we have that $B_i a_i \equiv B_i a_j \pmod{m_j}$, so $B_i a_i c_i = B_i a_j c_i \pmod{m_j}$. This leads to the equivalences $\sum_{i=1}^{k} B_i a_j c_i \pmod{m_j} \equiv \sum_{i=1}^{k} B_i a_i c_i = x \equiv (\underbrace{\sum_{i=1}^{k} B_i c_i}_{=1}) a_j \pmod{m_j}$. Therefore, $x \equiv a_j \pmod{m_j}$.

To see the uniqueness of x up to modulo M, suppose that $x \equiv a_j \pmod{m_j}$ and $y \equiv a_j \pmod{m_j}$ for $1 \leq j \leq k$. This implies that $y - x \equiv 0 \pmod{m_j},\ 1 \leq j \leq k$ which in turn gives us that $y - x \equiv 0 \pmod{M}$. So, $x \equiv y \pmod{M}$. Therefore, x is unique modulo M. Conversely, if x is a solution and $y \equiv x \pmod{M}$, then y is a solution as well. $\square$

CHAPTER THREE

Construction of $\widehat{F[x]}$

3.1 Introduction

In this chapter we discuss the construction of our ring, $\widehat{F[x]}$. We start simply by defining our ring then showing such a ring exists. We use our variant of the Chinese Remainder Theorem with respect to abelian groups (Proposition 2.1) as the basis for how we know that such a ring can and does exist.

3.2 The Elements of $\widehat{F[x]}$

In the case we discuss throughout this dissertation, we let F be a finite field with q elements where q is a power of a prime p. We represent the polynomial ring with its coefficients in F as $F[x]$. We define

$$\widehat{F[x]} = \{\alpha \in \mathrm{End}_F(F[x]) : \alpha(I) \subseteq I \text{ for all } I \trianglelefteq F[x]\},$$

$$\text{or more simply } \widehat{F[x]} = \{\alpha \in \mathrm{End}_F(F[x]) : \alpha(f) \in fF[x] \text{ for all } f \in F[x]\}.$$

If $\alpha, \beta \in \mathrm{End}_F F[x]$, then $\alpha\beta$ means α goes first. However, occasionally we let $\alpha(f)$ denote the image of f under α. This definition means that $\alpha \in \mathrm{End}_F(F[x])$ belongs to $\widehat{F[x]}$ if and only if f divides $\alpha(f)$ for all (monic) polynomials $f \in F[x]$.

Definition 3.1. Define $P_n = \{f \in F[x] : \deg(f) \leq n\}$ and $P_n^1 = \{f \in F[x] : \deg(f) = n \text{ and } f \text{ is monic}\}$. Note that for any $f \in P_n^1$, we have $f = x^n + f^-$ for some unique $f^- \in P_{n-1}$. Moreover, $|P_n| = q^{n+1}$ and $|P_n^1| = q^n$.

Before the next lemma, we need to clarify some notation. We will let $T = \{t_0, t_1, t_2, \dots\}$ represent a countable sequence of elements of $F[x]$. Note that with this definition, there exists a unique $\alpha = \alpha_T \in \mathrm{End}_F(F[x])$ such that $\alpha(x^i) = t_i$ for all $i \geq 0$. Let $f = \sum_{i=0}^{n} f_i x^i \in F[x]$, where f_i represent the coefficients in the field F of the i-th term of the polynomial.

Lemma 3.1. *The following are equivalent:*

(1) $\alpha = \alpha_T \in \widehat{F[x]}$.

(2) For each $n \in \mathbb{N}$ *and each monic polynomial* $f = \sum_{i=0}^{n} f_i x^i \in P_n^1$ *of degree* n, *we have* $t_n \equiv -\sum_{i=0}^{n-1} f_i t_i \pmod{f}$.

Proof. Let $\alpha = \alpha_T \in \widehat{F[x]}$. This implies that f divides $\alpha(f)$ for each $f \in F[x]$ which is true if and only if f divides $\alpha(f)$ for each monic $f \in F[x]$ if and only if f divides $\alpha(f) = t_n + \sum_{i=0}^{n-1} f_i t_i$ for each monic polynomial f of degree n if and only if $t_n \equiv -\sum_{i=0}^{n-1} f_i t_i \pmod{f}$. □

Remark 3.1. In other words, Lemma 3.1 shows that $\alpha = (t_i)_i \in \widehat{F[x]}$ if and only if $\alpha(x^n) \equiv \alpha(f^-) \pmod{f}$ for all $f \in P_n^1$.

Lemma 3.2. *Let* $n \in \mathbb{N}$ *and* $\beta \in Hom_F(P_{n-1}, F[x])$ *such that* f *divides* $\beta(f)$ *for all* $f \in P_{n-1}$. *Let* $\beta(x^i) = t_i$ *for all* $0 \le i \le n-1$. *Then there are infinitely many elements* $t_n \in F[x]$ *such that the map* $\gamma \in Hom(P_n, F[x])$ *defined by* $\gamma(x^i) = t_i$ *for all* $0 \le i \le n$ *has the property that* f *divides* $\gamma(f)$ *for all* $f \in P_n$. *Morever,* $\beta = \gamma|_{P_{n-1}}$.

Proof. For each of the q^n monic polynomials f in P_n^1, consider the system of congruences:

$$t \equiv -\sum_{j=0}^{n-1} f_i t_i \pmod{f} \text{ for all } f \in P_n^1 \tag{3.1}$$

and note that $\sum_{i=0}^{n-1} f_i t_i = \sum_{i=0}^{n-1} f_i \beta(x^i) = \beta(\sum_{i=0}^{n-1} f_i x^i) = \beta(f^-)$ where $f = x^n + f^-$ and $f^- \in P_{n-1}$. Let f, g be monic polynomials in P_n^1. Then,

$$-\sum_{i=0}^{n-1} f_i t_i - \left(-\sum_{i=0}^{n-1} g_i t_i\right) = -\beta(f^-) + \beta(g^-) = \beta(g^- - f^-) = \beta(g - f)$$

$$\text{and } g - f = -f^- + g^- \text{ divides } \beta(-f^- + g^-) = \beta(g - f).$$

It follows that $\gcd(f, g)$ divides $-f + g = -f^- + g^-$ which divides $-\beta(f^-) + \beta(g^-)$.

Now we apply the general Chinese Remainder Theorem and conclude that the system of congruences (3.1) has a solution for t, and in fact, it has infinitely many solutions. Let $t = t_n$ be one of these solutions and define $\gamma(x^n) = t_n$ and $\gamma|_{P_{n-1}} = \beta$. Let f be a monic polynomial of degree n. Then f divides $\gamma(f) = t_n + \sum_{j=0}^{n-1} f_i t_i$ because of the set of congruences. If $g \in P_n - P_{n-1}$, then there is a monic $f \in P_n$ and $k \in F$ such that $g = kf$ and g divides $\gamma(g) = k\gamma(f)$. Note that the choice of $x^n \in P_n^1$ implies $t_n = \rho_n x^n$ for some $\rho_n \in F[x]$. $\square$

Lemma 3.3. *The F-algebra $\widehat{F[x]}$ has cardinality* $2^{\aleph_0}$.

Proof. We get that $\widehat{F[x]}$ has at least cardinality continuum by Lemma 3.2. This is because in the inductive construction you have at least two choices of what the next one in the sequence will be. This, paired with the fact

$$\widehat{F[x]} \subseteq P := \{\alpha \in \mathrm{End}_F(F[x]) : \alpha(x^i) \in x^i F[x]\} \cong \prod_\omega F[x]$$

giving the inequality $|\widehat{F[x]}| \leq |\prod_\omega F[x]| = |F[x]|^{\aleph_0} = \aleph_0^{\aleph_0} = 2^{\aleph_0}$, yields the desired result. $\square$

Remark 3.2. Let $\Gamma_n = \mathrm{lcm}\{f : f \in P_n^1\}$. If $\alpha = (t_i)_i \in \widehat{F[x]}$, then t_n is determined by the initial sequence $(t_i)_{0 \leq i < n}$ only modulo Γ_n. You can pick any t_0, then by induction using Lemma 3.2, you can come up with a sequence t_i, so that you have a sequence in $\widehat{F[x]}$.

Now that we have that $\widehat{F[x]}$ exists and know its cardinality, when looking at $\alpha \in \widehat{F[x]}$, we have $\alpha(x^i) = x^i \alpha_i$ for some $\alpha_i \in F[x]$. Since the map α is uniquely determined by the α_i, we henceforth write $\alpha = (\alpha_i)_i$. For example, $\mathrm{id}_{F[x]} = (1)_i$.

