ABSTRACT

Limit Sets in Finitely-Generated Free Group and Monoid Actions

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For dynamical systems with the shadowing property, the omega-limit sets can be characterized by a condition of internal chain transitivity. In this thesis, we define four limit sets in the case of finitely-generated free group and monoid actions. We give a characterization for two of these limit sets in terms of a kind of internal transitivity under the condition that the group or monoid action has an asymptotic shadowing property.

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LIMIT SETS IN FINITELY-GENERATED FREE GROUP AND MONOID ACTIONS

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CHAPTER ONE

Introduction

The contents of this thesis are situated in the mathematical theory of topological dynamics. This field originated in the work of Henri Poincaré on the 3-body problem in physics, which asks to describe the motion of three celestial objects from various initial conditions [2]. While differential equations that model the forces acting on each of the bodies are known, giving an explicit solution is a devilishly hard problem. Indeed, while the 1-body and 2-body problems have exact solutions given by elementary functions, Poincaré proved no such solution exists for the 3-body problem. Poincaré's attention therefore turned away from finding the exact motion of the bodies and instead focused on the general *shape* that the system took as time went on. Under this new consideration, Poincaré answered questions on the periodicity and asymoptotic behavior of a system in general.

Poincaré's approach was further developed in the late 1800s and into the early 1900s by French mathematician Jacques Hadamard and American mathematician George Birkoff. Hadamard's study of geodesics on surfaces with negative curvature led to the creation of symbolic dynamics, a subfield of mathematics with applications in computer science and information theory [9]. On the other hand, Birkoff brought a rigorous topological framework to the study of dynamics. Birkoff introduced the notion of a system's recurrent states: states a system returns arbitrarily close to infinitely many times [2].

This present work draws upon the influence of these two mathematicians. Here, we consider the structure of several limit sets for systems consisting of a free group or monoid action on a compact metric space. These limit sets are reminiscent of Birkoff's recurrent states, as limit sets are exactly the points to which a system approaches arbitrarily close and infinitely often. Additionally, most of the examples given of these sorts of systems are generalizations of the symbolic dynamics instituted by Hadamard.

The structure of the thesis is as follows:

In the second chapter, we develop the principles of symbolic dynamics in their classical context of shift spaces over the integers. Definitions of subshifts and shifts of finite type are introduced, and shifts of finite type are first characterized by a special ability to concatenate words. After this, we move to notions of internal chain transitivity, pseudo-orbits and shadowing on shift spaces, and shifts of finite type are this time characterized by their ability to shadow certain pseudo-orbits. Finally, we study the most basic limit sets that arise in shift spaces and characterize their structure in the context of shifts of finite type.

In the third chapter, we generalize the ideas found in the second. In particular, we expand the definitions of pseudo-orbits, shadowing, and limit sets to the case of a dynamical system (X, f) consisting of a compact space X and a single continuous function f. After this, we generalize shift spaces over the integers to shift spaces over any group or monoid. Continuing in this direction, we import the properties of these shift spaces to any compact metric space by way of group and monoid actions. In this context, we redevelop our definition of a limit set. Finally, we present previous results in describing certain limit sets for $\mathbb{Z}^d$ actions.

In the final chapter, we present new results describing the structure of certain limit sets for finitely-generated free group and monoid actions. After defining four different limit sets, we introduce analogs of internal chain transitivity and shadowing. Finally, in the context of systems with a certain shadowing property, we characterize two of these limit sets in terms of types of internal transitivity.

CHAPTER TWO

General Results on Shift Spaces over $\mathbb{Z}$

A topological dynamical system is a pair (X, f) consisting of a compact (metric) space X and a continuous function $f : X \to X$. Some examples of dynamical systems are shift spaces over the integers. These systems can be useful in simplifying other dynamical systems while still preserving essential information. Suppose we have a dynamical system (X, f) and we partition X into 5 distinct pieces, labeling them 1 through 5. We can look at a point x of X and keep track of which regions x, $f(x)$, $f^2(x), \ldots$ are in by recording an infinite word $x_0x_1x_2x_3\ldots$ where x_i corresponds to the region that $f^i(x)$ is in. This word is called the *itinerary* of x. If we create the same record for $f(x)$, the resulting word will be $x_1x_2x_3\ldots$, which is the word obtained from x shifted over one space. Let A be the set of these itineraries for $x \in X$ and $\sigma : A \to A$ be the map that shifts a word over one space. If $\phi : X \to A$ maps a point x to its associated itinerary, then by the discussion above, $\sigma \circ \phi = \phi \circ f$. In the case that ϕ is a continuous function, ϕ is a *semi-conjugacy* of dynamical systems, and ϕ respects the action of the continuous functions in both systems. If ϕ is a homeomorphism, then (X, f) and (A, σ) are said to be *conjugate*. For all intents and purposes, these dynamical systems are viewed as the same. Even if ϕ is not a homeomorphism, it will still often preserve some attributes of (X, f) in the system (A, σ). Thus, we can approximate a dynamical system by discretizing the phase space (breaking it into regions) and tracking which regions a point's orbit visits. This is a fundamental motivation for shift spaces.

In this chapter, we introduce shift spaces over the integers and introduce the notions of pseudo-orbits, shadowing, and omega-limit sets in this context. The final chapter expands these notions to a more complicated, yet related, context. Much of what follows about shift spaces can be found in Lind and Marcus's seminal book on symbolic dynamics [5]. The subjects of shadowing and omega-limit sets can be found in [3, 4, 6].

Suppose that $\mathcal{A}$ is a finite set of symbols (these symbols can be anything, but integers are most commonly used). We call this our *alphabet*, from which we construct bi-infinite words

$\ldots x_{-3}x_{-2}x_{-1}x_0x_1x_2x_3\ldots$ where every "letter" x_i comes from the alphabet $\mathcal{A}$. The set of all of these bi-infinite words is denoted $\mathcal{A}^{\mathbb{Z}}$. For notation purposes, the n^{th} letter of a word x is represented as x_n. We denote the consecutive string of letters $x_kx_{k+1}\ldots x_{n-1}x_n$ as $x_{[k,n]}$ where $k < n$. In the case that $k = -n$, this is called the *central 2n + 1-block of x.*

We can place a metric on $\mathcal{A}^{\mathbb{Z}}$ by specifying that two bi-infinite words are close together if they match on their central blocks. More formally, the distance between $x, y \in \mathcal{A}^{\mathbb{Z}}$ is

$d(x,y) = \inf\left\{1 \cup \left\{2^{-n} : x_{[-n,n]} = y_{[-n,n]}\right\}\right\}$. This says that if the longest central block that x and y agree on is from index $-n$ to n, then the distance between x and y is 2^{-n}. This satisfies all three requirements for a metric: $d(x,y) \geq 0$ with equality exactly when $x = y$, $d(x,y) = d(y,x)$ for all x, y, and $d(x,z) \leq d(x,y) + d(y,z)$ for all x, y, z. The first two properties are obviously satisfied. Suppose that the largest central block x and y agree on is from $-n$ to n and the largest central block y and z agree on is from $-m$ to m. We can assume without loss of generality that $n \geq m$. This implies x and z must agree on the central block from $-m$ to m. Therefore $d(x,y) \leq 2^{-m} < 2^{-n} + 2^{-m} = d(x,y) + d(y,z)$.

For the shift space to be a dynamical system, it must comprise of a compact metric space (which will be the set $\mathcal{A}^{\mathbb{Z}}$) and a continuous function from and into that metric space. We reserve the proof that the shift space is compact for when we prove the same for subshifts. The continuous function we use in this situation is called the shift map σ. As the name suggests, this function works by "shifting" a bi-infinite word over one space. For example, if $x = \ldots x_{-3}x_{-2}x_{-1}x_0x_1x_2x_3\ldots$, σ shifts the center of the word from x_0 to x_1. Thus, $\sigma(x)_i = x_{i+1}$. The shift map is continuous because if two words x, y are close, then they remain close after the shift map is applied. In fact, we know that if $d(x,y) = 2^{-n}$, then $d(\sigma(x), \sigma(y)) \leq 2^{-n+1}$. This is because if $x_{[-n,n]} = y_{[-n,n]}$, then $\sigma(x)_{[-n-1,n-1]} = \sigma(y)_{[-n-1,n-1]}$.

Thus, the pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is a dynamical system. If $\mathcal{A}$ has n symbols, this dynamical system is called the *full shift on n letters*. This system is not the most interesting system to study, however. Returning to the motivation of modeling a dynamical system by way of a shift space, if the model is the full shift, this implies that a point can go from any one region to any other region; there are no restrictions. However, consider the following dynamical system $([0,1], f)$ where $f(t) = \frac{1}{2} + \frac{t}{2}$. We can model this with a shift space by dividing $[0,1]$ into 5 even pieces, $[0, \frac{1}{5}], (\frac{1}{5}, \frac{2}{5}], (\frac{2}{5}, \frac{3}{5}], (\frac{3}{5}, \frac{4}{5}], (\frac{4}{5}, 1]$ labeling them 1 through 5 respectively. However, if we look at all the possible infinite words in the shift space, we see that there is never a word with a 1 followed by a 1. In fact every 1 must be followed by a 3 as $f([0, \frac{1}{5}]) = [\frac{5}{10}, \frac{6}{10}] \subset (\frac{2}{5}, \frac{3}{5}]$. Therefore, to represent a dynamical system by way of shift spaces, we must define a way to forbid certain sequences of letters from appearing in a bi-infinite word of the shift space.

One way to do this is by having a list of forbidden blocks. A *block* is any $x_{[k,n]}$ of a bi-infinite word x where $k < n$. If a block w is m letters long, it is called an *m-block* and we write $|w| = m$. Now suppose we have a collection of forbidden blocks $\mathcal{F}$. We can use this to define a shift space that is a subsystem of $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ by first restricting to a subset of $\mathcal{A}^{\mathbb{Z}}$. This subset we denote $X_{\mathcal{F}}$ and define as the set of all bi-infinite words in $\mathcal{A}^{\mathbb{Z}}$ that do not contain a forbidden block (i.e. a block $w \in \mathcal{F}$). In this case, we see that σ is still a map from $X_{\mathcal{F}}$ to $X_{\mathcal{F}}$, as if $x \in X_{\mathcal{F}}$, x must not contain any forbidden block, so shifting x still means it contains no forbidden block. Therefore,

$\sigma(x) \in X_{\mathcal{F}}$.

It remains to see that $X_{\mathcal{F}}$ is compact. We show this by proving every infinite sequence in $X_{\mathcal{F}}$ converges to a point in $X_{\mathcal{F}}$. The conventional way to do this is through a diagonal argument. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of $X_{\mathcal{F}}$. As the alphabet $\mathcal{A}$ is finite, there must be a letter a_0 such that infinitely many of the x_n have a_0 as a central letter. We then pass to a subsequence $\{x_n^0\}_{n=1}^{\infty}$ consisting of all points with a_0 as the central letter. There are only finitely many blocks of three letters, so there must be a block $a_{-1}a_0a_1$ that is the central 3-block of infinitely many x_n^0. Again, we pass to a subsequence $\{x_n^1\}_{n=1}^{\infty}$ of all elements with $a_{-1}a_0a_1$ as the central 3-block. We continue this process inductively, finding a central 2n+1-block that appears infinitely often in a subsequence and then passing to the subsequence of all points with this as the central 2n+1-block and repeating the process for larger central blocks. In the end, we have arbitrarily long central 2n+1-blocks that all appear in $X_{\mathcal{F}}$. These blocks are also nested inside of each other. If we define $y \in \mathcal{A}^{\mathbb{Z}}$ so that $y_{[-n,n]}$ is the 2n+1-block we obtained, we see that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to y and that every block in y appears in a central block of some $x \in X_{\mathcal{F}}$ and thus as a non-forbidden block of $X_{\mathcal{F}}$. This implies that $y \in X_{\mathcal{F}}$ and therefore $X_{\mathcal{F}}$ is compact. Hence, we have a dynamical system $(X_{\mathcal{F}}, \sigma)$ that allows us to forbid certain patterns of letters from appearing. We call this a *subshift* of $\mathcal{A}^{\mathbb{Z}}$.

Shifts of Finite Type

If the set of forbidden blocks $\mathcal{F}$ is finite, the shift space exhibits some nice properties that allow us to easily construct words that are in the shift space. In this case, the shift space is said to be a *shift of finite type* (SFT). We should note that even if two sets of forbidden words $\mathcal{F}, \mathcal{F}'$ are different, their respective shift spaces $X_{\mathcal{F}}$ and $X_{\mathcal{F}'}$ may be equal. Consider for example $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{F} = \{12\}$, and $\mathcal{F}' = \{12, 121, 122, 123\}$. In this case, $\mathcal{F} \neq \mathcal{F}'$ but $X_{\mathcal{F}} = X_{\mathcal{F}'}$, the reason being that forbidding words $121, 122$, and 123 does not forbid any more words than forbidding 12. Thus, we ought to say that a subshift $X_{\mathcal{F}}$ is an SFT if $X_{\mathcal{F}} = X_{\mathcal{F}'}$ where $\mathcal{F}'$ is some finite set of forbidden blocks.

Perhaps the most important thing about an SFT is that it can be viewed as having a maximum size of minimal forbidden blocks. Therefore, in order to check if a bi-infinite word contains a forbidden block, it suffices to check only blocks of a certain length. We call an SFT with the largest forbidden block of size m an *m-step SFT*. This fact gives us the following property of shifts of finite type.

Theorem 2.1. *If X is an m-step SFT and uw, wv appear as blocks in some (not necessarily the same) element of X with $|w| \geq m$, then uwv appears as a block in some element of X.*

Proof. We prove this by constructing an element of X that contains the block uwv. As $uw = y_{[n,n+m+k-1]}$ for some $y \in X$ we can assume without loss of generality that $uw = y_{[0,m+k-1]}$. We can also assume that $wv = z_{[k,l]}$ for some $z \in X$. The result comes down to gluing the left part of y and the right part of z together at the block w. Thus, define x by $x_i = y_i$ for $i < k$ and $x_i = z_i$ for $i \geq k$. Note that for $k \leq i \leq l = k + m - 1$ $y_i = z_i$, as this is the block at which w occurs.

We claim that $x \in X$, and we prove this by showing that x contains no forbidden blocks. As X is by assumption an m-step SFT, we do this by showing that each m-block $x_{[i,i+m-1]}$ is not a forbidden block. By our definition of x, for $i < k$ the block $x_{[i,i+m-1]}$ is the block $y_{[i,i+m-1]}$. As $y \in X$, this block is not forbidden. For $i \geq k$ the block $x_{[i,i+m-1]}$ is the block $z_{[i,i+m-1]}$. As $z \in X$ this block is not forbidden. Therefore no block of x is forbidden, so $x \in X$. As uwv is a block of $x \in X$, the theorem is proven. □

We can also get the converse of this statement: any subshift with this property is an SFT. This allows us to completely characterize SFTs by this constructive property.

In order to prove the converse, it is first necessary to consider what it means for a subshift to *not* be an SFT. For this we use the contrapositive of our definition of an SFT. Thus, if X is not an SFT, then for every $\mathcal{F}$ with $X = X_{\mathcal{F}}$, $\mathcal{F}$ is infinite. This is equivalent to: If X is not an SFT, then for every positive integer n there is a forbidden block w of X with $|w| > n$ such that every subblock of w is *not* forbidden. We explain this fact by the following informal argument. Suppose X is not an SFT and $X_{\mathcal{F}} = X$. $\mathcal{F}$ is infinite, so it must contain arbitrarily long forbidden words, as there are only finitely many words with length less than n. Choose an arbitrary positive integer n and look at all blocks $w \in \mathcal{F}$ with $|w| > n$. If w already contains a forbidden word u, it does not forbid any new blocks of the shift space. Therefore we can remove w from $\mathcal{F}$, and the resulting set $\mathcal{F}'$ will still yield the same subshift. If this is true for all $w \in \mathcal{F}$ with $|w| > n$, then we reduce $\mathcal{F}$ to a finite set $\mathcal{F}'$ with $X_{\mathcal{F}} = X_{\mathcal{F}'}$. However, this is not the case, so there must be some block $w \in \mathcal{F}$ with $|w| > n$ that contains no forbidden word except itself. We use this fact to prove the following result which will get us a nice characterization of shifts of finite type.