When studying a new ring, it is frequently important to understand how the elements of the ring interact with each other. First, we look at the additive properties. Addition in $\widehat{F[x]}$ is such that if

$$\alpha = (\alpha_i)_i,\ \beta = (\beta_i)_i \in \widehat{F[x]},\ \alpha_i = \sum \alpha_{ij} x^j,\ \text{and } \beta_i = \sum \beta_{ij} x^j,$$
$$\text{then } \alpha + \beta = (\alpha_i + \beta_i)_i.$$

However, the multiplication is much more complicated. Let $a_i = \deg(\alpha_i)$ and $\alpha_i = \sum_{t=0}^{a_i} \alpha_{it} x^t$. Let $b_i = \deg(\beta_i)$ and $\beta_i = \sum_{s=0}^{b_i} \beta_{is} x^s$. We will find a formula for $\alpha\beta = (u_i)_i$. Note that

$$\begin{aligned}(x^i)(\alpha\beta) = (x^i\alpha_i)\beta &= \left(\sum_{t=0}^{a_i} \alpha_{it} x^{t+i}\right)\beta = \sum_{t=0}^{a_i} \alpha_{it}\beta_{i+t} x^{t+i} \\ &= \sum_{t=0}^{a_i} \alpha_{it}\left(\sum_{s=0}^{b_{i+t}} \beta_{i+t,s} x^s\right) x^{t+i} = \sum_{t,s} \alpha_{it}\beta_{i+t,s} x^{s+t+i}.\end{aligned}$$

Let $k = s+t$. We infer that $(x^i)(\alpha\beta) = \left(\sum_k \left(\sum_t \alpha_{it}\beta_{i+t,k-t}\right) x^k\right) x^i = u_i x^i$ where $u_i = \sum_t u_{it} x^t = \sum_t \left(\sum_j \alpha_{ij}\beta_{i+j,t-j}\right) x^t$ and thus $u_{it} = \sum_{j=0}^{a_i} \alpha_{ij}\beta_{i+j,t-j}$. We will do more with the degrees of these elements in Chapter 5.

Although the next lemma is not one of the main result, we are mentioning it here so that we have it for future reference.

Lemma 3.4. *Let $\sigma : F[x] \to F[x]$ be an automorphism. Then if F is finite, the order of σ is finite.*

Proof. Let σ be the substitution map defined by $\sigma : x \longmapsto ax + b$, where a and b are fixed, $a \neq 0$ and $a, b \in F$. We want to know the order of σ; in other words, we want to know the least n with $\sigma^n = \mathrm{id}_{F[x]}$. That means we need to find a formula $x\sigma^n = a_n x + b$, where $\sigma^n = \mathrm{id}_{F[x]}$ if and only if $x\sigma^n = x$, with n least. We do this by induction.

For $i = 1,\ 2,$ and 3, we have the following formulas:

$$\begin{aligned} x\sigma &= ax + b, \\ x\sigma^2 &= a(ax+b) + b = a^2x + ab + b, \text{ and} \\ x\sigma^3 &= a^2(ax+b) + ab + b \\ &= a^3x + a^2b + ab + b. \end{aligned}$$

Now, supposing that this pattern holds for $i = n$, we have

$$x\sigma^n \quad = a^n x + b\sum_{j=0}^{n-1} a^j = a^n x + \frac{b(a^n - 1)}{a - 1} = x \text{ if and only if } a^n = 1,\ (a \neq 1).$$

This gives us that for $i = n + 1$,

$$\begin{aligned} x\sigma^{n+1} &= (x\sigma^n)\sigma = (a^n x + b\sum_{j=0}^{n-1} a^j)\sigma \\ &= a^n(ax+b) + b\sum_{j=0}^{n-1} a^j \\ &= a^{n+1}x + a^n b + b\sum_{j=0}^{n-1} a^j \\ &= a^{n+1}x + b\sum_{j=0}^{n} a^j. \end{aligned}$$

Therefore, we have that $\mathcal{O}(\sigma) = \mathcal{O}(a)$ in $F - \{0\}$. Hence, when F is finite, the order of σ is finite. $\square$

One of the many things that makes the study of the ring $\widehat{F[x]}$ so interesting is that we can look at how elements react to one another, study elements in detail, show that it has cardinality continuum, yet we cannot look at specific elements of the ring. For this reason, we might call the study of the ring "endoscopic."

CHAPTER FOUR

Basic Properties of $\widehat{F[x]}$

4.1 Introduction

We dedicate this chapter to discussing basic properties of the ring $\widehat{F[x]}$. We cover properties such as units, zero divisors, and many other things that one would typically want to know about a ring. These properties serve as a strong basis on which to build the lemmas that we require for reaching our main result.

Throughout this chapter, we assume that $|F| = q$, and we define ψ as the Frobenius map, where $\psi : F[x] \to F[x]$ is defined by $f\psi = f^q = f(f^{q-1})$. Therefore, $\psi|_F = \mathrm{id}_F$. Thus, $\psi \in \widehat{F[x]}$ and $\psi = (x^{i(q-1)})_i$.

4.2 Basic Properties

Lemma 4.1. *$Z(\widehat{F[x]}) = F\cdot$. In other words, the center of the ring $\widehat{F[x]}$ consists only of the scalar multiplication by the field elements of F.*

Proof. ($\Rightarrow$) Let $\alpha = (\alpha_i)_i \in Z(\widehat{F[x]})$. Then $(x\alpha_{i+1})_i = (x\cdot)\alpha = \alpha(x\cdot) = (\alpha_i x)_i$. This implies that $\alpha_i = \alpha_{i+1}$ for all i. Thus, $\alpha = f\cdot$ for some $f \in F[x]$. Now, let $g \in F[x]$. Then we have $g(\psi\alpha) = g^q\alpha = g^q \cdot f$, but since α and ψ commute, $g(\psi\alpha) = (g\alpha)\psi = (gf)^q = g^q f^q$. Thus, $g^q \cdot f = g^q f^q$, and this implies that $f = f^q$. However, this only holds if the degree of f is zero. So, $f \in F\cdot$.

($\Leftarrow$) Let $f \in F\cdot$ and $\alpha \in \widehat{F[x]}$. Then, $f(\alpha_i)_i = f(\sum \alpha_{ij}x^j) = \sum f\alpha_{ij}x^j = (\sum \alpha_{ij}x^j)f = (\alpha_i)_i f$; hence, $f \in Z(\widehat{F[x]})$. $\square$

Lemma 4.2. *The group of units of $\widehat{F[x]}$, represented by $U(\widehat{F[x]})$, are equal to $F-\{0\}$.*

Proof. ($\Rightarrow$) Let $\alpha \in U(\widehat{F[x]})$ and $\alpha = (\alpha_i)_i$. Then, there exists a $\beta = (\beta_i)_i \in \widehat{F[x]}$ with $\alpha\beta = \mathrm{id}_{F[x]}$. For all $g \in F[x]$, there exists a $u_g \in F[x]$ with $g\alpha = gu_g$. In the same way, $g\beta = gv_g$. It follows that $g = g(\alpha\beta) = (g\alpha)\beta = (gu_g)\beta = (gu_g)v_{gu_g} =$

$g(u_g v_{gu_g})$. This implies $1 = u_g v_{gu_g}$, and that means $u_g \in F-\{0\}$. Thus, $\alpha_i \in F-\{0\}$ for all i. It follows that

$$\begin{aligned}
(1+x^i)\alpha = \alpha_0 + x^i\alpha_i &\equiv 0 \pmod{(1+x^i)} \\
\alpha_0 + x^i\alpha_i - \alpha_i(1+x^i) &\equiv 0 \pmod{(1+x^i)} \\
\alpha_0 - \alpha_i &\equiv 0 \pmod{(1+x^i)}
\end{aligned}$$

Thus, $\alpha_0 = \alpha_i$ for all $i \geq 1$. So, $\alpha = \alpha_0 \cdot \in F - \{0\}$.

($\Leftarrow$) Let $\alpha \in F - \{0\}$. We know that F is a field, so for any $\alpha \in F - \{0\}$, there is a mulitplicative inverse β also in $F - \{0\}$. Since $F - \{0\} \in \widehat{F[x]}$, we are done. $\square$

Lemma 4.3. *The Jacobson radical of $\widehat{F[x]}$, $J(\widehat{F[x]})$, is zero.*

Proof. Let $\alpha \in J(\widehat{F[x]})$. Since $1 \in \widehat{F[x]}$, we have that $1+\alpha \in U(\widehat{F[x]}) = (F-\{0\})\cdot$ by Hungerford. Thus, $\alpha \in F\cdot$ is not a unit because $J(\widehat{F[x]}) \subsetneq \widehat{F[x]}$. It follows that $J(\widehat{F[x]}) = \{0\}$. $\square$

Lemma 4.4. *If $0 \neq \alpha \in \widehat{F[x]}$, then dim$_F$(ker($\alpha$)) is finite.*

Proof. Assume $\dim_F(\ker(\alpha))$ is infinite. Then, $\{\deg(f) : f \in \ker(\alpha)\}$ is unbounded. Assume $f \in F[x] - \ker(\alpha)$. For all $g \in F[x]$, there exists $u_g \in F[x]$ with $g\alpha = gu_g$ by the definition of $\widehat{F[x]}$. There exists $k \in \ker(\alpha)$ with $\deg(k) > \deg(f) + \deg(u_f)$. Now $(f+k)u_{f+k} = (f+k)\alpha = f\alpha = fu_f$, and this implies that $ku_{f+k} = f(u_f - u_{f+k})$. (Note that $u_{f+k} \neq 0$ because $f + k \notin \ker(\alpha)$, since $f \notin \ker(\alpha)$.) Thus, $\deg(k) + \deg(u_{f+k}) \leq \deg(f) + \max\{\deg(u_f), \deg(u_{f+k})\}$.

If $\max\{\deg(u_f), \deg(u_{f+k})\} = \deg(u_{f+k})$, then $\deg(k) + \deg(u_{f+k}) \leq \deg(f) + \deg(u_{f+k})$. This implies that $\deg(k) \leq \deg(f)$. This is a contradiction to the choice of k. Thus, $\max\{\deg(u_f), \deg(u_{f+k})\} = \deg(u_f)$. This leads to the conclusion that $\deg(k) + \deg(u_{f+k}) \leq \deg(f) + \deg(u_f)$. This, too, is a contradiction to the choice of k. Thus, no such f exists. In other words, $F[x] = \ker(\alpha)$ which implies that $\alpha = 0$. $\square$

Lemma 4.5. *Let $\alpha, \beta \in \widehat{F[x]}$. If $\alpha\beta = 0$, then $\alpha = 0$ or $\beta = 0$.*

Proof. We break this proof up into two cases. In the first case, let $\alpha\beta = 0$ and $\alpha \neq 0$. Then, $\mathrm{im}(\alpha) \subseteq \ker(\beta)$. By the first isomorphism theorem, $\mathrm{im}(\alpha) \cong \frac{F[x]}{\ker(\alpha)}$, and $\dim_F(\ker(\alpha))$ is finite by Lemma 4.4. Thus, $\dim_F(\mathrm{im}(\alpha))$ is infinite, and this implies $\dim_F(\ker\beta)$ is infinite. Therefore, $\beta = 0$ by Lemma 4.4.

In the second case, let $\alpha\beta = 0$ with $\beta \neq 0$. Then, $\mathrm{im}(\alpha) \subseteq \ker(\beta)$. By Lemma 4.4, $\dim_F(\ker(\beta))$ is finite, and that implies $\dim_F(\mathrm{im}(\alpha))$ is finite. However, the first isomorphism theorem gives that $\mathrm{im}(\alpha) \cong \frac{F[x]}{\ker(\alpha)}$ has finite dimension. Thus, $\dim_F(\ker(\alpha))$ is infinite. By Lemma 4.4, $\alpha = 0$. $\square$

Lemma 4.6. *Let $\alpha = (\alpha_i)_i \in \widehat{F[x]}$ such that there is some $N \in \mathbb{N}$ such that $\deg(\alpha_i) < N$ for all $i \in \omega$. Then, $\alpha \in F[x]\cdot$.*

Proof. Let $n \geq N$ and $f \in P_n^1$. So, $x^n\alpha_n + \alpha(f^-) \equiv 0 \pmod f$. Pick $f = x^n + 1$. Then, $x^n + 1$ divides $x^n\alpha_n + \alpha_0 = (x^n+1)\alpha_n + \alpha_0 - \alpha_n$. Since $\deg(\alpha_0 - \alpha_n) < N$, we infer that $\alpha_0 = \alpha_n$ for all $n \geq N$. Subtracting $\alpha_0\cdot$ from α, we may assume that $\alpha_n = 0$ for all $n \geq N$. Fix some j with $0 \leq j < N$, and let $f = x^n + x^j$ where $n > N + j$. Then, $\alpha(f) = x^j\alpha_j \equiv 0 \pmod f$, and $\alpha_j = 0$ follows since $\deg(x^j\alpha_j) < n = \deg(f)$. This shows that $\alpha = \alpha_0\cdot$. $\square$

Lemma 4.7. *Let $f \in F[x]$ with $\deg(f) \geq 1$. Then, the centralizer, $C_{\widehat{F[x]}}(f\cdot)$, is $F[x]\cdot$.*

Proof. Let $f = \sum_{j=0}^{n} f_jx^j$ with $f_n \neq 0$ and $n \geq 1$. Let $\alpha = (\alpha_i)_i \in C_{\widehat{F[x]}}(f\cdot)$. Then, $\alpha(f\cdot) = (\sum_{j=0}^{n} f_jx^j\alpha_i)_i$, but $(f\cdot)\alpha = (\sum_{j=0}^{n} f_jx^j\alpha_{i+j})_i$, and it follows that $\sum_{j=0}^{n} f_jx^j\alpha_i = \sum_{j=0}^{n} f_jx^j\alpha_{i+j}$ for all $i \geq 0$. Without loss of generality, we may assume that $f_n = 1$. Then, $x^n\alpha_{i+n} = \sum_{j=0}^{n} f_jx^j\alpha_i - \sum_{j=0}^{n-1} f_jx^j\alpha_{i+j}$. It follows that $n + \deg(\alpha_{i+n}) \leq \max\{n +$

$\deg(\alpha_i), n-1+\deg(\alpha_{i+j}), 0 \leq j \leq n-1\}$ for all $i \geq 0$. In other words,

$$\deg(\alpha_{i+n}) \leq \max\{\deg(\alpha_i), \deg(\alpha_{i+j}) - 1, 0 \leq j \leq n-1\} \text{ for all } i \geq 0. \tag{4.1}$$

There exists some natural number N such that $\deg(\alpha_\ell) \leq N$ for all $0 \leq \ell \leq n-1$. Inequality (4.1) implies that for $i = 0$, that $\deg(\alpha_n) \leq N$ as well. Now, using induction, assume that $\deg(\alpha_j) \leq N$ for all $0 \leq j < k$. Write $k = mn + i$ for some $0 \leq i < n$. We may assume that $m \geq 1$. By inequality (4.1), we get

$$\begin{aligned} \deg(\alpha_k) &= \deg(\alpha_{(m-1)n+i+n}) \\ &\leq \max\{\deg(\alpha_{(m-1)n+i}), \deg(\alpha_{(m-1)n+i+j}) - 1, 0 \leq j \leq n-1\}. \end{aligned}$$

Note that $(m-1)n+i+j-1 \leq (m-1)n+i+n-1 = mn+i-1 = k-1 < k$ for all $0 \leq j \leq n-1$.

By the induction hypothesis, we get that all these degrees are no bigger than N, and so $\deg(\alpha_k) \leq N$ as well. Because of Lemma 4.6, we are done. $\square$

CHAPTER FIVE

Main Results about $\widehat{F[x]}$

5.1 Introduction

In this chapter, we present the results that follow from basic properties covered in Chapter Four. We first cover our preliminary lemmas that deal with elements in $\widehat{F[x]}$ and how they interact with polynomials, what their degree will be under certain circumstances, and much more. In section 5.3, we introduce our two main lemmas that lead to our main result, the main theorem itself, and some corollaries that follow. Lastly, we conclude Chapter Five with the only result we have for $\widehat{\widehat{F[x]}}^{r}$.