Theorem 2.2. *If there is a positive integer m such that whenever uw, wv appear as blocks in some (not necessarily the same) element of X with $|w| \geq m$, then uwv appears as a block in some element of X, then X is a SFT.*

Proof. We prove the contrapostive. Suppose that X is not an SFT. By our discussion above, this means that for every integer m there is some forbidden block of X such that each of its subwords is not forbidden. Now suppose for contradiction that there is some positive integer m such that if uw, wv appear as blocks in some (not necessarily the same) element of X with $|w| \geq m$ then uwv appears as a block in some element of X. By the fact that X is not an SFT we can choose a forbidden block b of X such that every subword of b is not forbidden and $|b| = k > m + 2$.

Let $u = b_0$, $w = b_{[1,k-2]}$, and $v = b_{k-1}$. Then $|w| = k - 2 > m$, and as uw and wv are subwords of b, they by assumption appear as blocks in X. However, uwv

is forbidden, so it never appears as a block in X. This contradicts our assumption, proving the theorem. □

Corollary 2.3. *A shift space X is a shift of finite type if and only if there is some positive integer m such that if uw and wv are blocks appearing in X with $|w| \geq m$, then uwv is a block appearing in X.*

This characterization of SFTs is important in proving another characterization of these subshifts involving shadowing.

Shadowing in Shift Spaces

Two related and notable notions in dynamical systems are pseudo-orbits and shadowing. Pseudo-orbits relate to how an orbit of a point appears when calculating with finite precision. Thus there is some error occurring between each iteration of the orbit. If the measure of the error is always bounded by some number (call it δ), the pseudo-orbit is called a *δ-pseudo-orbit*. Shadowing refers to how closely we can find an actual point of the space to follow the pseudo-orbit. If the distance between the orbit of a point and a pseudo-orbit are always within some length ϵ, we say that the point *ϵ-shadows* the pseudo-orbit. A dynamical system has the *shadowing property* if for every $\epsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo-orbit can be ϵ-shadowed by a point in the system. This is a very nice property for a dynamical system to have. In some sense, the shadowing property means that a computer approximation modeling the orbit of a point of the system actually models the orbit of a point (though perhaps not the intended point) in the system to some degree of precision. After formalizing these ideas in shift spaces, we will show that the subshifts that have the shadowing property are exactly the shifts of finite type.

Definition 2.4. Given $\delta > 0$, a *δ-pseudo-orbit* of a shift space X is a function $\mathcal{O} : \mathbb{N} \to X$ such that for every $i \in \mathbb{N}$, $d(\sigma(\mathcal{O}(i)), \mathcal{O}(i+1)) < \delta$.

Definition 2.5. Given $\epsilon > 0$ and a pseudo-orbit $\mathcal{O}$, a point $x \in X$ *ϵ-shadows* $\mathcal{O}$ if for every $n \in \mathbb{N}$ $d(\sigma^n(x), \mathcal{O}(n)) < \epsilon$.

Definition 2.6. A shift space X has the *shadowing property* if for every $\epsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo-orbit can be ϵ-shadowed by an element of X.

In the case of shift spaces over $\mathbb{Z}$ there is a nice way to visualize ϵ-shadowing (although this method breaks down in the last chapter when we generalize to shift spaces over free groups). Based on the way we defined the metric in a shift space, we can rephrase any requirement on the distance between two elements as a requirement on how big of a central block they have in common. Namely, specifying that x, y be distance less than δ apart is the same as requiring $x_{[-m,m]} = y_{[-m,m]}$ where m is the smallest integer so that $2^{-m} < \delta$. We will use this method of specifying distance throughout the rest of this section for simplicity. Suppose that we have a pseudo-orbit and have a point x that shadows the pseudo-orbit so that $\sigma^n(x)$ and $\mathcal{O}(n)$ have the same central 5-block.

$$\begin{array}{l}
\ldots 01002101\boxed{00\bar{1}02}1212020012\ldots \\
\quad\ldots 21102010\boxed{01\bar{0}21}2121020120\ldots \\
\qquad\ldots 00200000\boxed{10\bar{2}12}1012121111\ldots \\
\qquad\quad\ldots 12122020\boxed{02\bar{1}21}0100102012\ldots \\
\qquad\qquad\ldots 02001022\boxed{21\bar{2}10}1021110121\ldots
\end{array}$$

In this diagram, each line represents an element of the pseudo-orbit, starting with $\mathcal{O}(0)$ at the top and ending with $\mathcal{O}(4)$ at the bottom. The overlined number represents the central letter of each infinite word, and the boxed numbers represent the blocks with which $\sigma^n(x)$ must match. In this case, if x were to shadow this pseudo-orbit, $x_{[-2,6]}$ must be 001021210.

We note two things with this diagram. First, each pair of boxed blocks must agree on their overlap. If not, there would be no way a point could shadow the pseudo-orbit with this degree of precision; it would need two different symbols in the same location. Second, we note that the central block of the shadowing point is entirely determined by the central block of $\mathcal{O}(0)$, and each consecutive element of the pseudo-orbit defines the next letter that follows. Therefore, we know that if x ϵ-shadows a δ-pseudo-orbit with $\delta \leq \epsilon$ and ϵ is equivalent to necessitating that $\sigma^n(x)_{[-m,m]} = \mathcal{O}(n)_{[-m,m]}$, then $x_{[-m,\infty]}$ must be defined by $x_{[-m,m]} = \mathcal{O}(n)_{-m,m}$ and for $k > m$, $x_{m+k} = \mathcal{O}(k)_m$. The left side $x_{[-\infty,-m-1]}$ does not matter in shadowing; however, it is usually easiest to define the left side so as to match that of $\mathcal{O}(0)$. We will follow this convention.

It seems at first glance that every shift space has the shadowing property. For every $\epsilon > 0$ choose $\delta < \epsilon$, and then every δ-pseudo-orbit can be ϵ-shadowed. We even defined what that shadowing point must be. However, this misses a subtlety in the definition of the shadowing property for a shift space X: the shadowing point must itself be a point of the shift space X. Therefore, even though we know what the shadowing point *must* be, we do not know a priori if it contains any forbidden blocks of X. It is possible that the δ-pseudo-orbit necessitates the shadowing point to contain such a forbidden block. However, in the case of the full shift, there are no forbidden blocks, so this system does have the shadowing property.

In what follows, we will show that for shifts of finite type, we can show that the shadowing point will never contain a forbidden word, so long as we choose δ wisely. Furthermore, we can show that if X is not a shift of finite type, no matter what δ we choose, we can find a δ-pseudo-orbit that forces the shadowing point to contain a forbidden word. Therefore, we can completely characterize shifts of finite type as those which have the shadowing property.

Theorem 2.7. *A shift space X is a shift of finite type if and only if X has the shadowing property.*

Proof. Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that every δ-pseudo-orbit of X can be ϵ-shadowed by a point $x \in X$. Importantly, we must also choose δ so that x contains no forbidden words. We know that if X is a shift of finite type, we can choose a finite set of forbidden words for X, and thus there is a maximum size of a forbidden word, say m. The crux of the problem lies in noticing that if ϵ is such that $\sigma^n(x)_{[-m,m]}$ must be $\mathcal{O}(n)_{[-m,m]}$, then every $2m+1$-block of x is a block in $\mathcal{O}(n) \in X$, and thus contains no forbidden words. Therefore, if ϵ is small enough to require that $\sigma^n(x)_{[-m,m]}$ must be $\mathcal{O}(n)_{[-m,m]}$, then we can choose any arbitrary $0 < \delta \leq \epsilon$ and any δ-pseudo-orbit can be ϵ-shadowed. If ϵ is not small enough, we can find an $\epsilon' < 2^{-m} < \epsilon$ that is and a $0 < \delta < \epsilon'$. Then any δ-pseudo-orbit can be ϵ'-shadowed, and if it is ϵ'-shadowed, it is also ϵ-shadowed.

To prove the other direction, we recall the fact that if X is not an SFT we can find an arbitrarily long forbidden word with every subword not forbidden. Let $\epsilon = 1$. This means that if x ϵ-shadows a pseudo-orbit $\mathcal{O}$ then $\sigma^n(x)_0 = \mathcal{O}(n)_0$. Let $\delta > 0$ be given and say this requires that $\sigma(\mathcal{O}(i))_{[-m,m]} = \mathcal{O}(i+1)_{[-m,m]}$. This is equivalent to $\mathcal{O}(i)_{[-m+1,m+1]} = \mathcal{O}(i+1)_{[-m,m]}$. We want to construct a δ-pseudo-orbit that forces a forbidden word in any point that shadows.

First, note that for $0 \leq k \leq m$, $\mathcal{O}(n)_k = \mathcal{O}(n+k)_0$. This can be seen by repeatedly using that fact that $\mathcal{O}(n)_{[-m+1,m+1]} = \mathcal{O}(n+1)_{[-m,m]}$, so $\mathcal{O}(n)_k = \mathcal{O}(n-1)_{k-1} = \cdots = \mathcal{O}(n+k-1)_1 = \mathcal{O}(n+k)_0$. Therefore $\mathcal{O}(n)_{[0,m]} = \mathcal{O}(n)_0\mathcal{O}(n+1)_0 \ldots \mathcal{O}(n+m)_0 = x_{[n,n+m]}$. Hence, every m-block in the shadowing point is an m-block of an element in X. This means that if we want to force the shadowing to contain a forbidden word, it must contain one that is larger than m.

Choose w to be a forbidden word of X so that $|w| = k > 2m+3$ and so that every subword of w is not forbidden. In order to force the shadowing point x to contain

w, say at the block $x_{[0,k-1]}$, we need to define $w_i = \mathcal{O}(i)_0$ for $0 \leq i < k$. We also need to make sure there is sufficient overlap between $\mathcal{O}(i)$ and $\mathcal{O}(i+1)$. One way to achieve the overlap is just to let $\mathcal{O}(i+1) = \sigma(\mathcal{O}(i))$. However, this will imply that w is a word in some element of X. Thus, at some point we must change $\mathcal{O}(i+1)$ more drastically while still having overlap. One idea is to break w into three parts $uw'v$ so that $|w'| \geq 2m+1$ and u and v are single letters. Then uw' is a valid word, as is $w'v$. Therefore we can find words of X, y, z so that $y_{[0,k-2]} = uw'$ and $z_{[1,k-1]} = w'z$. We index in this way so that $y_{[1,k-2]} = w' = z_{[1,k-2]}$; this constitutes enough overlap for us to use. Now define $\mathcal{O}(i) = \sigma^i(y)$ for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$ and $\mathcal{O}(i) = \sigma^i(z)$ for $i > \lfloor \frac{k}{2} \rfloor$. We claim that this constitutes a δ-pseudo-orbit and thus that $d(\sigma(\mathcal{O}(i)), \mathcal{O}(i+1)) < \delta$. For $i \neq \lfloor \frac{k}{2} \rfloor$, this true as $d(\sigma(\mathcal{O}(i)), \mathcal{O}(i+1)) = 0$. For $i = \lfloor \frac{k}{2} \rfloor = n$, $\sigma(\mathcal{O}(i))_{[-n,k-n-2]} = \sigma^{n+1}(y)_{[-n,k-n-2]} = y_{[1,k-1]} = z_{[1,k-1]} = \sigma^{n+1}(z)_{[-n,k-n-2]} = \mathcal{O}(i+1)_{[-n,k-n-2]}$. As $k > 2m+3$, this means $\mathcal{O}$ is a δ-pseudo-orbit. Finally, we see that for $0 \leq i \leq k$, $\mathcal{O}(i)_0 = y_i = w_i$ and for $k+1 \leq i \leq 2k+2$, $\mathcal{O}(i)_0 = \sigma^i(z)_0 = z_i = w_i$. Therefore any point the ϵ-shadows $\mathcal{O}$ must contain the forbidden word w. Hence X does not have the shadowing property. $\square$

An interesting variant of the shadowing property is the *asymptotic shadowing property*. Whereas pseudo-orbits and shadowing are expressed in terms of fixed δ and ϵ, the aysmptotic variation considers the case when the pseudo-orbits and shadowing get more precise as time goes on.

Definition 2.8. An *asymptotic pseudo-orbit* of a shift space X is a function $\mathcal{O} : \mathbb{N} \to X$ such that for every $\delta > 0$ there is an $m \in \mathbb{N}$ such that $d(\sigma(\mathcal{O}(k)), \mathcal{O}(k+1)) < \delta$ for every $k > m$.

Definition 2.9. A psuedo-orbit $\mathcal{O}$ is *asymptotically shadowed* by x if for every $\epsilon > 0$ there is an $m \in \mathbb{N}$ such that $d(\sigma^i(x), \mathcal{O}(i)) < \epsilon$ for every $i > m$.

Definition 2.10. A shift space X has the *asymptotic shadowing property* if every asymptotic pseudo-orbit of X is asymptotically shadowed by a point of X.

Theorem 2.11. *If X is a shift of finite type, then X has the asymptotic shadowing property.*

Proof. Let X be an m-step SFT and $\mathcal{O}$ be an asymptotic pseudo-orbit. Therefore, there is an $N \in \mathbb{N}$ such that for $n > N$, $d(\sigma(\mathcal{O}(n)), \mathcal{O}(n+1)) < 2^{-m}$. We can define a new asymptotic pseudo-orbit $\mathcal{P}$ by $\mathcal{P}(i) = \mathcal{O}(i+N)$. This makes $\mathcal{P}$ an asymptotic pseudo-orbit that is also an 2^{-m} pseudo-orbit. As X is an m-step SFT, there is a point $x \in X$ that 2^{-1}-shadows $\mathcal{P}$. We claim that x actually asymptotically shadows $\mathcal{P}$. Let $2^{-m} > \epsilon > 0$ be given. There is thus an $M \in \mathbb{N}$ such that $\mathcal{P}$ is a ϵ-pseudo-orbit after index M. By the properties of shadowing that we demonstrated before, x will ϵ-shadow $\mathcal{P}$ after this index. Therefore x does indeed asymptotically shadow $\mathcal{P}$, so $\sigma^{-N}(x)$ asymptotically shadows $\mathcal{O}$ and the theorem is proved. $\square$

Omega-Limit Sets and Chain Transitivity

When looking at a point in a dynamical system, one useful thing to consider is the recurrent behavior of the point: places to which a point continues to draw near after arbitrarily many applications of the continuous function. These places constitute the *omega-limit set* of a point x. We write this as $\omega(x)$ and define it in the case of shift spaces as follows.

Definition 2.12. For x in some shift space X, the *omega-limit set of* x, $\omega(x) = \bigcap_{n\in\mathbb{N}} \overline{\{\sigma^k(x) : k > n\}}$

One of the consequences of any shift space X being compact is that the infinite intersection of nested closed sets of X is necessarily non-empty. Therefore, for any $x \in X$, $\omega(x)$ is non-empty. Moreover, $\omega(x)$ is also compact, as it is the intersection of closed sets in a compact space.

To see what kinds of points are in $\omega(x)$, recall that in a shift space, two points are close if they agree on a large central block. Therefore, if $y \in \overline{\{\sigma^k(x) : k > n\}}$, for every $\eta > 0$ there must be $z \in \{\sigma^k(x) : k > n\}$ with $d(x, z) \leq \eta$. In other words, the arbitrarily long central blocks of y must appear as central blocks of points in $\{\sigma^k(x) : k > n\}$. Central blocks in this set are simply blocks in x, as points in the set are all shifts of x. Therefore, points of $\omega(x)$ are those points of X whose (arbitrarily long) central blocks appear in $x_{[0,\infty]}$. We formalize this idea with the following theorem.

Theorem 2.13. *A point $y \in \omega(x)$ if and only if for every $n \in \mathbb{N}$ there is an $i > n$ such that $\sigma^i(x)_{[-n,n]} = y_{[-n,n]}$.*

Proof. Suppose that $y \in \omega(x)$. Then for $n \in \mathbb{N}$, $y \in \overline{\{\sigma^k(x) : k > n\}}$. Therefore, there is an $i > n$ such that $y_{[-n,n]} = \sigma^i(x)_{[-n,n]}$.

Now suppose that for every $n \in \mathbb{N}$ there is an $i > n$ such that $\sigma^i(x)_{[-n,n]} = y_{[-n,n]}$. It is immediate that $y \in \overline{\{\sigma^k(x) : k > n\}}$ for all $n \in \mathbb{N}$. Hence $y \in \omega(x)$. □

This characterization of $\omega(x)$ is useful for proving the following.

Theorem 2.14. *The set $\omega(x)$ is invariant; that is, $\sigma(\omega(x)) \subseteq \omega(x)$.*

Proof. We must show that if $y \in \omega(x)$, then $\sigma(y) \in \omega(x)$. Let $y \in \omega(x)$. By Theorem 2.13, for every $n \in \mathbb{N}$ there is an $i > n + 1$ such that $y_{[-n-1,n+1]} = \sigma^i(x)_{[-n-1,n+1]}$. Applying the continuous shift map yields $\sigma(y)_{[-n,n]} = \sigma^{i+1}(x)_{[-n,n]}$. Therefore, for every $n \in \mathbb{N}$ there is an $i > n$ such that $\sigma^i(x)_{[-n,n]} = \sigma(y)_{[-n,n]}$, and so $\sigma(y) \in \omega(x)$. □

Because $\omega(x)$ is compact and invariant under the shift map, $(\omega(x), \sigma)$ is a dynamical system. In fact, as a subset of the full shift, it is a subshift.

Another interesting characteristic of omega-limit sets is that we can get from one point of the omega-limit set to another using a finite portion of a δ-pseudo-orbit (called a δ-chain) of $\omega(x)$, no matter how small δ is. This property is called *internal chain transitivity (ICT).*

Definition 2.15. A subshift $A \subseteq X$ is *internally chain transitive (ICT)* if for every $\delta > 0$ and $x, y \in A$ there is a sequence of elements $\{a_i\}_{i=0}^{n} \subseteq A$ such that $a_0 = x$, $a_n = y$, and $d(\sigma(a_i), a_{i+1}) < \delta$ for $0 \leq i < n$. We call this sequence of elements a *δ-chain.*

It makes some intuitive sense that omega-limit sets have this property. Omega-limit sets are comprised of points that some point x gets arbitrarily close to during its orbit infinitely many times. Say that $y, z \in \omega(x)$ for some x. Then x gets close to y at $\sigma^i(x)$, and x gets close to z at $\sigma^{i+k}(x)$. Then the portion of the orbit of x, $\sigma^i(x), \sigma^{i+1}(x), \ldots, \sigma^{i+k}(x)$ connects y and z. However, we do not necessarily know if these shifts of x are elements of the omega-limit set. What we do know is that these shifts of x get close enough to $\omega(x)$ to connect y and z in a δ-chain. The following lemma shows a condition for which these elements of the δ-chain are suitably close to elements of the omega-limit set.

Lemma 2.16. *For $\epsilon > 0$ there is an $n > 0$ such that $d(\sigma^k(x), \omega(x)) < \epsilon$ for all $k > m$.*

Proof. First, what we mean by $d(\sigma^k(x), \omega(x)) < \epsilon$ is that we can find a point $y \in \omega(x)$ with $d(\sigma^k(x), y) < \epsilon$.

Suppose that the theorem does not hold. Then we can find an infinite sequence of integers $\{n_k\}$ such that $d(\sigma^{n_k}(x), \omega(x)) > \epsilon$. However, this sequence converges to a point $y \in X$ as X is compact. This also implies that $y \in \overline{\{\sigma^k(x) : k > n\}}$ for all n. Hence $y \in \omega(x)$. However, this means there must be some n_{k_0} with $d(\sigma^{n_{k_0}}(x), \omega(x)) < \epsilon$, which is a contradiction. $\square$

One way to think about this result in the context of shift spaces is that after a certain number of letters to the right of the central letter, x no longer contains blocks of a certain size which are not central blocks of points in $\omega(x)$. If this were not the case, we would be able to construct a point in $\omega(x)$ that has such a block as its central block using a diagonal argument similar to the one proving shift spaces are compact.

With this lemma, we can now show omega-limit sets are ICT.

Theorem 2.17. *For $x \in X$, $\omega(x)$ is ICT.*

Proof. Let $y, z \in \omega(x)$ and choose $\delta > 0$. By Lemma 2.16, there is some $m \in \mathbb{N}$ such that $d(\sigma^k(x), \omega(x)) < \frac{\delta}{6}$ for all $k > m$. As $y \in \overline{\{\sigma^k(x) : k > m\}}$ there is some $i > m$ so that $d(\sigma^i(x), y) < \frac{\delta}{2}$. As $z \in \overline{\{\sigma^k(x) : k > i\}}$ there is some $k > 0$ with $d(\sigma^{i+k}(x), z) < \delta$.

Now define $a_0 = y, a_k = z$ and for $0 < j < k$ choose $a_j \in \omega(x)$ with $d(a_j, \sigma^{i+j}(x)) < \frac{\delta}{6}$. Such an element exists by Lemma 2.16 and the choice of m. We claim that this is a δ-chain. By the continuity of the shift map, the requirement that $d(\sigma^{i+j}(x), a_j) < \frac{\delta}{6}$ means $d(\sigma^{i+j+1}(x), \sigma(a_j)) < \frac{\delta}{3}$. Therefore $d(\sigma(a_j), a_{j+1}) < d(\sigma(a_j), \sigma^{i+j+1}(x)) + d(\sigma^{i+j+1}(x), a_{j+1}) < \delta$. This is a δ-chain, implying $\omega(x)$ is ICT. $\square$

Thus, we have that every omega-limit set is ICT, but do we also have the converse? Is every ICT subset of a shift space X an omega-limit point of some $x \in X$? Surprisingly, this is true for shifts of finite type. Hence, we can characterize omega-limit sets as internally chain transitive sets in these subshifts.

Theorem 2.18. *Let X be a shift of finite type. If $A \subseteq X$ is ICT, then $A = \omega(x)$ for some $x \in X$.*

Proof. The idea here is to construct x so that the blocks that appear infinitely often in x are central blocks of elements of A and every central block of an element of A appears infinitely often. The only question is how to connect these central blocks together, but we can do this by way of the δ-chains guaranteed by ICT.

Suppose that X is an m-step shift of finite type. For $i > m$, let $U_i = \left\{x_j^i\right\}_{j=0}^{k_i}$ be a collection of elements of A such that every central $2i+1$-block of A appears as a central $2i+1$-block of an element of U_i. As the alphabet of the shift space is finite, each U_i can be chosen to be finite as there are only finitely many possibilities for different $2i+1$-blocks.

Between each x^i_j and x^i_{j+1} there is a 2^{-i} chain c^i_j. We can concatenate these chains to get another 2^i chain $d^i = c^i_0 c^i_1 \dots c^i_{k_i}$. Then we can find a 2^{-i} chain e^i between $x^i_{k_i}$ and x^{i+1}_0. Finally, by concatenating $d^m e^m d^{m+1} e^{m+1} \dots$ we obtain an asymptotic 2^{-1}-pseudo-orbit in A.

As X is an m-step SFT, there is some $\bar{x} \in X$ that shadows this pseudo-orbit. By the construction of the pseudo-orbit, every central block of an element of A appears infinitely often in $\bar{x}$. Furthermore, these central blocks are the only ones that appear in $\bar{x}$. Therefore, $\omega(\bar{x}) = A$. $\square$

Another historically important result about omega-limit sets is that they are *weakly incompressible.* This means that for any proper, closed $A \subseteq \omega(x)$, $A \cap \overline{\sigma(\omega(x) \backslash A)} \neq \emptyset$.

Theorem 2.19. *$\omega(x)$ is weakly incompressible.*

Proof. Let $A \subseteq \omega(x)$ be closed. Choose $x_0 \in \omega(x) \backslash A$ and $y \in A$. As $\omega(x)$ is ICT, for every integer n there is a $\frac{1}{n}$-chain in $\omega(x)$ from x_0 to y. As $x_0 \in \omega(x) \backslash A$ and $y \in A$, this implies there is an $x_n \in \omega(x) \backslash A$ in the $\frac{1}{n}$-chain with $x_{n+1} \in A$. Thus $d(\sigma(x_n), x_{n+1}) < \frac{1}{n}$. Thus $\{x_n\}_{n=1}^{\infty}$ is a sequence of points of $\omega(x) \backslash A$ such that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to a point of A. This point is also in $\overline{\sigma(\omega(x) \backslash A)}$ and therefore $A \cap \overline{\sigma(\omega(x) \backslash A)} \neq \emptyset$. $\square$

CHAPTER THREE

Group Actions and Dynamical Systems

In the last chapter, we developed the foundations of the study of shift spaces, specifically shifts of finite type, and the link between omega-limit sets and internal chain transitivity in these contexts. While this is a good basis for the study of limit sets in dynamical systems, there is still much room for generalization.

In this chapter, we provide the framework for many avenues of generalization in this area of study. Particularly, we begin by extending the results of shift spaces from the previous chapter into the context of a general dynamical system (X, f) where X is a compact metric space and f is a continuous function. After this, we expand the notion of a shift space from the classical example, which is shift spaces over $\mathbb{Z}$, to more exotic shift spaces over arbitrary groups. To extend these shift spaces into more general dynamical systems, we must then introduce group and monoid actions on a compact metric space. In this context, we give a general definition for limit sets. Finally, we summarize previous work which explores certain limit sets in the context of $\mathbb{Z}^d$ actions.

From Shift Spaces to Dynamical Systems

The categories of dynamical systems of the form $(X_{\mathcal{F}}, \sigma)$ and (X, f) share many similarities. For one, the former is a specific instance of the latter. Both of the ambient spaces are compact metric spaces, and there is one continuous function acting on the space (either σ or f). Because of the similarities of these dynamical systems, it is an easy task to transfer the definitions of omega-limit sets and shadowing from one category to the other. Indeed, most of the previous definitions and proofs can be emended simply by replacing instances of σ with f and converting statements of two points in the shift space sharing the same central n-block to statements of points being close with respect to the given metric. However, one serious difference between these contexts is present in this endeavor. For shift spaces, especially shifts of finite type, we can explicitly construct points of the space that shadow or asymptotically shadow pseduo-orbits; this is because the structure of shift spaces is explicitly given. When generalizing to more abstract dynamical systems, these known structures are lost in the generality. As such, instead of explicitly constructing points in the ambient space, we often must recourse to non-constructive existence proofs.

For the sake of completeness, we give the following generalizations of the results of the previous chapter in the context of general dynamical systems.

Definition 3.1. Given $\delta > 0$, a *δ-pseudo-orbit* is a function $\mathcal{O} : \mathbb{N} \to X$ such that for every $i \in \mathbb{N}$, $d(f(\mathcal{O}(i)), \mathcal{O}(i+1)) < \delta$.

Definition 3.2. Given $\epsilon > 0$ and a pseudo-orbit $\mathcal{O}$, a point $x \in X$ *ϵ-shadows* $\mathcal{O}$ if for every $n \in \mathbb{N}$ $d(f^n(x), \mathcal{O}(n)) < \epsilon$.

Definition 3.3. A dynamical system (X, f) has the *shadowing property* provided for every $\epsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo-orbit can be ϵ-shadowed by an element of X.

Definition 3.4. An *asymptotic pseudo-orbit* is a function $\mathcal{O} : \mathbb{N} \to X$ such that for every $\delta > 0$ there is an $m \in \mathbb{N}$ such that $d(f(\mathcal{O}(k)), \mathcal{O}(k+1)) < \delta$ for every $k > m$.

Definition 3.5. A pseudo-orbit $\mathcal{O}$ is *asymptotically shadowed* by x if for every $\epsilon > 0$ there is an $m \in \mathbb{N}$ such that $d(f^i(x), \mathcal{O}(i)) < \epsilon$ for every $i > m$.

Definition 3.6. A shift space X has the *asymptotic shadowing property* if every asymptotic pseudo-orbit of X is asymptotically shadowed by a point of X.

Definition 3.7. For $x \in X$, the *omega-limit set of* x, $\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k > n\}}$

Theorem 3.8. *A point $y \in \omega(x)$ if and only if for every $n \in \mathbb{N}$ there is an $i > n$ such that $d(f^i(x), y) < \frac{1}{n}$.*

Proof. Suppose that $y \in \omega(x)$. Then for $n \in \mathbb{N}$, $y \in \overline{\{f^k(x) : k > n\}}$. Therefore, there is an $i > n$ such that $d(f^i(x), y) < \frac{1}{n}$.

Now suppose that for every $n \in \mathbb{N}$ there is an $i > n$ such that $d(f^i(x), y) < \frac{1}{n}$. It is immediate that $y \in \overline{\{f^k(x) : k > n\}}$ for all $n \in \mathbb{N}$. Hence $y \in \omega(x)$. □

Theorem 3.9. *The set $\omega(x)$ is invariant; that is, $f(\omega(x)) \subseteq \omega(x)$.*

Proof. We must show that if $y \in \omega(x)$ then $f(y) \in \omega(x)$. Let $y \in \omega(x)$. By Theorem 3.8, for every $n \in \mathbb{N}$ there is an $i > n+1$ such that $d(f^i(x), y) < \frac{1}{n}$. Applying the function f yields $d(f^{i+1}(x), y) < \frac{1}{n}$. Therefore, for every $n \in \mathbb{N}$ there is an $i > n$ such that $d(f^i(x), y) < \frac{1}{n}$, and so $f(y) \in \omega(x)$. □

Definition 3.10. A closed subset $A \subseteq X$ is *internally chain transitive (ICT)* if for every $\delta > 0$ and $x, y \in A$ there is a sequence of elements $\{a_i\}_{i=0}^n \subseteq A$ such that $a_0 = x$, $a_n = y$ and $d(f(a_i), a_{i+1}) < \delta$ for $0 \leq i < n$. We call this sequence of elements a *δ-chain*.

Lemma 3.11. *For $\epsilon > 0$ there is an $n > 0$ such that $d(f^k(x), \omega(x)) < \epsilon$ for all $k > m$.*

Proof. Suppose that the theorem does not hold. Then we can find an infinite sequence of integers $\{n_k\}$ such that $d(f^{n_k}(x), \omega(x)) > \epsilon$. However, this sequence converges to a point $y \in X$ as X is compact. This also implies that $y \in \overline{\{f^k(x) : k > n\}}$ for all n. Hence $y \in \omega(x)$. However, this means there must be some n_{k_0} with $d(f^{n_{k_0}}(x), \omega(x)) < \epsilon$, which is a contradiction. □

Theorem 3.12. *For $x \in X$, $\omega(x)$ is ICT.*

Proof. Fix y, $z \in \omega(x)$ and $\epsilon > 0$. By the uniform continuity of f we have $\delta > 0$ and $\delta < \frac{\epsilon}{3}$ with $d(f(p), f(q)) < \frac{\epsilon}{3}$ whenever $d(p, q) < \delta$. By Lemma 3.11 we have a positive integer N with $d(f^k(x), \omega(x)) < \delta$ for $k > N$.

Choose $m > N$ with $d(f^m(x), y) < \delta$ and $k > m$ with $d(f^k(x), z) < \delta$. Construct $\{x_i\}_{i=0}^{k-m}$ in $\omega(x)$ by $x_0 = y$, $x_{k-m} = z$, and choosing x_i so $d(x_i, f^{m+i}(x)) < \delta$ otherwise. For $i < w - k$, $d(x_i, f^{m+i}(x)) < \delta$ so $d(f(x_i), f^{m+i+1}(x)) < \frac{\epsilon}{3}$. Therefore $d(f(x_i), x_{i+1}) < \epsilon$. □

Theorem 3.13. *For (X, f) with the asymptotic shadowing property, if $A \subseteq X$ is ICT, then $A = \omega(x)$ for some $x \in X$.*

Proof. As in the proof for the shift space case, we will construct an asymptotic psuedo-orbit such that any point x which asymptotically shadows it will have $\omega(x) = A$.