5.2 Preliminary Lemmas

Lemma 5.1. *If $\alpha \in P$ and $n \in \mathbb{N}$ such that $\alpha(x^n \cdot) \in \widehat{F[x]}$, then $\alpha \in \widehat{F[x]}$.*

Proof. Let $f = \sum_{j=s}^{k} f_j x^j$, $f_s \neq 0 \neq f_k$. There exists $w_f \in F[x]$ such that $fw_f = f((x^n \cdot)\alpha) = \sum_{j=s}^{k} f_j(x^j((x^n \cdot)\alpha)) = \sum_{j=s}^{k} f_j(x^{n+j}\alpha) = \sum_{j=s}^{k} f_j x^{n+j}\alpha_{n+j}$, where $f_j \in x^{n+s}F[x]$. Since x^s is the highest x-power dividing f, $x^n | w_f$ and this implies that there exists a $u_f \in F[x]$ with $w_f = x^n u_f$. This gives $fx^n u_f = x^n(\sum_{j=s}^{k} f_j x^j \alpha_{n+j}) = x^n(f\beta)$, where $\beta = (\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \dots) = (\alpha_{n+i})_i$ and $fu_f = f\beta$ for all $f \in F[x]$. Therefore, $\alpha \in \widehat{F[x]}$. $\square$

Lemma 5.2. *Let $n \in \mathbb{N}$ and $\alpha_j \in F[x]$ for all $j \geq n$ such that $(x^n \cdot)\alpha = (x^n\alpha_{n+i})_i = (x^n\alpha_n, x^n\alpha_{n+1}, x^n\alpha_{n+2}, \dots) = \gamma \in \widehat{F[x]}$. Let $\beta = (\beta_i)_i$ where $\beta_i = \alpha_{n+i}$ for all $i \geq 0$. Then, $\beta \in \widehat{F[x]}$.*

Proof. Let $f = \sum_{j=s}^{k} f_j x^j$ with $f_s \neq 0$. Then, $fw_f = f\gamma = \sum_{j=s}^{k} f_j x^j x^n \alpha_{n+j} = x^{n+s}\sum_{j=s}^{k} f_j x^{j-s}\alpha_{n+j} \in x^{n+s}F[x]$ for some $w_f \in F[x]$. This implies that $w_f = x^n u_f$

for some $u_f \in F[x]$. As a result, $fx^n u_f = x^n \sum_{j=s}^{k} f_j x^j \alpha_{n+j} = x^n(f\beta)$ and $fu_f = f\beta$ follows. Hence, $\beta \in \widehat{F[x]}$. $\square$

Remark 5.1. Lemma 5.2 shows that if $\alpha = (\alpha_i)_i \in \widehat{F[x]}$, then for any $n \in \mathbb{N}$, we have $\alpha^{\leftarrow n} := (\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \dots) \in \widehat{F[x]}$ as well. It also shows that if $\alpha = (\alpha_i)_i \in \widehat{F[x]}$ and $\alpha^{\leftarrow n} = (\alpha_{i+1})_i$, then $(x\cdot)\alpha = \alpha^{\leftarrow n}(x\cdot)$. This implies that $\widehat{F[x]}\alpha \cap \widehat{F[x]}(x\cdot) \neq \{0\}$.

Lemma 5.3. *Let $f \in F[x] - \{0\}$ and $\beta \in \widehat{F[x]}$. Then, there exists some unique $\gamma \in \widehat{F[x]}$ such that $(f\cdot)\beta = \gamma(f\cdot)$.*

Proof. Let $\beta \in \widehat{F[x]}$ and $(f\cdot)\beta = (u_i)_i$, so there exists $\gamma_i \in F[x]$ with $(x^i f)\beta = x^i f \gamma_i$. Thus, we have

$$\begin{aligned} x^i(f\cdot)\beta &= \left(\sum_{j=0}^{n} f_j x^{i+j}\right)\beta \\ &= \left(\sum_{j=0}^{n} f_j x^{i+j}\right)\gamma_i \\ &= \sum_{j=0}^{n} f_j x^{i+j}\gamma_i \\ &= x^i\left(\textstyle\sum_{j=0}^{n} f_j x^j \gamma_i\right). \end{aligned}$$

This gives $f \cdot \beta = (\sum_{j=0}^{n} f_j x^j \gamma_i)_i = (\gamma_i \cdot f)_i$. Let $\gamma = (\gamma_i)_i$. This gives $f\beta = \gamma f$.

We need to show $\gamma \in \widehat{F[x]}$, so let $g \in F[x]$. Then, $g((f\cdot)\beta) = (g\gamma)\cdot f = (gf)\beta = gfw_{gf}$ for some $w_{gf} \in F[x]$ since $\beta \in \widehat{F[x]}$. Thus, we have $(gf)w_{gf} = g(f\cdot)\beta = g(\gamma(f\cdot)) = (g\gamma)f$. Now, we may cancel f and get $g\gamma = gw_{gf}$ for all $g \in F[x]$. This shows that $\gamma \in \widehat{F[x]}$. The uniqueness of γ follows from Lemma 4.5. $\square$

Remark 5.2. Note that this shows that $(f\cdot)\widehat{F[x]} \subseteq \widehat{F[x]}(f\cdot)$ for all $f \in F[x]$.

Now, let $0 \neq f \in F[x]$. By Lemma 5.3, we have a map $\eta(f) : \widehat{F[x]} \to \widehat{F[x]}$ defined by $(f\cdot)\alpha = \alpha^{\eta(f)}(f\cdot)$ where $\alpha^{\eta(f)} = \beta$ denotes the image of α under the map $\eta(f)$.

Lemma 5.4. *Let $0 \neq f \in F[x]$. Then, $\eta(f)$ is an injective ring endomorphism of $\widehat{F[x]}$. In other words, $\eta(f) \in End(\widehat{F[x]})$.*

Proof. We will repeatedly use the fact that $\widehat{F[x]}$ has no zero divisors (Lemma 4.5). Let $\alpha, \beta \in \widehat{F[x]}$. Then, $(\alpha+\beta)^{\eta(f)}(f\cdot) = (f\cdot)(\alpha+\beta) = (f\cdot)\alpha + (f\cdot)\beta = \alpha^{\eta(f)}(f\cdot) + \beta^{\eta(f)}(f\cdot) = (\alpha^{\eta(f)} + \beta^{\eta(f)})(f\cdot)$, and the map $\eta(f)$ is additive. Also, $(\alpha\beta)^{\eta(f)}(f\cdot) = (f\cdot)(\alpha\beta) = ((f\cdot)\alpha)\beta = (\alpha^{\eta(f)}(f\cdot))\beta = \alpha^{\eta(f)}((f\cdot)\beta) = \alpha^{\eta(f)}\beta^{\eta(f)}(f\cdot)$ and this shows that $\eta(f)$ preserves multiplication as well. If $\alpha^{\eta(f)} = 0$, then $\alpha(f\cdot) = 0$, and it follows that $\alpha = 0$. $\square$

Lemma 5.5. *Given $f \in F[x]$, $\deg(f) = d$, and $n \in \mathbb{N}$, there exists $q \in F[x]$ such that $qf = x^{n+d} + g$, $\deg(g) < d$.*

Proof. Divide x^{n+d} with remainder by f, in order to get $x^{n+d} = qf + r$, $\deg r < d$. This implies $qf = -r + x^{n+d}$, $\deg(-r) < d$. $\square$

Lemma 5.6. *Let $F[x]^1$ be the monoid of all monic polynomials in $F[x]$. Then, $\eta : F[x]^1 \to End(\widehat{F[x]})$ is an injective homomorphism of monoids.*

Proof. Let $f, g \in F[x]^1$ and $\alpha \in \widehat{F[x]}$. Then, $(fg\cdot)\alpha = \alpha^{\eta(fg)}(fg\cdot) = (f\cdot)((g\cdot)\alpha) = ((g\cdot)\alpha)^{\eta(f)}(f\cdot) = (g\cdot)\alpha^{\eta(f)}(f\cdot) = (\alpha^{\eta(f)})^{\eta(g)}(g\cdot)(f\cdot)$. It follows that $\eta(fg) = \eta(gf) = \eta(f)\eta(g)$, and this shows that η is a monoid homomorphism. Now, suppose that $\eta(f) = \eta(g)$ for some $f, g \in F[x]^1$. This means that for all $\alpha \in \widehat{F[x]}$, we have $f\alpha g = \alpha^{\eta(f)}gf = \alpha^{\eta(g)}fg = g\alpha f$ for all $\alpha \in \widehat{F[x]}$. Let $f = \sum_{j=0}^{n} f_j x^j$ have degree n and $g = \sum_{j=0}^{m} g_j x^j$ have degree m. We need to construct an element $\alpha = (\alpha_i)_i \in \widehat{F[x]}$ that defeats the equation $f\alpha g = g\alpha f$ in the case where $f \neq g$. This equation means that

$$\left(\sum_{j=0}^{n} f_j \alpha_{i+j} x^j\right)_i g = \left(\sum_{j=0}^{m} g_j \alpha_{i+j} x^j\right)_i f. \tag{5.1}$$