To construct such a pseudo-orbit, for $n \in \mathbb{N}$, let $\{a_i^n\}_{i=1}^{k_n} \subseteq A$ be a $\frac{1}{n}$ cover of A. As A is ICT, we can find a $\frac{1}{n}$-chain between a_i^n and a_{i+1}^n for every i and a $\frac{1}{n+1}$-chain between $a_{k_n}^n$ and a_1^{n+1} for every n. If we concatenate all these chains in the obvious way, then we get an asymptotic pseudo-orbit $\mathcal{O}$.

By the asymptotic shadowing property, there is some $x \in X$ that asymptotically shadows $\mathcal{O}$. We claim that $\omega(x) = A$. If $y \in \omega(x)$, then for every $m \in \mathbb{N}$ there is some $l > m$ with $d(f^l(x), y) < \frac{1}{m}$. Finding l large enough, as $\mathcal{O}$ is an asymptotic pseudo-orbit, we have $d(f^l(x), \mathcal{O}(l)) < \frac{1}{m}$. Thus we have $d(\mathcal{O}(l), y) < \frac{2}{m}$. As we can do this with arbitrary m, y then must converge to a point in A. Hence, $\omega(x) \subseteq A$.

Now suppose $y \in A$. By the construction, for every $m \in \mathbb{N}$ there is a k_m such that $d(y, \mathcal{O}(k_m) < \frac{1}{m}$. We can also assume without loss of generality that $k_m > m$. As x asymptotically shadows $\mathcal{O}$ there is an N_m such that for all $i > N_m$, $d(f^{k_i}(x), \mathcal{O}(k_i)) < \frac{1}{m}$. Choosing n large so that $k_n > N_{2m}$ and $n > 2m$, we have $d(f^{k_n}(x), y) < \frac{1}{m}$. This then implies $y \in \omega(x)$. Hence $A = \omega(x)$. □

Theorem 3.14. *$\omega(x)$ is weakly incompressible.*

Proof. Let $A \subseteq \omega(x)$ be closed. Choose $x_0 \in \omega(x) \backslash A$ and $y \in A$. As $\omega(x)$ is ICT, for every integer n there a $\frac{1}{n}$-chain in $\omega(x)$ from x_0 to y. As $x_0 \in \omega(x) \backslash A$ and $y \in A$, this implies there is an $x_n \in \omega(x) \backslash A$ in the $\frac{1}{n}$-chain with $x_{n+1} \in A$. Thus

$d(f(x_n), x_{n+1}) < \frac{1}{n}$. Thus $\{x_n\}_{n=1}^{\infty}$ is a sequence of points of $\omega(x)\backslash A$ such that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to a point of A. This point is also in $\overline{f(\omega(x)\backslash A)}$ and therefore $A \cap \overline{f(\omega(x)\backslash A)} \neq \emptyset$. □

Shift Spaces over Groups

Another avenue to extend the results of Chapter 2 is to consider different kinds of shift spaces. To this point, the shift spaces we have considered consist of a set of points which are constrained by some rules. These rules dictate that the words be formed of symbols from some fixed, finite alphabet; the rules also determine which finite pattens of symbols can or cannot appear in an infinite word. There is also a continuous shift map, σ, acting on this space which takes a word and shifts all the symbols over one spot. It was easy to make this shift space into a metric space by calling infinite words close if they shared a large central block. Specifically, for any infinite words x or y, if the largest central block they shared was size n, then $d(x, y) = 2^{-n}$ (or if they do not share any central block $d(x, y) = 1$).

Suppose instead of arranging the symbols of the word in an infinite line, we arranged symbols in a finite circle:

1
1 2
3 2
2 1
1 3

Again, we can create a set of forbidden patterns of symbols and then create a space of all these circular words which do not contain these patterns. Define the *central n-block* of a word to be the n symbols beginning at the top of the circle and going clockwise. We can make this set of circular words into a metric space by again calling points close if they agree on a set block of symbols. More specifically, if two words x, y agree on an n-block, but not an n+1-block, then $d(x, y) = \frac{1}{2^n}$, and if the words do not agree on any n-block, $d(x, y) = 1$.

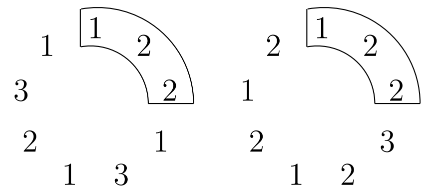

Figure 3.1: Two words with their central 3-blocks outlined. The distance between these words is $\frac{1}{8}$.

There is also a natural shift map σ on this space. Instead of shifting the word linearly as in the classic case of a shift space, we can instead rotate the word one letter counter-clockwise. It is easy to see that this function is continuous. Furthermore, by the same diagonal argument used in the classic shift space case, for any collection of forbidden block $\mathcal{F}$, the associated subshift $X_{\mathcal{F}}$ consisting of all words not containing a block of $\mathcal{F}$ is compact. Hence, $(X_{\mathcal{F}}, \sigma)$ is a dynamical system.

Another natural way to extend the idea of a shift space is to arrange the symbols into an infinite grid.

Just as previously, if we define a set of forbidden patterns of symbols $\mathcal{F}$, we get a set of allowed "square" words $X_{\mathcal{F}}$. We can then define a central n-block by associating the position of letter in a word to the set of lattice points $\mathbb{Z} \times \mathbb{Z}$ and then define the n-block to consist of all symbols whose maximal coordinate is less than or equal to n.

We define a metric on $X_{\mathcal{F}}$ by $d(x, y) = 2^{-n}$ where n is the largest integer for which x and y have the same central n-block. If x, y do not share a central n-block for any n, then $d(x, y) = 1$. This makes $X_{\mathcal{F}}$ a metric space. In fact, by a similar argument

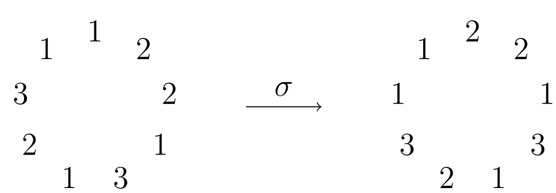

Figure 3.2: The shift map for circular words

$$
\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 2 & 3 & 1 & 3 & 3 & \cdots \\
\cdots & 2 & 1 & 2 & 3 & 1 & \cdots \\
\cdots & 1 & 2 & 1 & 2 & 2 & \cdots \\
\cdots & 3 & 2 & 3 & 3 & 1 & \cdots \\
\cdots & 3 & 1 & 1 & 1 & 3 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Figure 3.3: An example of a square word with its central 1-block outlined

to the other two shift spaces, $X_{\mathcal{F}}$ is compact.

In this shift space, there are two obvious shift maps, and each is continuous. One of the maps, σ_x, shifts the word one letter to the left. The other map, σ_y shifts the word one letter down. An important observation to make about these two maps is that they commute with each other. Thus, given a word $z \in X_{\mathcal{F}}$, $\sigma_x(\sigma_y(z)) = \sigma_y(\sigma_x(z))$ (see Figure 3.4).

To this point, we have given three examples of shift spaces. In order to reach a generalization of shift spaces, it is helpful to take a step back and see what these examples have in common. Each shift space primarily consists of a set of "words" whose symbols are arranged in a particular pattern and some functions which shift the symbols of a word. On top of this, we defined a metric on the shift space based on a sort of geometric center of the configuration of the symbols. The metric was defined in such a way that the shift maps were continuous.

What then is the best way to generalize this? We can think of each word in the shift space as a set of symbols in a specific configuration. If we place each point of the configuration in a set X, we can think of a word as a function from the base configuration X into a finite alphabet $\mathcal{A}$. Thus, every point in the configuration is assigned a letter. For the examples, we can view the configurations as $\mathbb{Z}$ for the classic

$$
\begin{array}{ccc}
\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 2 & 3 & 1 & 3 & 3 & \cdots \\
\cdots & 2 & 1 & 2 & 3 & 1 & \cdots \\
\cdots & 1 & 2 & 1 & 2 & 2 & \cdots \\
\cdots & 3 & 2 & 3 & 3 & 1 & \cdots \\
\cdots & 3 & 1 & 1 & 1 & 3 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
& \xrightarrow{\sigma_x} &
\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 3 & 1 & 3 & 3 & 2 & \cdots \\
\cdots & 1 & 2 & 3 & 1 & 1 & \cdots \\
\cdots & 2 & 1 & 2 & 2 & 1 & \cdots \\
\cdots & 2 & 3 & 3 & 1 & 1 & \cdots \\
\cdots & 1 & 1 & 1 & 3 & 2 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
\\
\Big\downarrow \sigma_y & & \Big\downarrow \sigma_y
\\
\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 3 & 2 & 3 & 2 & 1 & \cdots \\
\cdots & 2 & 3 & 1 & 3 & 3 & \cdots \\
\cdots & 2 & 1 & 2 & 3 & 1 & \cdots \\
\cdots & 1 & 2 & 1 & 2 & 2 & \cdots \\
\cdots & 3 & 2 & 3 & 3 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
& \xrightarrow{\sigma_x} &
\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 2 & 3 & 2 & 1 & 1 & \cdots \\
\cdots & 3 & 1 & 3 & 3 & 2 & \cdots \\
\cdots & 1 & 2 & 3 & 1 & 1 & \cdots \\
\cdots & 2 & 1 & 2 & 2 & 1 & \cdots \\
\cdots & 2 & 3 & 3 & 1 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
\end{array}
$$

Figure 3.4: The commuting shift maps

shift space case, $\mathbb{Z}_n$ for the circular shift space case (where n is the number of symbols in the word), and $\mathbb{Z} \times \mathbb{Z}$ for the infinite grid case. Note that these configurations are all groups. The reason for this will be seen shortly when considering shift maps. Thus, the set of words of a shift space consists of a set of function from some fixed group G to some fixed, finite alphabet $\mathcal{A}$.

Having established the general nature of the words in a shift space, it is time to turn our attention to the shift maps. If $f : G \to \mathcal{A}$ is a word in the shift space, a shift map σ must take f to another function $\sigma(f) : G \to \mathcal{A}$ by means of shifting the symbols in the configuration. Because of this, it seems like σ must act on the symbols themselves. However, there is no apparent way for this to be defined. Another option is to view σ as shifting the underlying group G. For the classic shift space over $\mathbb{Z}$, consisting of words of the form $x : \mathbb{Z} \to \mathcal{A}$, the shift map worked by $\sigma(x(n)) = x(n+1)$ for all $n \in \mathbb{N}$. The shift map worked in the same way for the circular word case, and for the infinite grid case, $\sigma_x(x)(m,n) = x(m+1,n)$ and $\sigma_y(x)(m,n) = x(m,n+1)$.

At this point, it is still difficult to understand what is happening in the abstract. In order to assuage this problem, we recourse to a common technique in algebra. Let G be a group and Y a finite set. We denote the set of all functions from G to Y by $\mathcal{F}(G)$. Then for every $g \in G$, there is a function $h_g : \mathcal{F}(G) \to \mathcal{F}(G)$ by $h_g(f)(g') = f(gg')$ for all g'. It turns out, this is the exact way shift maps act on words of a shift space. For the classic and the circular word case, σ acts like h_1, and for the infinite grid case, σ_x acts like $h_{(1,0)}$ and σ_y acts like $h_{(0,1)}$.

There is, however, one crucial difference between the example from algebra and the shift maps: in the example from algebra, there was a map defined from every element of the underlying group, while there are only a few shift maps in each case. This difference is easily dismissed when one considers the composition of the shift maps and their inverses. For the classical case, any h_n is equivalent to σ^n for any $n \in \mathbb{Z}$. Hence it suffices to just consider the composition of the one shift map and its inverse. The same is true for the other cases. This leads us to the formal definition of a shift space over any arbitrary group over any arbitrary group.

Definition 3.15. Given a group G and a finite alphabet $\mathcal{A}$, the *full shift over G to $\mathcal{A}$* consists of X, the set of all functions from $G \to \mathcal{A}$, and a set of shift maps $\{\sigma_g\}_{g \in G}$ where $\sigma_g(x)(h) = x(gh)$ for all $x \in X$ and $h \in G$.

If the group G is countable, the full shift over G is a metric space. To construct the metric, fix an ordering $\{g_k\}_{k \in \mathbb{N}}$ of the elements of G, and define $d(x,y) = 2^{-n}$ where $n = \inf\{\{1\} \cup \{k : x(g_k) \neq y(g_k)\}\}$. By the same diagonal argument used in the classic case, this metric makes the full shift compact. Moreover, every shift map σ_g is continuous. To see this, let x be an element of the full shift and $\epsilon > 0$. We must find a $\delta > 0$ so that if $d(x,y) < \delta$, then $d(\sigma_g(x), \sigma_g(y)) < \epsilon$. Find k large so that $2^{-k} < \epsilon$. Hence, we need $\sigma_g(x)(g_n) = \sigma_g(y)(g_n)$ for every $n \leq k$. This is equivalent to saying $x(gg_n) = y(gg_n)$ for $n \leq k$. Finding the maximum m such that $gg_n = g_m$ for

$n \leq k$, this condition is satisfied when $d(x, y) < 2^{-m}$. Thus, whenever $d(x, y) < 2^{-m}$, $d(\sigma_g(x), \sigma_g(y)) < \epsilon$. Thus σ_g is continuous.

In the cases we have seen before, we can also define a *subshift* by forbidding a certain set of patterns. Just as before, these subshifts will be invariant under the shift maps and will be compact.

Definition 3.16. For a group G and alphabet $\mathcal{A}$, a *pattern* is a finite subset $\{(g_i, a_i)\}$ of $G \times \mathcal{A}$ such that $g_i \neq g_j$ for $i \neq j$. A word x does not contain a pattern $\{(g_i, a_i)\}$ if for every $g \in G$ there is an i such that $x(gg_i) \neq a_i$.

Given a set of forbidden patterns $\mathcal{F}$, the *subshift* $X_{\mathcal{F}}$ consists of all words of X which do not contain a pattern in $\mathcal{F}$. In the case that $\mathcal{F}$ is finite, $X_{\mathcal{F}}$ is a *shift of finite type*.

With this definition, it is possible to study a shift space over any group, even those not as easily visualized like the previous examples. Moreover, because none of the definitions rely on the existence of inverses, simply replacing the word "group" with "monoid" leads to the definition of shift spaces over monoids.

Group Actions and Dynamical Systems

Having defined shift spaces over arbitrary groups and monoids, our goal now turns to placing these spaces into the context of more general dynamical systems. At the beginning of the chapter, we noted how many of the definitions and results of limit sets in shift spaces over $\mathbb{Z}$ carried over directly to a dynamical system (X, f). This is because there was a direct analogue between the dynamics of the shift maps in shift spaces over $\mathbb{Z}$ and iterates of the continuous map f in an arbitrary dynamical system. As noted in the previous chapter, shift spaces over $\mathbb{Z}$ are examples of dynamical systems. While we to this point have no results about limit sets in shift spaces over arbitrary groups, our goal is to find what sort of system over a general, compact metric space is analogous to these shift spaces.

Because we know the ambient space of the analogous system ought to be a compact metric space, the only question is how to generalize the functions acting on the space. For any group G, there are $|G|$ many functions acting on the space, where $|G|$ represents the size of the group G. The group structure also importantly defines how these functions interact with one another. Given two elements $g, h \in G$ and an element x in a shift space over G, the compostion of shift maps $\sigma_g \circ \sigma_h$ is the same as the shift map σ_{hg}, as for any $g' \in G$, $\sigma_g(\sigma_h(x))(g') = \sigma_h(x)(gg') = x(hgg') = \sigma_{hg}(x)(g')$. Thus, our generalized system must have a set of continuous functions that interact similarly, according to the group structure. In order to formalize this, we need the definition of a *group action*.

Definition 3.17. Given a group G and a set X, a *left group action* is a function $\phi : G \times X \to X$ such that

1. $\phi(e, x) = x$ for all $x \in X$
2. $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $g, h \in G$ and $x \in X$

A *right group action* is a function $\psi : X \times G \to X$ such that

1. $\psi(x, e) = x$ for all $x \in X$
2. $\psi(\psi(x, g), h) = \psi(x, gh)$ for all $x \in X$ and $g, h \in G$

Often, we write $g \cdot x$ and $x \cdot g$ for $\phi(g, x)$ and $\psi(x, g)$ respectively. Also, the functions $\phi_{|\{g\} \times X}$ and $\psi_{|X \times \{g\}}$ are written ϕ_g and ψ_g where $\phi_g = \phi(g, \cdot)$ and $\psi_g = \psi(\cdot, g)$.

In the case that X is a topological space, and G is endowed with the discrete topology, a *continuous left (resp. right) group action* is a left (resp. right) group action ϕ (resp. ψ) such that ϕ (resp. ψ) is continuous. This is equivalent to $\phi_g : X \to X$ (resp. ψ_g) being continuous for every $g \in G$.

For the most part, we will not be particular about the difference between left and right group actions. In fact, we can easily convert between a left and right group action by simply defining $\phi_g = \psi_{g^{-1}}$ for all $g \in G$.

We have already seen an example of a continuous right group action, namely the shift maps in a shift space over any group G. Therefore, continuous group actions acting on a compact metric space is the desired generalization of shift spaces over groups. We can further extend this to generalize shift spaces over monoids by defining a *monoid action* in the same way as a group action but replacing the word "group" with the word "monoid." It is important to note that in group actions, the continuous functions are necessarily homeomorphisms because of the existence of a continuous inverse. This further implication is not present in monoid actions.

For the rest of the paper, we will use f_g instead of ϕ_g or ψ_g. This is in order to be more congruent with normal function notation.

Limit Sets in Group and Monoid Actions

In the case of shift spaces over $\mathbb{Z}$, the definition of a limit set was a simple one, as there was essentially only one function. Therefore, the only "future" of a point was the point under iterations of the single function. However, when considering a general G action, there is more than one function to consider. Thus, there is no way to define a single limit set for a general group or monoid action. This being the case, there is a general form a limit set ought to take. This form was first defined by Souza in [8].

Definition 3.18. Given a group or monoid G, a family of subsets $\mathcal{F}$ of G, and a G action on a (compact metric) topological space X, the *ω-limit set for the family $\mathcal{F}$* is the intersection

$$\omega(x) = \bigcap_{A \in \mathcal{F}} \overline{\{f_g(x) : g \in A\}}$$

As the intersection of closed sets, $\omega(x)$ is closed. Furthermore, in the case that X is compact and $\mathcal{F}$ is a filter basis (i.e. $\emptyset \notin \mathcal{F}$ and for every $A, B \in \mathcal{F}$ there is a $C \in \mathcal{F}$ with $C \subseteq A \cap B$) then $\omega(x)$ is non-empty. Importantly, the limit set defined in the classical case of a $\mathbb{Z}$ action is of this form. Taking $\mathcal{F} = \{A_n\}_{n\in\mathbb{N}}$ where $A_n = \mathbb{N} \setminus \{1, \ldots, n\}$, the ω-limit set for the family $\mathcal{F}$ is exactly the omega-limit set previously defined.

In the case that G is finite, the limit sets are largely uninteresting. The ω-limit sets are exactly the sets $\{f_g(x) : g \in A \text{ and } A \subseteq G\}$. This is because for every family of subsets $\mathcal{F}$ of G and every $A \in \mathcal{F}$, $\{f_g(x) : g \in A\}$ is finite, so $\overline{\{f_g(x) : g \in A\}} = \{f_g(x) : g \in A\}$. Therefore, $\omega(x)$ is the intersection of shifts of x.

On the other hand, these limit sets become more interesting when G is infinite. In these instances, limit sets may consist of more than shifts of a point. This is because the closures $\overline{\{f_g(x) : g \in A\}}$ may not be trivial. The complexity of studying ω-limit sets therefore increases when G is infinite. In general, the internal structure of arbitrary ω-limit sets is not well-known. In the previous section, we saw ω-limit sets in $\mathbb{Z}$ actions are characterized by the property of internal chain transitivity. Whether there is an analogous internal property which characterizes arbitrary ω-limit set is still an open question. However, characterizations have been given for certain ω-limit sets in certain G actions.

Limit Sets in $\mathbb{Z}^d$ actions

In [7], the authors consider four limit sets for $\mathbb{Z}^d$ actions and, in the case of shifts of finite type or systems with a type of shadowing property, characterize these limit sets by a generalization of internal chain transitivity called *internal mesh transitivity.* Each of the limit sets have an underlying geometric interpretation for the shift space case. Points in these limit sets have their central blocks occuring in some defined region of $\mathbb{Z}^d$ lying arbitrarily far away from the origin. The four limit sets are defined formally as follows:

Let $\mathcal{D}^d = \left\{\eta \in \mathbb{Z}^d : \gcd\{\eta_i\} = 1\right\}$.

Definition 3.19. For $\eta \in \mathcal{D}^d$ and $x \in X$,

$L_\eta(x) = \left\{y \in X : \forall M \in \mathbb{N}, \epsilon > 0\ \exists\, t \in \mathbb{Z}^d \text{ such that } t \cdot \eta > M \text{ and } d(y, \sigma_t(x)) < \epsilon\right\}$

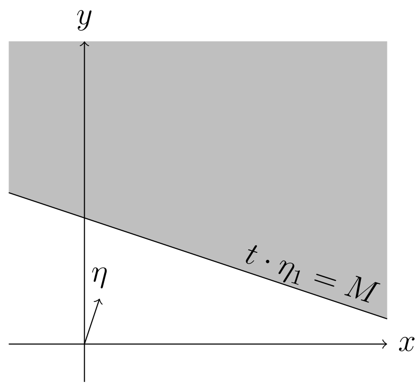

Figure 3.5: The underlying geometry for $L_\eta(x)$

Figure 3.5 captures the "futures" of the group action considered by $L_\eta(x)$-limit sets. In paticular, the futures following the direction of the vector η. It is also easy to see that this limit set is also a limit set in the sense of Souza. Taking $A_M = \left\{t \in \mathbb{Z}^d : t \cdot \eta > M\right\}$ and $\mathcal{F} = \{A_M\}_{M \in \mathbb{N}}$, then $L_\eta(x) = \bigcap_{A \in \mathcal{F}} \overline{\{f_g(x) : g \in A\}}$. Finally, there is an explicit geometric interpretation of this limit set in the case of shift spaces: $L_\eta(x)$ consists of all points whose middle M-block appears in the shaded region of Figure 3.5 for every $M \in \mathbb{N}$.

Definition 3.20. For $E \subseteq \mathcal{D}^d$ and $x \in X$,

$L_E^+(x) = \left\{y \in X : \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z}^d \text{ s.t.} \min_{\eta \in E}\{t \cdot \eta\} > M,\ d(y, \sigma_t(x)) < \epsilon\right\}$

$L_E^-(x) = \left\{y \in X : \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z}^d \text{ s.t.} \max_{\eta \in E}\{t \cdot \eta\} > M,\ d(y, \sigma_t(x)) < \epsilon\right\}$

Given a subset $E \subseteq \mathcal{D}^d$, L_E^+ considers the intersection of the regions considered in L_η for $\eta \in E$, and L_E^- considers the union of such regions. Again, these are limit sets in the sense of Souza. For L_E^+ take $\mathcal{F} = \{B_M\}_{M \in \mathbb{N}}$ where $B_M = \left\{t \in \mathbb{Z}^d : \min_{\eta \in E}\{t \cdot \eta\} > M\right\}$. Similarly, for L_E^-, take $\mathcal{F} = \{C_M\}_{M \in \mathbb{N}}$ where $C_M =$

$\left\{t \in \mathbb{Z}^d : \max_{\eta \in E} \{t \cdot \eta\} > M\right\}$. Once again there is a geometric interpretation for the shift space case: both L_E^+ and L_E^- consist of points whose middle M-block appears in the corresponding shaded regions for every $M \in \mathbb{N}$.

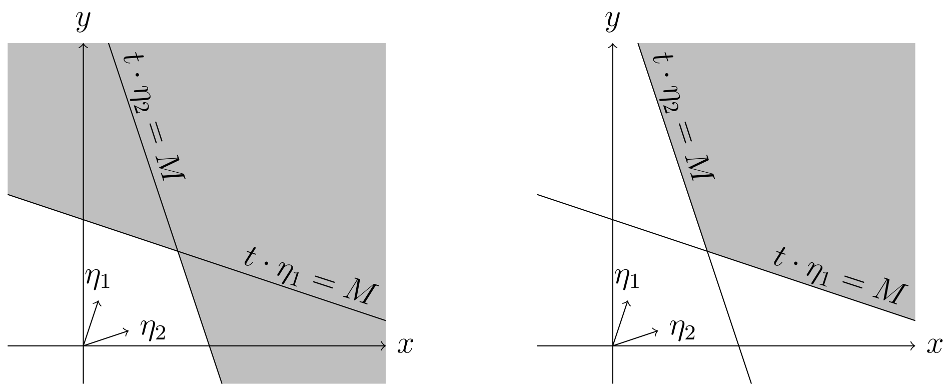

Figure 3.6: The associated regions for L_E^- and L_E^+ respectively

Definition 3.21. For $x \in X$,
$\omega(x) = \left\{y \in X : \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z}^d \text{ s.t.} \max_{1 \le i \le d} \{|t_i|\} > M,\ d(y, \sigma_t(x)) < \epsilon\right\}$

This limit set captures all "futures" of a point occuring arbitrarily far away from the origin. Because this limit set does not consider any particular direction, it is the most general of the considered limit sets. By taking $\mathcal{F} = \{D_M\}_{M \in \mathbb{Z}}$ with $D_M = \left\{t \in \mathbb{Z}^d : \max_{1 \le i \le d} |t_i| > M\right\}$, this is a limit set in the sense of Souza. In the 2-dimensional shift case, there is an explicit geometric interpretation: points in $\omega(x)$ are exactly those whose middle M-blocks appear in the shaded region of the below figure for every $M \in \mathbb{N}$. Finally, it is important to note that $\omega(x) = L_E^-(x)$ when E is the set of all canonical basis vectors and their inverses.

Because of the strong interelations between the geometries of the corresponding described regions of the definitions, it is easy to get the following relation of limit sets:

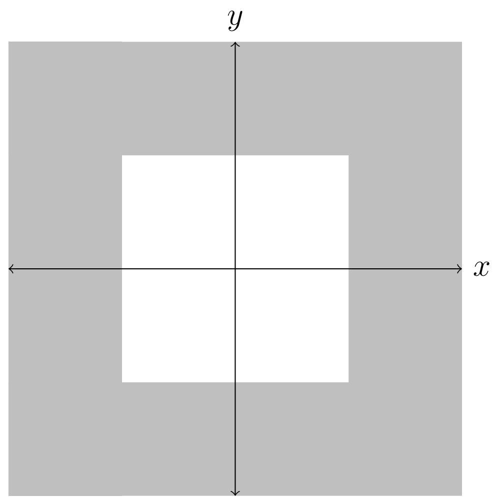

Figure 3.7: The underlying geometry for $\omega(x)$

Theorem 3.22. *For $E \subseteq \mathcal{D}^d$ finite and $x \in X$, for any $\eta \in E$ we have*

$$L_E^+(x) \subseteq \bigcap_{\beta \in E} L_\beta(x) \subseteq L_\eta(x) \subseteq \bigcup_{\beta \in E} L_\eta(x) = L_E^-(x) \subseteq \omega(x)$$

To characterize these limit sets, the authors of [7] define an analog of internal chain transitivity for higher dimensions. Recall that for $\epsilon > 0$ an ϵ-chain is a subset $\{x_1, \ldots, x_n\} \subseteq X$ such that $d(\sigma(x_k), x_{k+1}) < \epsilon$. For the higher dimensional analog, we need to consider relations between points under the shift maps in any dimension, and we would like the underlying geometry of the collection of points to mimic the underlying geometry of the limit sets.

For $E \subseteq \mathcal{D}^d$, let F denote either E^+ or E^-. Also, let $\|t\|$ denote the respective norms for these limit sets, that is, $\min_{\eta \in E} \{t \cdot \eta\}$ for E^+ and $\max_{\eta \in E} \{t \cdot \eta\}$ for E^-.

Definition 3.23. For $\epsilon > 0$, an $\epsilon - F$*-mesh* is a collection $\{p_t\}_{M \leq \|t\| \leq K} \subseteq X$ for some $M \leq K \in \mathbb{Z}$ such that for every $|s| = 1$ and $M \leq \|t\| \leq \|t + s\| \leq K$, $d(\sigma_s(p_t), p_{s+t}) < \epsilon$.

In the case that $M = K$, this is called an $\epsilon - F$*-band*.

The idea behind this definition can be easily visualized. Suppose we have $M \leq$

$K \in \mathbb{Z}$ and corresponding shaded regions for these two integers as in Figure 3.6. The set $\{p_t\}_{M \leq \|t\| \leq K}$ corresponds to choosing an element for every lattice point in the symmetric difference of the two regions. Furthermore, these elements must be chosen such that whenever we move within this region from one point to another in a direction away from the origin and of distance 1 (i.e. movement along a canonical basis vector away from the origin) the shift maps the starting point close to the ending point.

Unlike chains in one dimenion, meshes are more difficult to concatenate. The primary reason for this difficulty is that there is a much greater overlap required in order to concatenate. Particularly, to concatenate two ϵ-chains, only the final point and the initial point of the chains need to correspond. For meshes to concatenate, it is not even enough that the outer and inner bands of the two meshes correspond. Some meshes simply cannot be extended.

However, the following definitions give sufficient conditions for when meshes can be expanded to form a situation analogous to internal chain transitivity.

Definition 3.24. For $\epsilon > 0$, an $\epsilon - F$-band C and $y \in X$, an $\epsilon - F$-*mesh* from C to y is an $\epsilon - F$-mesh $P = \{p_t\}_{K \leq \|t\| \leq M}$ such that $\{p_t\}_{\|t\|=K} = C$ and there exists $\|t\| = M$ with $d(p_t, y) < \epsilon$.

Definition 3.25. A set $\Lambda \subseteq X$ is *internally mesh transitive with respect to F* if there exist collections $\{\mathcal{C}_N\}$ of F-bands in Λ such that for every $\epsilon > 0$ there is N_ϵ such that for every $C \in \mathcal{C}_{N_\epsilon}$ and $y \in X$ there exists an $\epsilon - F$-mesh $\{p_t\}_{K \leq \|t\| \leq M}$ in Λ from C to y and $C' = \{p_t\}_{\|t\|=M} \in \mathcal{C}_{N_{\frac{\epsilon}{2}}}$.

This definition clearly lends itself to the construction of a sort of asymptotic pseudo-orbit in which one can specify that certain points appear arbitrarily often. Indeed, by this method of construction, it is shown that in cases with an asymptotic shadowing property, closed sets which are internally mesh transitive with respect to F are exactly those L_F limit sets.

Conclusion

In the final chapter, we present results in a similar fashion to those in [7], looking at a few specific limit sets and providing a characterization for such sets within a context of asymptotic shadowing. However, instead of considering $\mathbb{Z}^d$ actions, we consider actions of finitely generated free groups and monoids.

CHAPTER FOUR

Limit Sets in Free Group and Monoid Actions

Introduction

In the previous chapter, we mentioned a generalized notion of limit sets in any monoid action. In this understanding, limit sets are defined by first choosing a family $\mathcal{F}$ of subsets of the monoid H and a point x; then the limit set is the intersection $\cap_{S\in\mathcal{F}}\overline{\{f_u(x) : u \in S\}}$. While this definition has the benefit of being defined in every context of a monoid action, it also creates an astounding number of limit sets for any system. Furthermore, without knowing any structure of the monoid acting on the topological space, it is difficult to investigate any properties of the limit sets. To alleviate these hindrances, we will fix the monoids in our investigation to be finitely-generated free groups and monoids. Also, we will constrain ourselves to looking at only four different families of subsets of these monoids.