Assume $m \neq n$. Then, we may also assume that $m > n$. Now, construct $\alpha = (\alpha_i)_i$ such that $n + m + \deg(\alpha_{i+m}) > \max\{n + m + \deg(\alpha_\ell) : 0 \leq \ell < i + m\}$. We infer that the polynomial on the right hand side of (5.1) has a degree larger than the one on the left hand side. Thus, we may assume that $m = n$. Let $\Gamma_n = \mathrm{lcm}(P_n^1)$ and $L_n = \deg(\Gamma_n) = \sum_{1\leq j\leq n} q^j = q\dfrac{q^n - 1}{q - 1} \geq q^n - 1$ [5]. We construct $\alpha_i = \beta_i + q_i \Gamma_i x^{-i}$ where $\deg(\beta_i) < L_i - i$ and $\deg(q_i) = m_i$ is such that $\alpha_{i+n} = x^{e_i} + \gamma_{i+n}$ and $e_i - n > \max\{\deg(\gamma_{i+n}), j + \deg(\alpha_{i+j}) : 0 \leq j \leq n\}$, where $e_i = \deg(\alpha_{i+n})$. This implies that $f g_n x^{n+e_i} = g f_n x^{n+e_i}$ and so $f = g$ since $f_n = 1 = g_n$. □

Definition 5.1. [10] A subset S of a ring R is called a left denominator set if it satisfies the following three conditions for every a, b in R, and s, t in S:

(1) st in S; i.e., S is multiplicatively closed.

(2) $Sa \cap Rs$ is not empty.

(3) If $as = 0$, then there is some u in S with $ua = 0$.

This definition is a left version modification of Lam's definition of a right denominator set. It is also helpful to note that if S is a left denominator set, then one can construct the ring of left fractions $S^{-1}R$ similarly to the commutative case. If S is taken to be the set of regular elements (those elements a in R such that if b in R is nonzero, then ab and ba are nonzero), then the left Ore condition is simply the requirement that S be a left denominator set [10].

Lemma 5.7. *Let $S = (F[x] - \{0\})\cdot$. Then, S is a left denominator set for $\widehat{F[x]}$.*

Proof. S is closed with respect to multiplication. By Lemma 5.3, we know that for all $a \in \widehat{F[x]}$ and all $s \in S$, $aS \cap s\widehat{F[x]} \neq \emptyset$. Lastly, if $as = 0$, this implies that $a = 0$, and property (3) from Definition 5.1 follows. □

In order to see that S is not a right denominator set, we need to develop several more results. In the process, we introduce a new topic, slowness.

Definition 5.2. $\beta \in \widehat{F[x]}$ is called slow if there exists an i_0 such that for all $i \geq i_0$, $\deg\beta_i < L_i - i$.

Lemma 5.8. *The set* $T = \{\beta \in \widehat{F[x]} : \beta \text{ slow}\}$ *is countable.*

Proof. Assume β is slow. Then, there exists an i_0 with $\deg \beta_i < L_i - i$ for all $i > i_0$. We can write $\beta = (\beta_0, \beta_1, \ldots, \beta_{i_0}, \beta_{i_0+1}, \beta_{i_0+2}, \ldots)$, and there are only countably many choices for i_0 and $\beta_0, \ldots, \beta_{i_0}$. Each β_j for $j > i_0$ is uniquely determined by $\beta_0, \ldots, \beta_{i_0}, \ldots, \beta_{j-1}$ because $x^j \beta_j = -\sum_{i=0}^{j-1} g_i x^i \beta_i \pmod{g}$, $g \in P_j^1$ and β_j is the unique solution of this system of congruences with least degree $< L_j - j$. $\square$

Lemma 5.9. *Given* $t, f \in F[x]$, $\alpha = (\alpha_i)_i, \beta = (\beta_i)_i \in \widehat{F[x]}$, *and* $\deg(f) = n \geq 1$ *such that* $\alpha t = f\beta$, *there exists some* $\kappa \in \mathbb{N}$ *such that* $deg(x^n \beta_{n+i}) \leq max\{\kappa, deg(\alpha_j t), 0 \leq j \leq i\}$ *for all* $i \geq 0$.

Proof. We prove this by induction. When $i = 0$, we have $\alpha_0 t = \sum_{j=0}^{n} f_j x^j \beta_j$ and this implies $x^n \beta_n = \alpha_0 t - \sum_{j=0}^{n-1} f_j x^j \beta_j$. Let $\kappa = \max\{\deg(x^n \beta_j) : 0 \leq j \leq n-1\}$. Then, $\deg(x^n \beta_n) \leq \max\{\deg(\alpha_0 t), \kappa\}$.

Our induction hypothesis is $\deg(x^n \beta_{n+j}) \leq \max\{\kappa, \deg(\alpha_0 t), \ldots, \deg(\alpha_j t)\}$ for all $0 \leq j < i$. Then, we have $x^n \beta_{n+i} = \alpha_i t - \sum_{j=0}^{n-1} f_j x^j \beta_{i+j}$, as a result,

$$\begin{aligned}
\deg(x^n \beta_{n+i}) &\leq \max\{\deg(\alpha_i t), \deg(x^j \beta_{i+j}), 0 \leq j \leq n-1\} \\
&\leq \max\{\deg(\alpha_i t), \deg(x^n \beta_{i+j}), 0 \leq j \leq n-1\} \\
&= \max\{\deg(\alpha_i t), \deg(x^n \beta_{i+j}), 0 \leq i+j < n, \deg(x^n \beta_{i+j}), i+j \geq n\} \\
&\leq \max\{\deg(\alpha_i t), \deg(x^n \beta_a), 0 \leq a < n, \deg(x^n \beta_{n+b}), 0 \leq b \leq i-1\} \\
&\leq \max\{\deg(\alpha_i t), \kappa, \deg(x^n \beta_{n+b}), 0 \leq b \leq i-1\} \\
&\leq \max\{\deg(\alpha_i t), \kappa, \deg(\alpha_0 t), \ldots, \deg(\alpha_{i-1} t)\} \\
&= \max\{\kappa, \deg(\alpha_b t), 0 \leq b \leq i\}.
\end{aligned}$$

$\square$

Lemma 5.10. *Given* $f \in F[x]$ *and* $\deg f = n \geq 1$*, there exists an uncountable subset* $W^{(f)} \subseteq W_n$ *where* $W_n = \{\alpha = (\alpha_i) \in \widehat{F[x]} : \forall N\ \exists i_0\ \forall i > i_0,\ (deg(\alpha_i) + N < L_{i+n} - (i+n))\}$ *such that* $\alpha S \cap f\widehat{F[x]} = \emptyset$ *for all* $\alpha \in W^{(f)}$.

Proof. By way of contradiction, assume that for each $\alpha \in W_n$ there exists $t^{(\alpha)} \in S$ with the property

$$\alpha t^{(\alpha)} \in f\widehat{F[x]}. \tag{5.2}$$

Pick $t^{(\alpha)} = t$ of the least degree that satisfies Property (5.2). We know that t is unique, for if t' is another such polynomial, then $\alpha(t - t') \in f\widehat{F[x]}$ and $\deg(t - t') < \deg(t)$. Thus, $t = t'$ because otherwise we could make $t - t'$ monic and contradict the minimality of $\deg t$. Therefore, $t^{(\alpha)} = t$ is unique. This implies that $\alpha \mapsto t^{(\alpha)}$ is a function from W_n into the countable set S. Thus, there exists $t \in S$ such that $W_n^{\#} = \{\alpha \in W_n : t^{(\alpha)} = t\}$ is uncountable. By Property (5.2), there exists $\beta^{(\alpha)} \in \widehat{F[x]}$ with $\alpha t = f\beta^{(\alpha)}$ for all $\alpha \in W_n^{\#}$. This implies that $\alpha \mapsto \beta^{(\alpha)}$ is a map.

Let β' be such that $f\beta^{(\alpha)} = f\beta'^{(\alpha)}$. Then, $f(\beta^{(\alpha)} - \beta'^{(\alpha)}) = 0$ and so $\beta^{(\alpha)} = \beta'^{(\alpha)}$. If $\beta^{(\alpha_1)} = \beta^{(\alpha_2)}$, then $\alpha_1 t = \alpha_2 t$, giving $(\alpha_1 - \alpha_2)t = 0$ and thus $\alpha_1 = \alpha_2$. This implies that $\alpha \mapsto \beta^{(\alpha)}$ is an injective map.