There is good reason to choose finitely-generated free groups and monoids as the groups acting on a space. For one, the structure of these monoids is easily understood: they can be viewed as the set of finite words over a finite alphabet (which includes inverses in the free group case), and multiplication is concatenation of these words (with cancellation of inverses for the free groups). Thus, multiplication can be done iteratively by multiplying by single letters of the alphabet. Also, these monoids are non-abelian, and there is not much interaction between elements. In particular, this means it is not difficult to extend a finite portion of an element of a shift space over a free monoid to a complete element in the shift space. This is a crucial fact for some of the constructions.

The general goal of this chapter is to generalize some of the properties of limit sets in $\mathbb{Z}$ actions to the context of free group and monoid actions. The structure of the chapter is as follows. First, preliminary definitions and notations are given. Then, four different limit sets are defined, and general properties like compactness and invariance are proven. Next, different types of internal transitivity of sets are defined. Finally, we prove that these types of internal transivity can characterize two of the limit sets.

Preliminaries

Let G be a group or monoid and X a compact metric space. As we've seen before, a continuous left G action on X is a function $f : G \times X \to X$ that satisfies the following conditions: for each $g \in G$, the function f_g defined by $f_g(x) = f(g, x)$ is continuous, for the identity $e \in G$, f_e is the identity on X, and for $g, h \in G$, $f_{gh} = f_h \circ f_g$.

For $n \in \mathbb{N}$, let F denote the free group on n generators $\{s_0, \dots s_{n-1}\}$ and $S = \{s_0, \dots s_{n-1}, \dots s_{n-1}^{-1}\}$ be the set of all generators and their inverses. The set of reduced words of F is the set $W = \{e\} \cup \{w_0 w_1 \cdots w_k \in S^\omega : w_i \neq w_{i+1}^{-1} \text{ for } i < k\}$ with e denoting the empty word. Each element of F has a unique representative in W, and the group operation of F is realized in W by concatenation followed by cancellation.

We define the *length* of an element $u \in F$ to be the number of letters in its reduced representation. We denote this by $|u|$. The identity of F is associated with the empty word e and has length 0. Finally, we will say for two elements of F with reduced representations $u = u_0 \dots u_n$ and $v = v_0 \dots v_{n+k}$, that u is a *prefix* of v if $u_i = v_i$ for $0 \leq i \leq n$. Additionally, we will take the identity e to be a prefix of every element of F.

It will also be necessary to consider the infinite words of F. In particular, we define the set $W_\infty = \{\langle w_i \rangle_{i \in \omega} : w_i \neq w_{i+1}^{-1} \text{ for } i \in \omega\}$. Let u, w be two words (either finite or infinite) with length at least n. We say $u|_n = w|_n$ if $u_i = w_i$ for $0 \leq i < n$.

For $n \in \mathbb{N}$, let H denote the free monoid on the n generators $P = \{s_0, \dots s_{n-1}\}$.

In this case, H coincides with the set of words of $\{e\} \cup \{w_0 w_1 \cdots w_k \in P^\omega\}$ with e denoting the empty word, as every element of H has a unique representation, and the binary operation is simply concatenation. The collection of infinite words of H is the set $H_\infty = \{\langle w_i \rangle_{i\in\omega} : w_i \in P\}$. The length of elements, prefixes, and restrictions $w_{|n}$ are defined the same as in the free group case.

For the sake of generality, we take G as either a free group or monoid with W the set of words, W_∞ the set of infinite words, and S the set of generators (with inverses in the group case).

For a finite alphabet $\mathcal{A}$, *the full shift of $\mathcal{A}$ over G* (denoted $\mathcal{A}^G$) is the set of all functions $x : G \to \mathcal{A}$. There is a natural G action σ on $\mathcal{A}^G$ defined as follows. For every $s \in S$ there is an associated *shift map* $\sigma_s : \mathcal{A}^G \to \mathcal{A}^G$ defined by $\sigma_s(x)(u) = x(su)$. For $v = v_0 \ldots v_n \in G$, we define $\sigma_v(x) = \sigma_{v_n} \circ \cdots \circ \sigma_{v_0}(x)$. Thus $\sigma_v(x)(u) = x(vu)$.

For fixed G, define $\Sigma^n = \{u \in G : |u| < n\}$. We place a metric on $\mathcal{A}^G$ defined by $d(x, y) = \inf\left(\{2^{-n} : x|_{\Sigma^n} = y|_{\Sigma^n}\} \cup \{1\}\right)$. It is easy to see that under the topology induced by this metric, $\mathcal{A}^G$ is compact and the action σ is continuous. In particular, if $d(x, y) = 2^{-n}$ then $d(\sigma_i(x), \sigma_i(y)) \leq 2^{-n+1}$. Furthermore, for $u \in G$, if $d(x, y) = 2^{-n}$ then $d(\sigma_u(x), \sigma_u(y)) \leq \min\left\{2^{-m+|u|}, 1\right\}$.

An *n-block* is a function $B_n : \Sigma^n \to \mathcal{A}$. An element $x \in \mathcal{A}^G$ is said to contain B_n if there exists a $u \in G$ such that $\sigma_u(x)|_{\Sigma^n} = B_n$. Let $\mathcal{F}$ be a collection of m-blocks where m is allowed to range over the integers. We create a subspace of $\mathcal{A}^G$ defined as $X_{\mathcal{F}} = \left\{x \in \mathcal{A}^G : x \text{ contains no } B \in \mathcal{F}\right\}$. $X_{\mathcal{F}}$ is compact and invariant under the shift maps. In this case, $\mathcal{F}$ is called a *set of forbidden blocks.* Any invariant and compact subspace of $\mathcal{A}^G$ can be expressed as $X_{\mathcal{F}}$ for some set $\mathcal{F}$. This is called a *shift space.* In the case that $\mathcal{F}$ is finite, the shift space is a *shift of finite type* (SFT). If M is the maximal integer such that some $B \in \mathcal{F}$ is an M-block, Y is called an *M-step shift of finite type.* In this case, we can assume without loss of generality that every element

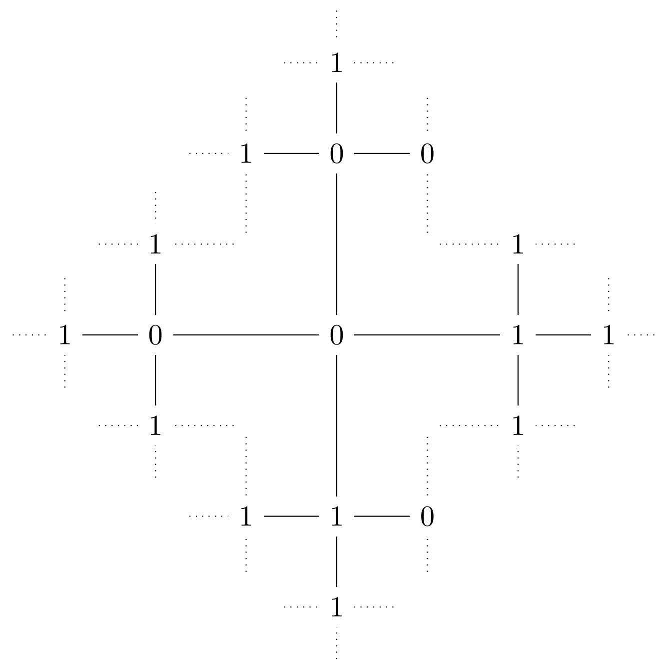

Figure 4.1: A representation of an element of $\{0,1\}^{F_2}$ with the middle 3-block filled in.

of $\mathcal{F}$ is an M-block.

In a metric space X, the distance between a point $x \in X$ and closed subset $A \subseteq X$ we define as $d(x, A) = \inf_{a \in A} d(x, a)$. This gives rise to the *Hausdorff distance* between two closed subsets $A, B \subseteq X$ defined as

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

Limit Sets for Free Group and Monoid Actions

Here, we define four different limit sets in this context. Importantly, each limit set is a limit set in the sense of Souza. Moreover, each limit set has an underlying geometric interpretation.

The first limit set we consider is perhaps the most general in that there is no inherent "direction" or "future" involved.

Definition 4.1. For $x \in X$, $\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f_u(x) : |u| > n\}}$.

This can be seen to be a limit set under Souza's definition by taking $\mathcal{F} = \{A_n\}_{n\in\mathbb{N}}$ where $A_n = \{u \in F : |u| > n\}$. From the definition, it is immediate to see $\omega(x)$ is compact and non-empty, as it is the intersection of nested, closed subsets of a compact space. An equivalent metric interpretation of the set is given in the following lemma.

Lemma 4.2. $\omega(x) = \left\{y \in X : \forall\, n \in \mathbb{N}\ \ \exists\ |u_n| > n\ \ s.t.\ \ d(f_{u_n}(x), y) < \frac{1}{n}\right\}$.

Proof. Let $y \in \omega(x)$. For $n \in \mathbb{N}$ there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq G$ with $u_i > n$ such that $\{f_{u_i}(x)\}$ converges to y, as $y \in \overline{\{f_u(x) : |u| > n\}}$. Thus we can choose $|u_n| > n$ with $d(f_{u_n}(x), y) < \frac{1}{n}$.

Now let $y \in \left\{y \in X : \forall\, n \in \mathbb{N}\ \ \exists\ |u_n| > n\ \ s.t.\ \ d(f_{u_n}(x), y) < \frac{1}{n}\right\}$. For $n \in \mathbb{N}$ choose $|u_n| > n$ with $d(f_{u_n}(x), y) < \frac{1}{n}$. Thus, for $m \geq n$, $f_{u_m}(x) \in \{f_u(x) : |u| > n\}$, and $\{f_{u_i}(x)\}_{i=m}^{\infty}$ converges to y. Therefore $y \in \overline{\{f_u(x) : |u| > n\}}$. □

Lemma 4.3. *The set $\omega(x)$ is invariant.*

Proof. Let $y \in \omega(x)$ and $i \in S$. By the uniform continuity of f_i, there exists $\delta_n > 0$ such that $d(f_i(x), f_i(y)) < \frac{1}{n}$ if $d(x, y) < \delta_n$. As $y \in \omega(x)$, there is a sequence $\{u_j\}$ increasing in length such that $d(y, f_{u_j}(x)) < \delta_j$. Hence, $d(f_i(y), f_{u_j i}(x)) < \frac{1}{j}$ for all j so $y \in \omega(x)$. □

Compared to the first limit set, the following limit set has a strong directionality, consisting of all points that occur along one particular trajectory of a point's orbit.

Definition 4.4. For $x \in X$ and $w \in W_\infty$, $\omega_w(x) = \bigcap_{n\in\mathbb{N}} \overline{\{f_{w_0\ldots w_k}(x) : k > n\}}$.

Defining $A_n = \left\{u : u = w_{|m} \text{ for some } m > n\right\}$ and $\mathcal{F} = \{A_n\}_{n\in\mathbb{N}}$, this is a limit set in the sense of Souza. Again, $\omega_w(x)$ is automatically compact and non-empty. However, in gaining directionality, we also lose general invariance, although a remnant is still present.

Theorem 4.5. *In the case of a free group action, for $y \in \omega_w(x)$ there exists $i \neq j \in S$ such that $f_i(y),\ f_j(y) \in \omega_w(x)$.*

Proof. Let $y \in \omega_w(x)$ be given. By the uniform continuity of the maps f_i, for every $n \in \mathbb{N}$ there is k_n such that if $d(x,z) < \frac{1}{k_n}$, then $d(f_i(x), f_i(z)) < \frac{1}{n}$ for all $i \in S$. For $n \in \mathbb{N}$ choose $m_n > n+1$ such that $d(f_{w_0 \dots w_{m_n}}(x), y) < \frac{1}{k_m}$. Choose $i \in S$ such that $i = w_{m_n+1}$ for infinitely many n. We can then choose j so $j = w_{m_n}^{-1}$ for infinitely many n with $w_{m_n+1} = i$. Thus $i \neq j$. By passing to a subsequence if necessary, we can assume $w_{m_n} = i^{-1}$ and $w_{m_n+1} = j$. Notice then that $d(f_{w_0 \dots w_{m_n}}(x), y) < \frac{1}{k_n}$. Thus $d(f_j(f_{w_0 \dots w_{m_n}}(x)), f_j(y)) = d(f_{w_0 \dots w_{m_n-1}}(x), f_j(y)) < \frac{1}{n}$. Similarly, $d(f_{w_0 \dots w_{m_n+1}}(x), f_i(y)) < \frac{1}{n}$. As $m_n - 1 > n$, $f_i(y)$, $f_j(y) \in \omega_w(x)$. $\square$

Because free monoids do not have inverses, the same formulation of invariance does not hold. Indeed, each point in the free monoid context may only have one direction of invariance. However, there is a direction in which the pre-image of a point remains in the limit set.

Corollary 4.6. *In the case of a free monoid action, for $y \in \omega_w(x)$ there exists $i \in S$ such that $f_i(y) \in \omega_w(x)$, and there exists $z \in \omega_w(x)$ and $j \in S$ with $y = f_j(z)$.*

Proof. The proof is essentially the same as the preceeding. Let $y \in \omega_w(x)$. By the uniform continuity of the maps f_i, for every $n \in \mathbb{N}$ there is k_n such that if $d(x,z) < \frac{1}{k_n}$, then $d(f_i(x), f_i(z)) < \frac{1}{n}$ for all $i \in S$. For $n \in \mathbb{N}$ choose $m_n > n+1$ such that $d(f_{w_0 \dots w_{m_n}}(x), y) < \frac{1}{k_m}$. Choose $i \in S$ such that $i = w_{m_n+1}$ for infinitely many n. Likewise, choose j such that $w_{m_n} = j$ for infinitely many m_n. By passing to a subsequence if necessary, we can assume this is the case for every m_n. Note then that $d(f_{w_0} \dots f_{w_m}(x), y) < \frac{1}{k_n}$. Hence, $d(f_{w_0} \dots f_{w_m+1}(x), f_i(y)) < \frac{1}{n}$, and $f_i(y) \in \omega_w(x)$. Now the set $\left\{ f_{w_0} \dots f_{w_{m_n-1}}(x) \right\} \subseteq X$ converges to some $z \in X$ as X is compact. This implies $z \in \omega_w(x)$ as we can find $m_{n'} > n$ with $d(z, f_{w_0} \dots f_{w_{m_{n'}-1}}(x)) < \frac{1}{n}$. Then, $f_j(z) = f_j(\lim_{n \to \infty} f_{w_0} \dots f_{w_{m_n-1}}(x)) = \lim_{n \to \infty} f_{w_0} \dots f_{w_{m_n}-1} f_j(x) = y$. $\square$

In the third limit set we consider, there is still a strong sense of direction. Instead of looking at the orbit of a point under one trajectory, we look at the trajectories that follow one direction for the first n steps. Even though this loses some directionality compared to the previous limit sets, it does recover general invariance.

Definition 4.7. For $x \in X$ and $w \in W_\infty$, $\omega_{F_w}(x) = \bigcap_{n\in\mathbb{N}} \overline{\{f_u(x) : u|_n = w|_n\}}$

By taking $\mathcal{F} = \{A_n\}_{n\in\mathbb{N}}$ where $A_n = \left\{u \in F : u_{|n} = w_{|n}\right\}$, this is considered a limit set under Souza's definition.

Lemma 4.8. $\omega_{F_w}(x) = \left\{y \in X : \forall\, m\ \exists\, u \in G \text{ s.t. } d(f_u(x), y)) < \frac{1}{m} \text{ and } u_{|m} = w_{|m}\right\}$.

Theorem 4.9. *$\omega_{F_w}(x)$ is invariant.*

Proof. For $m \in \mathbb{N}$ choose $k_m > m$ such that for $i \in S$, if $d(y, z) < \frac{1}{k_m}$ then $d(f_i(y), f_i(z)) < \frac{1}{m}$. Find $u \in G$ such that $d(f_u(x), y) < \frac{1}{k_m}$. Thus $d(f_{ui}(x), f_i(y)) < \frac{1}{m}$. As $k_m > m$, $ui_{|m} = w_{|m}$. Therefore $f_i(y) \in \omega_{F_w}(x)$. □

The final limit set we consider is the most restrictive and consists of points that occur in a point's orbit in *every* direction. Because of this strict requirement, not much structure is apparent. Even more, it is not guaranteed the limit set is non-empty.

To formalize the idea of occurence along every trajectory of an orbit, we define the following.

Definition 4.10. A *prefix set* P is a collection of words in F such that no word is a prefix of another. The *size* of P is $|P| = \sup\{|x| : x \in P\}$. The *minimum size* of P is $\lfloor P \rfloor = \min\{|x| : x \in P\}$ A finite prefix set P is a *complete prefix set* (CPS) if every $x \in X$ with $|x| \geq |P|$ has a prefix in P.

Definition 4.11. Let $x \in X$.
$\omega_{CPS}(x) = \left\{y \in X : \forall n\ \exists \text{ CPS } P_n \text{ with } \lfloor P_n \rfloor \geq n \text{ s.t. } \forall w \in P_n\ d(f_w(x), y) < \frac{1}{n}\right\}$.

Although difficult to see at this moment, this is a limit set under Sousa's definition when we take $\mathcal{F} = \cup_{w\in W_\infty} \mathcal{F}_w$ where $\mathcal{F}_w$ is the family taken to define $\omega_w(x)$. Once we prove that $\omega_{CPS}(x) = \bigcap_{w\in W_\infty} \omega_w(x)$, this will easily follow.

Lemma 4.12. *The set $\omega_{CPS}(x)$ is compact.*

Proof. Suppose $\omega_{CPS}(x) \neq \emptyset$ and $y \in \overline{\omega_{CPS}(x)}$. For $n \in \mathbb{N}$ choose $y_n \in \omega_{CPS}(x)$ with $d(y, y_n) < \frac{1}{2n}$. Choose a CPS P_n with $\lfloor P_n \rfloor > n$ so that for $w \in P_n$ $d(y_n, f_w(x)) < \frac{1}{2n}$. Then $d(y, f_w(x)) < \frac{1}{n}$. Hence, $y \in \omega_{CPS}(x)$, and $\omega_{CPS}(x)$ is closed. As X is compact, $\omega_{CPS}(x)$ is compact. $\square$

It is not necessary that $\omega_{CPS}(x)$ is non-empty. Consider $x \in \{0,1\}^F$ where for some fixed $i \in S$ $x(iu) = 0 = x(1)$ for $u \in F$, and $x(v) = 1$ otherwise. However, under the condition of mixing in a shift space, the set of all $x \in X$ with non-empty $\omega_{CPS}(x)$ is dense in X when in the context of free monoid actions. The following definition is the natural generalization of mixing in tree-shifts explored by Ban and Chang [1].

Definition 4.13. A shift space X over H is *mixing* if for every $i \in S$ there exists CPS P_i such that for any n-block A and m-block B there is an $x \in X$ such that the central n-block of x is A and for every word $u = u_1 \dots u_{n-1} \in H$ and for every $v = v_1 \dots v_k \in P_{u_{n-1}}$ with $v_1 \neq u_{n-1}^{-1}$, the m-block centered at uv is B.

Theorem 4.14. *If X is mixing, the set $\{x \in X : \omega_{CPS}(x) \neq \emptyset\}$ is dense in X.*

Proof. Fix $n > 1$ and $x \in X$. Let $A_1 = x_{|\Sigma^n}$. We will construct $y \in X$ with $d(x, y) = 2^{-n}$ and $y \in \omega_{CPS}(y)$. Let P_i be given by the definition of mixing, and let $M = \max\{|P_i|\}_{i \in S}$ and $m = \min\{\lfloor P_i \rfloor\}_{i \in S}$. By mixing, choose $y_1 \in X$ with $y_{1|\Sigma^n} = A_1$ and for for every word $u = u_1 \dots u_{n-1} \in H$ and for every $v = v_1 \dots v_m \in P_{u_{n-1}}$, the n-block centered at uv is A_1. We claim that $P_1 = \left\{uv : |u| = n-1 \text{ and } v \in P_{u_{n-1}}\right\}$ is a CPS. First, note that $|P_1| = M + n - 1$ and $\lfloor P_1 \rfloor > 1$. Suppose for contradiction that P_1 is not a CPS. Then for some $|t| > |P_1|$ every prefix of t is not in P_1. Write $t = t_1 \dots t_n \dots t_m$ and look at the word $t_n \dots t_m$. The length of this subword is greater than M so there exists $r \in P_{t_{n-1}}$ that is a prefix of $t_n \dots t_m$. Therefore, $t_1 \dots t_{n-1} r \in P_1$ and is a prefix of t. Hence P_1 is a CPS. Now set $n_1 = M + 2n$.

For the inductive step of the construction, suppose we have sequences $y_1, \ldots, y_k \in X$, integers $n_1, \ldots, n_k$, and CPS $P_1, \ldots, P_k$ such that

1. $n_{i+1} = M + 2n_i > |P_{i+1}|$.

2. $d(y_i, y_{i+1}) \leq 2^{-(n_i)}$.

3. For $t \in P_i$, $d(\sigma_t(y_{i+1}), y_i) \leq 2^{-(n_i)}$.

4. $\lfloor P_i \rfloor > i$.

Let $A_i = y_{i|\Sigma^{n_i}}$. Set $P_{i+1} = \{uv : |u| = n_i - 1 \text{ and } v \in P_{u_{n_i-1}}\}$. By the same argumentation as above, this is a CPS. Setting $n_{i+1} = M + 2n_i$ we see $|P_{i+1}| = n_i - 1 + M < n_{i+1}$ Furthermore, $\lfloor P_i \rfloor = n_i + m > i$. Finally, mixing gives the existence of a $y_{i+1} \in X$ such that for $t \in P_k$, $d(\sigma_t(y_{i+1}), y_i) \leq 2^{-(n_i)}$ and $d(y_i, y_{i+1}) \leq 2^{-(n_i)}$. Therefore our inductive hypotheses are correct.

By induction we have sequences $y_1, y_2, \dots \in X$, integers $n_1, n_2, \ldots,$ and CPS $P_1, P_2, \ldots$ matching our inductive hypothesis. As X is compact, $\{y_i\}$ converges to some $y \in X$. By nature of the inductive hypothesis, we get the following properties of y:

1. $d(y_i, y) \leq 2^{-(n_i)}$.

2. For $t \in P_i$, $d(\sigma_t(y), y_i) \leq 2^{-(n_i)}$.

We claim that $y \in \omega_{CPS}(y)$. Note that for $i > 0$, $\frac{1}{n} < 2^{-(n-1)}$. Properties (1) and (2) yield for $t \in P_i$, $d(\sigma_t(y), y) \leq 2^{-(n_i-1)}$. As $\lfloor P_i \rfloor > i$, $y \in \omega_{CPS}(y)$. Finally, $d(y, y_1) \leq 2^{-n_1} = 2^{-(M+2n)}$ so $d(x, y) \leq 2^{-n}$. □

Lemma 4.15. $\bigcup_{w \in W_\infty} \omega_w(x) \subseteq \omega(x)$.

Proof. Suppose $y \in \omega_w(x)$ for some $w = w_0 \cdots \in W_\infty$. For all $n > m$, $|w_0 \cdots w_{n-1}| > m$. Hence $\overline{\{f_{w_1 \ldots w_n}(x) : n > m\}} \subseteq \overline{\{f_u(x) : |u| > m\}}$ and $\omega_w(x) \subseteq \omega(x)$. □

Theorem 4.16. $\omega_{CPS}(x) = \bigcap_{w \in W_\infty} \omega_w(x)$.

Proof. Choose $w = w_0 w_1 \cdots \in W_{infty}$. Suppose $y \in \omega_{CPS}(x)$. For $n \in \mathbb{N}$ there is an $u_n \in P_n$ that is a prefix of w. Hence $u_n = w_0 \dots w_{k_n}$ and $d(f_{w_1 \dots w_{k_n}}(x), y) = d(f_{u_n}(x), y) < \frac{1}{n}$. As $k_n > n$, $y \in \omega_w(x)$. So $\omega_{CPS}(x) \subseteq \bigcap_{w \in W_\infty} \omega_w(x)$.

Now suppose that $y \in \bigcap_{w \in W_\infty} \omega_w(x)$. Let $n \in \mathbb{N}$. For each $w = w_1 w_2 \cdots \in W_\infty$, let $n_w > n$ be the minimal integer with $d(f_{w_1 \dots w_{n_w}(x)}, y) < \frac{1}{n}$. Under the same metric used for words in a shift space over $\mathbb{Z}$ (more specifically, $\mathbb{N}$), for each w, $U = B_{2^{-(n_w - 1)}}(w)$ satisfies $n_w = n_{w'}$ for $w' \in U$. W_∞ is compact so finitely many of these balls cover w. We can assume without loss of generality that these open sets are disjoint, as if not, one set will contain the other. For each open set in the finite subcover, the common central block of each open set forms a CPS with minimal length greater than n. Hence, $y \in \omega_{CPS}(x)$. □

Recall these defined limit sets were based on a sort of underlying directionality based on the orbit of a point under certain trajectories: ω_{CPS} under each trajectory, ω_w under a specific trajectory, ω_{F_w} under trajectories agreeing on the first few steps, and ω under any trajectory. Unsurprisingly, because of the underlying similarities of these trajectories, there are significant relationships between the limit sets.

Lemma 4.17. $\omega_{F_w}(x) \subseteq \omega(x)$

Proof. Let $y \in \omega_{F_w}(x)$. Thus there is a sequence of finite words w_n with $w_{n|n} = w_{|n}$ and $d(f_{w_n}(x), y) < 2^{-n}$. Note then that $|w_n| \geq n$. Therefore $y \in \omega(x)$. □

Lemma 4.18. $\omega_w(x) \subseteq \omega_{F_w}(x)$

Proof. This is immediate after noting $\{f_{w_0 \dots w_k}(x) : k > n\} \subseteq \left\{f_u(x) : u_{|n} = w_{|n}\right\}$. □

Theorem 4.19. $\omega_{CPS}(x) \subseteq \bigcap_{w \in W_\infty} \omega_{F_w}(x)$.

Proof. This follows immediately from the previous lemma. □

For an instance in which equality does not hold, consider $x \in \{0,1\}^{F_2}$ where for some fixed $i \in S$, $x(i^k) = 0$ for all $k \in \mathbb{Z}$, and $x(v) = 1$ elsewhere. Along $w_i = i^\omega$, $\omega_{w_i}(x) = x_1$ the constant element of all 1s. On the other hand, for $w_j = j^\omega$ for $j \neq i, i^{-1}$, $\omega_{w_j}(x) = x_0$ the constant element of all 0s. Therefore $\omega_{CPS}(x) = \emptyset$. However, it is easily seen that $x_1 \in \omega_{F_w}(x)$ for any $w \in W_\infty$.

These relationships are summarized in the following.

Theorem 4.20. $\omega_{CPS}(x) \subseteq \omega_w(x) \subseteq \omega_{F_w}(x) \subseteq \omega(x)$ *for all* $w \in W_\infty$.

Analogues of Internal Chain Transitivity

We have shown before that in the case of $\mathbb{Z}$ actions, there is a notion of a set of points being *chain transitive*. In these chain transitive sets, we can get from one point to another by way of a finite section of an ϵ-pseudo orbit, for any $\epsilon > 0$. In the context of $\mathbb{Z}$ actions, there is a definite direction the chain travels from one point to another, and it is the forward direction under the shift map. However, in the case of free group actions, there are many ways to go "forward." Thus, there are many directions a chain can go and needs to be specified.

Furthermore, an important property of chain transitivity in $\mathbb{Z}$ actions with the shadowing property is that they are precisely the limit sets of the action. A reasonable question to ask is whether there is a definition of internal transitivity which also characterizes limit sets in free group actions. The answer to this question, as we will see towards the end of the chapter, is yes for some of the limit sets we defined previously. However, due to the variety of limit sets defined, it is only natural there must also be a variety of types of internal transitivity. In what follows, we define these types of internal transitivity, prove properties of the collection of sets exhibiting these transitivities, and demonstrate limits sets display these transitivities. For notation, let $\mathfrak{W}_w$ denote all ω_w-limit sets and $\mathfrak{W}_{F_w}$ the set of all ω_{F_w}-limit sets in X.

The first internal transitivity we define is like in the context of $\mathbb{Z}$ actions in that a chain between two points follows a "linear" path, but the direction of the path depends on the end points and the precision of the path.

Definition 4.21. Given $\epsilon > 0$ and an element $u = u_1 \dots u_n \in W$, an *ϵ-chain indexed by u* is a sequence $\{x_1, \dots, x_{n+1}\}$ of X such that $d(f_{u_i}(x_i), x_{i+1}) < \epsilon$.

Definition 4.22. A closed subset Y of X is *internally chain transitive* ($Y \in ICT$) if for every $x, y \in Y$ and $\epsilon > 0$ there is a $u \in W$ and ϵ-chain indexed by u with $x_1 = x$ and $x_{n+1} = y$.

Theorem 4.23. *ICT is closed.*

Proof. Let $Y \in \overline{ICT}$, $x, y \in Y$, and $\epsilon > 0$. By the uniform continuity of f_i there is a $\frac{\epsilon}{3} > \delta > 0$ such that $d(a, b) < \delta$ implies $d(f_i(a), f_i(b)) < \frac{\epsilon}{3}$. Choose $B \in ICT$ with $d_H(A, B) < \delta$ and $x', y' \in B$ with $d(x, x') < \delta$ and $d(y, y') < \delta$. There is a δ-chain indexed by u from x' to y' in B. Say this chain is $\{x_1, \dots, x_{n+1}\}$. For $2 \leq i \leq n$ choose $z_i \in Y$ with $d(z_i, x_i) < \delta$, and let $z_1 = x$, $z_{n+1} = y$. It follows that $\{z_1, \dots, z_{n+1}\}$ is an ϵ-chain from x to y, as $d(f_{u_i}(z_i), z_{i+1}) \leq d(f_{u_i}(z_i), f_{u_i}(x_i)) + d(f_{u_i}(x_i), x_{i+1}) + d(x_{i+1}, z_{i+1}) < \epsilon$. Therefore $Y \in ICT$. □

In order to show that ω_w-limit sets are ICT, we utilize the following lemma.

Lemma 4.24. *For every $\epsilon > 0$, $w = w_1 w_2 \cdots \in W_\infty$, and $x \in X$ there exists $N_\epsilon \in \mathbb{N}$ such that for $n > N_\epsilon$ $d(f_{w_1 \dots w_n}(x), \omega_w(x)) < \epsilon$.*

Proof. Suppose to the contrary. Then we have an increasing sequence of integers $\{m_n\}$ with $d(f_{w_1 \dots w_{m_n}}(x), \omega_w(x)) > \epsilon$. By passing to a subsequence if necessary, $\left\{f_{w_1 \dots w_{m_n}}(x)\right\}$ converges to a point $y \in \omega_w(x)$. However, $d(y, \omega_w(x)) \geq \epsilon$, a contradiction. □

Theorem 4.25. $\mathfrak{W}_w \subseteq ICT$.

Proof. Let $\omega_w(x) \in \mathfrak{W}_w$, $y, z \in \omega_w(x)$, and $\epsilon > 0$. By the uniform continuity of f_i, there is a $\frac{\epsilon}{3} > \delta > 0$ such that if $d(p,q) < \delta$ then $d(f_u(p), f_u(q)) < \frac{\epsilon}{3}$ for $u \in S$. Let N_δ be given by the previous lemma. Find $n > m > N_\delta$ such that $d(f_{w_1 \dots w_m}(x), y) < \delta$ and $d(f_{w_1 \dots w_n}(x), z) < \delta$.