We have from Lemma 5.9 that $\beta^{(\alpha)}$ is slow. However, this is a contradiction. Now, W_1 is uncountable. Let $\alpha^{(i)}$ be a set of representatives for the equivalence class of $\sim$, $i < 2^{\aleph_0}$, where $\sim$ is defined by $\alpha \sim \beta$ if there exists $f, g \in F[x] - \{0\}$ with $\alpha f = g\beta$. The equation $\alpha^{(i)} t^{(i)} = f\beta^{(i)}$, where $t^{(i)}$ is of minimum degree, defines a function $i \mapsto \beta^{(i)}$. This function is injective because $\alpha^{(i)} \mid \widetilde{\alpha}^{(i)}$ for $i \neq j < 2^{\aleph_0}$. Thus, we have a contradiction. Therefore, there exist many $\alpha \in W_1$ with $\alpha S \cap f\widehat{F[x]} = \emptyset$. □

Remark 5.3. Note that by Lemma 5.10, it has now been shown that S is not a right denominator set of $\widehat{F[x]}$.

Lemma 5.11. *Let* $f \in F[x]^1$ *with* $f(0) = 0$. *Then,* $\eta(f)$ *is an injective but not surjective ring endomorphism of* $\widehat{F[x]}$.

Proof. Let $\alpha = (\alpha_i)_i \in \widehat{F[x]}$. By Lemma 5.3, we have $(x\cdot)\alpha = \alpha^{\leftarrow}(x\cdot)$ where $\alpha^{\leftarrow} = (\alpha_{i+1})_i \in \widehat{F[x]}$ is the "left shift" of α. Now, assume that $\deg(\alpha_i) < L_{i+1} - (i+1)$. There are uncountably many such elements. Then, if there exist some $\gamma_0 \in F[x]$ such that $\beta = (\gamma_0, \alpha_0, \alpha_1, \alpha_2, \dots) \in \widehat{F[x]}$ is a "right shift" of α, then β is slow and there are only countably many of those elements. This implies that the ring endomorphism $\eta(x) =^{\leftarrow}$ is not surjective. Now, let $f = \sum_{j=1}^{n} f_j x^j \in F[x]^1$ be a polynomial without a constant term. Then, $\eta(f) = \sum_{j=1}^{n} f_j \eta(x)^j = \eta(x)(\sum_{j=1}^{n} f_j \eta(x)^{j-1})$ is not surjective because $\eta(x)$ is not. □

Lemma 5.12. *If* $f \in F[x]$, *then the left ideal* $\widehat{F[x]}(\cdot f)$ *is also an ideal of* $\widehat{F[x]}$.

Proof. $\widehat{F[x]}(\cdot f)$ is a left ideal. Let $\alpha, \beta \in \widehat{F[x]}$. Then, $\alpha f \beta = \alpha \beta^{\eta(f)} f \in \widehat{F[x]}(\cdot f)$. Therefore, $\widehat{F[x]}(\cdot f) \trianglelefteq \widehat{F[x]}$. □

Lemma 5.13. *The right ideal* $x^n\widehat{F[x]}$ *is only a right ideal.*

Proof. If $x^n\widehat{F[x]} \trianglelefteq_\ell \widehat{F[x]}$, then $\widehat{F[x]}x^n \subseteq x^n\widehat{F[x]}$. This means that for all $\alpha \in \widehat{F[x]}$, we have $\alpha x^n = x^n\beta$ so $\alpha = \beta^{\leftarrow n}$. This would imply that ${}^{\leftarrow n}$ is surjective, but ${}^{\leftarrow 1}$ is not surjective. However, this implies that $\mathrm{Im}({}^{\leftarrow n}) \subseteq \mathrm{Im}({}^{\leftarrow 1}) \subsetneq \widehat{F[x]}$. This is a contradiction. □

5.3 Main Result

Before stating our main result, we state and prove two lemmas. When used together along with a slight substitution, Lemmas 5.14 and 5.15 directly give the proof of our main result, Theorem 5.1. These two lemmas give us a result about whether $\widehat{F[x]}$ is an Ore domain as well. This result is stated after our main result in order not to break the line of thought that leads to the theorem itself.

Lemma 5.14. *Given* $\alpha = (\alpha_i)_i \in \widehat{F[x]}$ *and* $a(i) = \deg(\alpha_i)$ *such that the* $a(i+1) \geq a(i)$ *for all* i*,* $\beta = (\beta_i)_i \in \widehat{F[x]}$ *with* $b_i = \deg(\beta_i)$*. Then,* $\beta\alpha = (u_i)_i$ *and deg*$(u_i) = b_i + a(i+b_i)$ *unless* $\beta_i = 0$*. If* $\beta_i = 0$*, then* $u_i = 0$*.*

Proof. Recall that $u_{it} = \sum_{j=0}^{t} \beta_{ij}\alpha_{i+j,t-j}$. Also, $a(i+b_i) \geq a(i+j)$ for all $0 \leq j \leq b_i$ and for all i. This implies that $b_i + a(i+b_i) \geq j + a(i+j)$ for all $0 \leq j \leq b_i$. So, if $t > b_i + a(i+b_i)$, we have $t > j + a(i+j)$ for all $0 \leq j \leq b_i$. This implies that $u_{it} = \sum_{j=0}^{b_i} \beta_{ij}\alpha_{i+j,t-j}$ and $\alpha_{i+j,t-j} = 0$ for $t-j > a(i+j)$ if and only if $t > j+a(i+j)$. However, $t > b_i + a(i+b_i) \geq j + a(i+j)$. Therefore, $\deg(u_i) \leq b_i + a(i+b_i)$. So, $u_{i,b_i+a(i+b_i)} = \sum_{j=0}^{b_i} \beta_{ij}\alpha_{i+j,b_i+a(i+b_i)-j}$.

Let $0 \leq j < b_i$. Then, $b_i + a(i+b_i) - j > b_i + a(i+b_i) - b_i = a(i+b_i) \geq a(i+j) = \deg(\alpha_{i+j})$. As a result, $u_{i,b_i+a(i+b_i)} = \underbrace{\beta_{i,b_i}}_{\neq 0} \cdot \underbrace{\alpha_{i+b_i,a(i+b_i)}}_{\neq 0}$. Therefore, $\deg(u_i) = b_i + a(i+b_i)$. ☐

Definition 5.3. Define $(\widehat{\widehat{F[x]}})^{\ell} = \{\varphi \in \mathrm{End}_F(\widehat{F[x]}) : \varphi(J) \subseteq J \text{ all } J \trianglelefteq_{\ell} \widehat{F[x]}\}$.

Lemma 5.15. *Let* $a(n), n \geq 0$*, be a strictly increasing sequence of natural numbers such that* $a(n) \geq \delta_n = L_n - n$ *for all* $n \geq 0$*. Let* $\alpha = (\alpha_i)_i, \alpha' = (\alpha'_i)_i \in \widehat{F[x]}$ *such that deg*$(\alpha_i) = a(i)$*, and deg*$(\alpha'_i) = a(i)+1$*. Then,* $\widehat{F[x]}\alpha \cap \widehat{F[x]}\alpha' = \{0\}$*.*

Proof. Assume that there exists $\beta = (\beta_i)_i$ and $\beta' = (\beta'_i)_i$ in $\widehat{F[x]}$ with $\alpha\beta = \alpha'\beta' = (u_i)_i$. Let $b_i = \deg(\beta_i)$ and $b'_i = \deg(\beta'_i)$ for all $i > 0$. Then, by Lemma 5.14, we have $\deg(u_i)_i = b_i + a(i+b_i) = b'_i + a'(i+b'_i) = b'_i + a(i+b'_i) + 1$. Assume $b_i > b'_i$. Then, $a(i+b_i) - a(i+b'_i) = b'_i - b_i + 1 \leq 0$, but this is not possible since the $a(n)$'s are strictly increasing. On the other hand, if $b'_i > b_i$, then $a(i+b'_i) - a(i+b_i) = b_i - b'_i - 1 < 0$. This is another contradiction and we infer $b_i = b'_i$ for all $i \geq 0$ for which $\beta_i \neq 0$. This implies that $a(i+b_i) = a(i+b_i) + 1$, and this is clearly impossible, so we infer that $\beta = 0$. ☐

Theorem 5.1. $\widehat{(\widehat{F[x]})}^{\ell} = (\widehat{F[x]})\cdot$.

Proof. Let $\psi \in \widehat{(\widehat{F[x]})}^{\ell}$ and $\gamma = (\gamma_i)_i \in \widehat{F[x]}$. Pick elements $\alpha = (\alpha_i)_i, \alpha' = (\alpha'_i)_i \in \widehat{F[x]}$ as in Lemma 5.15 such that $\deg(\gamma_i) < \deg(\alpha_i)$ for all $i > 0$. Then, $\deg(\alpha_i) = \deg(\alpha_i + \gamma_i)$, and by the lemma, we have that $\widehat{F[x]}\alpha \cap \widehat{F[x]}\alpha' = \{0\}$ and $\widehat{F[x]}(\alpha+\gamma) \cap \widehat{F[x]}\alpha' = \{0\}$. There exist elements $u_\alpha, u_{\alpha'}, u_{\alpha+\alpha'} \in \widehat{F[x]}$ such that $\psi(\alpha) = u_\alpha \alpha, \psi(\alpha') = u_{\alpha'}\alpha'$, and $\psi(\alpha+\alpha') = u_{\alpha+\alpha'}(\alpha+\alpha') = \psi(\alpha) + \psi(\alpha') = u_\alpha\alpha + u_{\alpha'}\alpha'$, and we infer $(u_{\alpha+\alpha'} - u_\alpha)\alpha = (u_{\alpha'} - u_{\alpha+\alpha'})\alpha' \in \widehat{F[x]}\alpha \cap \widehat{F[x]}\alpha' = \{0\}$ by the lemma. Since $\widehat{F[x]}$ has no zero divisors, we get $u_\alpha = u_{\alpha+\alpha'} = u_{\alpha'} =: u$. Now, replace α by $\widetilde{\alpha} = \alpha + \gamma$. Then, $\widehat{F[x]}\widetilde{\alpha} \cap \widehat{F[x]}\alpha' = \{0\}$ as well. The same argument shows that $\psi(\widetilde{\alpha}) = u_{\widetilde{\alpha}}\widetilde{\alpha} = u_{\alpha'}\widetilde{\alpha} = u_\alpha\widetilde{\alpha} = u_\alpha\alpha + u_\alpha\gamma$, and this is the same as $\psi(\alpha+\gamma) = u_\alpha\alpha + \psi(\gamma)$; thus $\psi(\gamma) = u_\alpha\gamma$. Now, consider $1 = (1)_i$, the identity element of $\widehat{F[x]}$. Letting 1 play the role of γ, we get that $\psi(1) = u_\alpha 1 = u_\alpha$, and so $\psi(\gamma) = u_\gamma\gamma = \psi(1)\cdot\gamma$. This shows that $\psi = \psi(1)\cdot \in (\widehat{F[x]})\cdot$. $\square$

Before stating our result on whether $\widehat{F[x]}$ is an Ore domain, we should define what an Ore domain is.

Definition 5.4. A ring R is a left Ore domain if and only the set S of nonzero divisors is mulitplicative and has the property that for $a \in R$ and $s \in S$, $Sa \cap Rs \neq \emptyset$.

Lemma 5.16. *$\widehat{F[x]}$ is not a left Ore domain.*

Proof. Let $S = \widehat{F[x]} - \{0\}$. Suppose that $S\beta \cap S\beta' \neq \emptyset$, where $\beta, \beta' \in \widehat{F[x]}$. Since S is not equal to 0, we know $S\beta \neq 0$. By Lemma 5.15, we know $S\beta \cap S\beta' = 0$. This gives us that $\widehat{F[x]}\beta \cap \widehat{F[x]}\beta' = \{0\}$. Since we know $0 \notin S\beta$, $S\beta \cap \widehat{F[x]}\beta' = \emptyset$. Thus, $\widehat{F[x]}$ is not a left Ore domain. $\square$

Although we have not been able to prove that $\widehat{(\widehat{F[x]})}^{r} = (\widehat{F[x]})\cdot$, we have come to a conclusion about $\widehat{(\widehat{F[x]})}^{r}$ which we include below.

The algebra $\mathrm{End}_F(F[x])$ carries the finite topology which induces a topology on $\widehat{F[x]}$, which we call the finite topology of $\widehat{F[x]}$. Let $J_n = \{\alpha \in \widehat{F[x]} : \alpha(x^i) = 0 \text{ for all } 0 \leq i \leq n\}$ and $\mathcal{O} = \{J_n : n \geq 0\}$. Then $\mathcal{O}$ is a basis of the finite topology and $J_n = \{\alpha \in \widehat{F[x]} : \alpha(P_n) = 0\}$.

Lemma 5.17. *(1) $\widehat{F[x]}$ is complete in the finite topology.*

(2) The countable, additive group $T = \{\alpha \in \widehat{F[x]} : \alpha \text{ slow}\}$ is dense in $\widehat{F[x]}$.

(3) $(F[x]\cdot)$ is a closed, nowhere dense subalgebra of $\widehat{F[x]}$.

(4) Let $\Lambda \in \widehat{(\widehat{F[x]})}^r$. Then, Λ is a continuous function with respect to the finite topology of $\widehat{F[x]}$.

Proof. Let $\{\gamma_n\}_n$ be a Cauchy sequence in $\widehat{F[x]}$. We may assume that $\gamma_{n+1} - \gamma_n \in J_n$ for all $n \geq 1$. It follows that $\gamma_m - \gamma_k \in J_n$ for all $m, k \geq n$. Define $\alpha \in \mathrm{End}_F(F[x])$ by $(f)\alpha = (f)\gamma_n$ whenever $f \in P_n$. Since $\gamma_n \in \widehat{F[x]}$, assertion (1) follows. (2) follows from Lemma 3.2 and Lemma 4.1. Since each J_n is uncountable, $(F[x]\cdot)$ contains no open subset. If $\gamma_n = g_n\cdot$ is a Cauchy sequence with $g_n \in F[x]$, then $\gamma_n - \gamma_m \in J_1$ for all $n, m \geq 1$, and we have $g_n = g_n 1 = \gamma_n(1) = \gamma_m(1) = g_m 1 = g_m$. This shows that the sequence is constant and (3) follows. Let $\alpha \in J_n$. Then, $\Lambda(\alpha) \in \alpha\widehat{F[x]} \subseteq J_n$ since J_n is a right ideal of $\widehat{F[x]}$. This implies (4). $\square$

5.4 Corollaries

Corollary 5.1. *Let $\psi : \widehat{F[x]} \to \widehat{F[x]}$ be the map $\psi(\alpha) = \alpha^q$. Then, ψ is not additive.*

Proof. By way of contradiction, assume that ψ is additive. Then, $\psi \in \widehat{\widehat{F[x]}}^{\ell} = \widehat{F[x]}\cdot$ by Theorem 5.1. This implies that there exists a $\gamma \in \widehat{F[x]}$ with $\alpha\psi = \gamma\alpha$ for all $\alpha \in \widehat{F[x]}$. Consider $\alpha = x\cdot$. Then, $x^q\cdot = (x\cdot)\psi = \gamma(x\cdot) \implies ((x^{q-1})\cdot - \gamma)(x\cdot) = 0$, thus $\gamma = x^{q-1}\cdot$. We then have that $(1 + x\cdot)\psi = x^{q-1}(1 + x) = x^{q-1} + x^q$. However, $(1 + x\cdot)\psi = 1\psi + x\psi = 1^q + x^q = 1 + x^q$. This implies that $1 = x^{q-1}$. This is a contradiction to $q \geq 2$. $\square$

Corollary 5.2. *Consider the map* $\theta : \widehat{F[x]} \to \widehat{F[x]}$ *with* $\alpha\theta = \alpha^p, p = charF$. *Then,* θ *is not additive.*

Proof. Assume θ is additive and $q = p^m$. Then, $\theta^m = \psi$ and compositions of additive maps are additive. By Corollary 5.1, ψ is not additive, a contradiction. $\square$

CHAPTER SIX

Conclusion

6.1 Introduction

In conclusion, we have learned many things about the ring $\widehat{F[x]}$. With the properties that we learned about the ring as well as the behavior of its elements, there is much more that can now be pursued on the topic.

6.2 Further Results

At the end of working on these results, we were able to tie some of our research into other topics in Algebra. The results that we found are included below. We felt that the way our research tied into these concepts was too beautiful to be left out.

For this section, let F be any field and A some F-algebra. We define $\widehat{A}^{\ell} = \{\varphi \in \mathrm{End}_F(A) : \varphi(x) \in Ax \text{ for all } x \in A\}$ and $\widehat{A}^{r} = \{\varphi \in \mathrm{End}_F(A) : \varphi(x) \in xA \text{ for all } x \in A\}$. Then $\widehat{A}^{\ell}$ and $\widehat{A}^{r}$ are F-algebras with $A\cdot \subseteq \widehat{A}^{\ell}$ and $\cdot A \subseteq \widehat{A}^{r}$. Of course, if A is commutative, then $\widehat{A}^{\ell} = \widehat{A}^{r}$.

Definition 6.1. Let C be a commutative ring and A some C-algebra. Let $\langle \_|\_ \rangle : A \times A \to X$ be any bilinear map into some C-module X. The bilinear map $\langle \_|\_ \rangle$ is called zero-preserving if $\langle x|y \rangle = 0$ whenever $x, y \in A$ with $xy = 0$.

To get to our results, we used a result from a paper by Brešar, et al [1] which we will include here for convenient reference.

Lemma 6.1. *[1] Let C be a commutative ring and A some C-algebra with $1 \in A$. The following are equivalent:*

(1) If $\langle \_|\_ \rangle : A \times A \to X$ is a zero preserving C-bilinear map, then there exists some $T \in Hom_C(A, X)$ with $T(xy) = \langle x|y \rangle$ for all $x, y \in A$.

(2) If $\langle \_|\_ \rangle : A \times A \to X$ *is a zero preserving* C*-bilinear map and* $x_i, y_i \in A$, *for* $1 \leq i \leq m$, *with* $\sum_{1\leq i\leq m} x_i y_i = 0$, *then* $\sum_{1\leq i\leq m} \langle x_i | y_i \rangle = 0$.

(3) Same as (2), *but with* m *restricted to* $m = 2$.

Proof. We start by showing that (1) implies (2). If $\sum_{1\leq i\leq m} x_i y_i = 0$, then $\sum_{1\leq i\leq m} \langle x_i | y_i \rangle = \sum_{1\leq i\leq m} T(x_i y_i) = T(\sum_{1\leq i\leq m} x_i y_i) = T(0) = 0$. Note that (2) trivially implies (3). Now, we show that (3) implies (1). Consider the equation $xy + (xy)(-1) = 0$. By (3), we have $\langle x|y\rangle + \langle xy| -1\rangle = 0$. Now, define $T(z) = \langle z|1\rangle$ for all $z \in A$ and observe that $\langle x|y\rangle = T(xy)$ for all $x, y \in A$. □

Definition 6.2. [1] Let C be a commutative ring and A some C-algebra with $1 \in A$. The algebra A is called zero product determined if A satisfies one and thus all of the conditions of Lemma 6.1.

Remark 6.1. Note that if A has no zero divisors, then all bilinear maps preserve zero.

Lemma 6.2. *The following hold for any field* F*:*

(1) $F[x]$ *is not zero product determined and*

(2) $\widehat{F[x]}$ *is not zero product determined.*

Proof. (1) Assume that $F[x]$ is zero product determined. Let the bilinear map $\langle \_|\_ \rangle : F[x] \times F[x] \to F[x]$ be defined by $\langle f|g\rangle = f'g$, where $f' = \frac{d}{dx} f$. Note that $fg + g(-f) = 0$, and since $F[x]$ is zero product determined, we get $0 = \langle f|g\rangle + \langle g| - f\rangle = \langle f|g\rangle - \langle g|f\rangle$. This implies that $\langle f|g\rangle = \langle g|f\rangle$ for all f, g, i.e., all bilinear maps are symmetric. To see that this is not possible, look at $x^2 = \langle x|x^2\rangle = \langle x^2|x\rangle = 2x \cdot x = 2x^2$, which implies that $x^2 = 0$. This is a contradiction.

(2) If F is infinite, then $\widehat{F[x]} = F[x] \cdot$ by Bucker and Dugas's result [3]. Then, we are done by repeating part (1).

Assume F is finite. Now define $\langle\_|\_\rangle : F[x] \times F[x] \to F[x]$ by $\langle\alpha|\beta\rangle = \alpha^{\leftarrow}\beta$. Again, assume that $\widehat{F[x]}$ is zero product determined. Let $\alpha \in \widehat{F[x]} - F[x]$. Recall the identity $x\alpha = \alpha^{\leftarrow}x$ for all $\alpha \in \widehat{F[x]}$. Then, $x\alpha + \alpha^{\leftarrow}(-x) = 0$ implies that $\langle x|\alpha\rangle + \langle\alpha^{\leftarrow}| - x\rangle = 0$. By the fact that $\widehat{F[x]}$ is zero product determined, we get that $\langle x|\alpha\rangle + \langle\alpha^{\leftarrow}| - x\rangle = 0$, but this gives $0 = \langle x|\alpha\rangle + \langle\alpha^{\leftarrow}| - x\rangle = x^{\leftarrow}\alpha - \alpha^{\leftarrow\leftarrow}x = x\alpha - \alpha^{\leftarrow\leftarrow}x = \alpha^{\leftarrow}x - \alpha^{\leftarrow\leftarrow}x = (\alpha^{\leftarrow} - \alpha^{\leftarrow\leftarrow})x$, and this leads to $\alpha^{\leftarrow} = \alpha^{\leftarrow\leftarrow}$. This means that $\alpha_{i+1} = \alpha_{i+2}$ for all $i \geq 0$. This implies that $\alpha \in F[x]\cdot$ by Lemma 4.6, a contradiction to the choice of α. $\square$

Lemma 6.3. *Let A be a zero product determined ring with identity. Then, $\widehat{A}^{\ell} = A\cdot$ and $\widehat{A}^{r} = \cdot A$.*

Proof. Let $\alpha \in \widehat{A}^{\ell}$, and let $\langle\_|\_\rangle : A \times A \to A$ be defined by $\langle x|y\rangle = \alpha(x)y$. Then, if $xy = 0$, we have that $\langle x|y\rangle = \alpha(x)y = u_x xy = 0$. So, $\langle\_|\_\rangle$ preserves zero. Note that $1 \cdot y + y(-1) = 0$ implies that $\langle 1|y\rangle + \langle y| - 1\rangle = 0$. This gives $\alpha(1)y + \alpha(y)(-1) = 0$. In other words, $\alpha y = \alpha(1)y$ for all $y \in A$, and that means $\alpha = \alpha(1)\cdot$. The proof for $\widehat{A}^{r} = \cdot A$ is done similarly with defining $\langle x|y\rangle = x\beta(y) = xyv_y$, and so we exclude it from this epilogue. $\square$

Lemma 6.4. *Let A be an F-algebra. The following are equivalent:*

(1) $\widehat{A}^{\ell} = \widehat{A}^{r}$.

(2) $\alpha A = A\alpha$ for all $\alpha \in A$.

(3) A is a duo ring, i.e., all one-sided ideals of A are two-sided ideals of A.

Proof. Assume (1) holds. Let $a \in A$. Then, the left multiplication $a\cdot \in \widehat{A}^{\ell} = \widehat{A}^{r}$, and for each $a \in A$, there exists some $\beta_{\alpha} \in A$ with $a\alpha = \alpha\beta_{\alpha}$. It follows that $A\alpha \subseteq \alpha A$ for all $\alpha \in A$. By symmetry, we get the other inclusion and we have (2).

Clause (2) trivially implies (3). Finally, assume that (3) holds. Note that $\widehat{A}^{\ell} = \{\varphi \in \mathrm{End}_F(A) : \varphi(J) \subseteq J$ for all right ideals J of $A\}$. Thus, both sets are equal to $\{\varphi \in \mathrm{End}_F(A) : \varphi(J) \subseteq J$ for all ideals J of $A\}$. $\square$

Corollary 6.1. *Let A be a zero product determined duo ring with identity. Then, A is commutative.*

Proof. By Lemmas 6.2 and 6.3, we have $A\cdot = \widehat{A}^{\ell} = \widehat{A}^{r} = \cdot A$. We infer that for all $a \in A$ there exists some $b \in A$ such that $at = tb$ for all $t \in A$. For $t = 1$, we get $a = b$, and so $at = ta$ for all $a, t \in A$. $\square$

This concludes our extra findings. We now have an easy way to test what $\widehat{A}$ is when dealing with zero product determined rings. Hopefully these results will further other mathematicians's studies about rings in general.

6.3 Future Work

I would love to attempt to work on the right side of the proof for Theorem 5.1. We have a hunch that it works, but despite much effort and many methods of approach, we keep getting stuck on the right side. I would love to be able to fully show that $\widehat{\widehat{F[x]}}^{r} = \widehat{F[x]}$.

It would also be quite interesting to see what other characteristics can help determine $\widehat{R}$ as easily as the characteristic of being zero product determined. The way the proofs worked so beautifully makes one wonder what else might work in such a way with the concept of $\widehat{R}$.

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