Let $k = n - m$ and fix $t_1 \dots t_k = w_{m+1} \dots w_n$. Set $x_0 = y$, $x_k = z$. For $1 < i < k$ choose x_i so $d(x_i, \sigma_{w_1 \dots w_{n+i-1}}(x)) < \delta$. We claim that $d(f_{t_i}(x_i), x_{i+1}) < \epsilon$. $d(x_i, f_{w_1 \dots w_{n+i-1}}(x)) < \delta$ so $d(f_{t_i}(x_i), f_{w_1 \dots w_{n+i}}(x)) < \frac{\epsilon}{3}$. Thus $d(f_{t_i}(x_i), x_{i+1}) < d(f_{t_i}(x_i), f_{w_1 \dots w_{n+i}}(x)) + d(x_{i+1}, f_{w_1 \dots w_{n+i}}(x)) < \epsilon$. Therefore $\omega_w(x)$ is internally chain transitive. $\square$

Likewise, we use the following lemma to show $\mathfrak{W}_{F_w} \subseteq ICT$ in the case of free group actions.

Lemma 4.26. *Given $x \in X$ and $w \in W$, for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $u \in G$ with $u_{|n} = w_{|n}$ for $n > N_\epsilon$, $d(\sigma_u(x), \omega_{F_w}(x)) < \epsilon$.*

Proof. Suppose not. Then for every $n > N$ there is a $u_n \in G$ with $u_{n|n} = w_{|n}$ such that $d(f_{u_n}(x), \omega_{F_w}(x)) > \epsilon$. By passing to a subsequence if necessary, these $f_{u_n}(x)$ converge to a point $y \in \omega_{F_w}(x)$. However, $d(y, \omega_{F_w}(x)) > \epsilon$, a contradiction. $\square$

Theorem 4.27. *For free group actions, $\mathfrak{W}_{F_w} \subseteq ICT$.*

Proof. Let $\omega_{F_w}(x) \in \mathfrak{W}_{F_w}, y, z \in \omega_{F_w}(x)$, and $\epsilon > 0$. By the uniform continuity of the f_i, there is a $\frac{\epsilon}{3} > \delta > 0$ such that if $d(p,q) < \delta$ then $d(f_i(p), f_i(q)) < \frac{\epsilon}{3}$ for $i \in S$. Let N_δ be given by the previous lemma. We can then find $u, v \in F$ such that for $n > m > N_\delta$ $u_{|m} = w_{|m}$, $v_{|n} = w_{|n}$,$u_{|n} \neq w_{|n}$, $d(f_u(x), y) < \delta$, and $d(f_v(x), z) < \delta$.

Fix $t_1 \dots t_k = u^{-1}v$. Set $x_0 = y$ and $x_k = z$. Note then that for any $l \leq k$, $ut_1 \dots t_{l|N_\delta} = w_{|N_\delta}$. Therefore, we can find $x_i \in \omega_{F_w}(x)$ such that $d(x_i, f_{ut_1} \dots t_i(x)) < \delta$. This choice then makes an ϵ chain from y to z in $\omega_{F_w}(x)$. Therefore, $\omega_{F_w}(x) \in ICT$. $\square$

Note that the previous proof relies heavily on the existence of inverses in the free group. Therefore, the proof does not hold in the strictly free monoid case, and is indeed false for some cases.

Even though ICT seems to be the most natural generalization of internal chain transitivity from $\mathbb{Z}$ actions, it is deficient in that it cannot characterize $\mathfrak{W}_w$.

Example 4.28. Let X be the full-shift on the alphabet $\{0, 1, 2\}$ over F_2. Let $Y \subset X = \{x_0, x_1, \dots\}$ where x_0 is 1 for elements of the form $a^m b^n$ $(n > 0)$ and 0 elsewhere, $x_1 = \sigma_{b^{-1}}(x_0)$, and finally we define x_2 the same as x_0 except replacing each 1 with 2. For $i > 2$, $x_{2+i} = \sigma_{a^i}(x_2)$. It is not hard to see that this is ICT. Getting to and from x_0 and x_1 is just a shift as is x_2 to x_{2+i}. To get from x_1 to x_2 we just need to get i sufficiently large so that $d(\sigma_a(x_1), x_{2+i}) < \epsilon$ and then shift to get to x_2.

However, Y cannot be expressed as $\omega_w(x)$ for any w, x. Note that for $\epsilon < 2^{-2}$ any ϵ-chain ending at x_0 must be indexed by a word ending in b and any ϵ-chain beginning at x_0 and going to x_i $(i > 0)$ must begin with b^{-1}. However, we claim this can never be the case for an element in $\omega_w(x)$.

Let $x \in X$, $2^{-2} > \epsilon > 0$, $w = w_1 w_2 \cdots \in W_\infty$ be given, and choose $N_\epsilon \in \mathbb{N}$ such that for $m > N_\epsilon$, $d(\sigma_{w_1 \dots w_m}(x), \omega_w(x)) < \epsilon$. Let $\frac{\epsilon}{3} > \delta > 0$ so that if $d(x, y) < \delta$, $d(\sigma_i(x), \sigma_i(y)) < \frac{\epsilon}{3}$. Find $k > N_\epsilon$ such that $d(\sigma_{w_1 \dots w_k}(x), x_1) < \delta$, $m > k$ with $d(\sigma_{w_1 \dots w_m}(x), x_0) < \delta$, and $l > m$ with $d(\sigma_{w_1 \dots w_l}(x), x_1) < \delta$. As in the proof that $\omega_w(x)$ is ICT, we can get an ϵ-chain between x_1 and x_0 indexed by the word $w_{m+1} \dots s_k$ and a ϵ-chain between x_0 and x_1 indexed by $w_{k+1} \dots w_l$. As w_k, w_{k+1} are in a reduced word, $w_k \neq w_{k+1}^{-1}$. This then shows that our example cannot be expressed as an ω_w-limit set.

In a sense, the failure of ICT to characterize $\mathfrak{W}_w$ is due to the lack of consistent structure or direction in the ϵ-chain between two points. In particular, if there is an ϵ-chain between x and y, and an ϵ-chain between y and z, it is not necessarily the case we can concatenate these chains to get an ϵ-chain between x and z. For instance, the ϵ-chain between y and z could backtrack on the direction of the chain from x to y and not agree on the overlap.

Thus, our goals necessitate a definition of internal transitivity that has sufficient structure and consistency in the ϵ-chain between points. This leads to the following definitions.

Definition 4.29. Let Y be a closed subset ot X. Y is *consistently internally chain transitive* ($Y \in CICT$) if for every $x \in Y$ there are $i(x), t(x) \in S$ with $i(x) \neq t(x)^{-1}$ such that for every $\epsilon > 0$ and $x, y \in Y$ there is an ϵ-chain between x and y indexed by a word starting with $i(x)$ and ending with $t(y)$.

It is easy to see that because monoids have no inverses, CICT is exactly the same as ICT for free monoid actions.

In the case of ICT, it was relatively easy to show the set is closed. However, the analogous result for $CICT$ has an added subtlety. Not only must there be δ-chains, but also each point x has an associated initial and terminal index $i(x)$ and $t(x)$. For $Y \in \overline{CICT}$ and $x \in Y$, it is intuitive to define $i(x), j(x)$ to match those of points in $CICT$ sets that converge to x. However, defining $i(x), j(x)$ for all x in this manner may not imply $i(x), t(y)$ interact in the necessary way for any pair x, y. In order to properly choose $i(x), t(x)$, we must appeal to properties of X being a compact metric space, namely that X is separable, viz., that X has a countable, dense subset.

Theorem 4.30. *$CICT$ is closed.*

Proof. Let $A \in \overline{CICT}$, and choose a countable, dense subset of A, $\Lambda = \{x_1, x_2, \dots\}$. Choose a sequence $\{B_i\}_{i \in \mathbb{N}} \subseteq CICT$ so that $d_H(A, B_n) < \frac{1}{n}$. For each x_n and $k \in \mathbb{N}$ choose $x_n^k \in B_k$ with $d(x_n, x_n^k) < \frac{1}{k}$. For x_1 we can choose $i(x_1), j(x_1)$ such that $i(x_1) = i(x_1^k)$ and $j(x_1) = j(x_1^k)$ for infinitely many k. By passing to a subsequence $\{k_m(1)\}$ of $\mathbb{N}$, we can assume $i(x_1) = i(x_1^{k_m(1)})$ and $j(x_1) = j(x_1^{k_m(1)})$ for all $k_m(1)$ in the sequence. By induction, we can choose $i(x_n), t(x_n)$ and a subsequence $\{k_m(n)\}$ of

$\{k_m(n-1)\}$ so that $i(x_n) = i(x_n^{k_m(n)})$ and $j(x_n) = j(x_n^{k_m(n)})$. For $x \in A \backslash \Lambda$, choose a sequence $\{x_{n_k}\} \subseteq \Lambda$ that converges to x. Choose $i(x), j(x)$ so that $i(x) = i(x_{n_k})$ and $j(x) = j(x_{n_k})$ for infinitely many k.

Let $x, y \in A$ and $\epsilon > 0$ be given. By the uniform continuity of the maps f_i there is a $\frac{\epsilon}{3} > \delta > 0$ such that $d(p,q) < \delta$ implies $d(f_i(p), f_i(q)) < \frac{\epsilon}{3}$. Choose $x_n, x_l \in \Lambda$ with $d(x, x_n), d(y, x_l) < \frac{\delta}{2}$, and $i(x_n) = i(x)$, $j(x_k) = j(y)$. Without loss of generality, assume $l > n$. Choose $k_m(l) > \frac{2}{\delta}$. By our choice, $i(x_n) = i(x_n^{k_m(l)})$, $j(x_l) = j(x_l^{k_m(l)})$, $d(x, x_n^{k_m(l)}), d(y, x_l^{k_m(l)}) < \delta$, and there is a δ-chain in $B_{k_m(l)}$ from $x_n^{k_m(l)}$ to $x_l^{k_m(l)}$ indexed by a word beginning with $i(x_n^{k_m(l)})$ and ending with $j(x_l^{k_m(l)})$. Replacing the first $x_n^{k_m(l)}$ with x and the last $x_l^{k_m(l)}$ with y, and every other element of the chain with an element of A within δ, gives an ϵ-chain from x to y indexed by a word beginning with $i(x)$ and ending with $t(y)$. Thus $A \in CICT$. $\square$

The following gives one direction of the characterization of $\mathfrak{W}_w = CICT$.

Lemma 4.31. $\mathfrak{W}_w \subseteq CICT$.

Proof. Fix $w \in W_\infty$ and $x \in X$. Choose $\frac{1}{3n} > \delta_n > 0$ so that for $d(y,z) < \delta_n$, $d(f_i(y), f_i(z)) < \frac{1}{3n}$ and an increasing sequence of integers $\{N_n\}$ so that if $m > N_n$ $d(f_{w_1 \ldots w_m}(x), \omega_w(x)) < \delta_n$.

For $y \in \omega_w(x)$, choose an increasing sequence of integers $\{k_n(y)\}$ with $d(f_{w_1 \ldots w_{k_n(y)}}(x), y) < \delta_n$. By passing to a subsequence if necessary, we can assume $w_{k_i(y)} = w_{k_{i+1}(y)}$ and $w_{1+k_i(y)} = w_{1+k_{i+1}(y)}$ for $i \in \mathbb{N}$. Set $t(y) = w_{k_i(y)}$ and $i(y) = w_{1+k_i(y)}$. Note that $i(y) \neq t(y)^{-1}$, as both are consecutive letters in a reduced word.

For given $y, z \in \omega_w(x)$ and $\epsilon > 0$, find $\frac{1}{n} < \epsilon$. Choose $N_n < k_l(y) < k_m(z)$. Just as in the proof that $\omega_w(x)$ is ICT we can find a ϵ-chain in $\omega_w(x)$ between y and z indexed by $w_{1+k_l(y)} \ldots w_{k_m(z)}$. $\square$

In the case of X being a shift of finite type over a free group or monoid, we have the necessary machinary in place to prove the other direction of the characterization of $\mathfrak{W}_w$. The following lemma helps to verify our construction is indeed correct.

Lemma 4.32. *Let X be a shift space and fix a function $\mathcal{O} : G \to X$. Suppose $u \in G$ and $m \in \mathbb{N}$ with the property that for $v \in \Sigma^{m-1}$ and $i \in S$, we have*

$$d(\sigma_i(\mathcal{O}(uv)), \mathcal{O}(uvi)) < 2^{-m}.$$

Then $\mathcal{O}(u)(v) = \mathcal{O}(uv)(e)$.

Proof. Fix $v = v_0 \dots v_n \in \Sigma^{m-1}$. As $d(\sigma_{v_0}(\mathcal{O}(u)), \mathcal{O}(uv_0)) < 2^{-m}$, the uniform continuity of the shift maps gives $d(\sigma_{v_0 v_1}(\mathcal{O}(u)), \sigma_{v_1}(\mathcal{O}(uv_0))) < 2^{-m+1}$. Because $d(\sigma_{v_1}(\mathcal{O}(uv_0)), \mathcal{O}(uv_0v_1)) < 2^{-m}$ we have $d(\sigma_{v_0v_1}(\mathcal{O}(u)), \mathcal{O}(uv_0v_1)) < 2^{-m+1}$. By induction we see $d(\sigma_{v_0 \dots v_i}(\mathcal{O}(u)), \mathcal{O}(uv_0 \dots v_i)) < 2^{-m+i}$.

Hence $d(\sigma_{v_0 \dots v_n}(\mathcal{O}(u)), \mathcal{O}(uv_0 \dots v_n)) < 2^{-m+n} \leq 2^{-1}$. This implies $\sigma_v(\mathcal{O}(u))(e) = \mathcal{O}(uv)(e)$. The left-hand side of this equation is equal to $\mathcal{O}(u)(v)$, so our lemma holds. □

Theorem 4.33. *If X is a shift of finite type and $Y \in CICT$, then $Y = \omega_w(x)$ for some $x \in X$ and $w \in W_\infty$.*

Proof. Let X be M-step. For $k \geq M+1$, let $\{x_i^k\}_{i=0}^{n_k} \subseteq Y$ be sequence that 2^{-k} covers Y. As $Y \in CICT$, there is a 2^{-k}-chain from x_i^k to x_{i+1}^k indexed by u_i where u_i begins with $i(x_i^k)$ and ends with $t(x_{i+1}^k)$. By concatenating these chains, we can get a 2^{-k}-chain $\{y_0^k, \dots, y_{n_k}^k\}$ from x_0^k to $x_{n_k}^k$ indexed by $w_k = v_1^k \dots v_{m_k}^k$ such that $v_1^k = i(x_0^k)$, $v_{m_k}^k = t(x_{n_k}^k)$, and for every i there is an n such that $x_i^k = y_n^k$. We also have a 2^{-k-1}-chain $\{z_0^k, \dots, z_{l_k}^k\}$ from $x_{n_k}^k$ to x_0^{k+1} indexed by $w_k' = v_1' \dots v_{l_k}'$ with $v_1' = i(x_{n_k}^k)$ and $v_{l_k}' = i(x_0^{k+1})$.

Concatenating both the words $w = w_{M+1}w'_{M+1}w_{M+2}w'_{M+2}\dots$ and the chain $\left\{y_0^{M+1}, \dots, y_{n_{M+1}}^{M+1}, z_1^{M+1}, \dots, z_{l_{M+1}}^{M+1}, y_1^{M+2}, \dots, y_{n_{M+2}}^{M+2}, \dots\right\}$ yields a sequence of points $\{z_0, z_1, \dots\} \subseteq Y$ and $w = t_1 t_2 \dots \in W_\infty$ such that for all $i \in \mathbb{N}$ $d(\sigma_{t_{i+1}}(z_i), z_{i+1}) < 2^{-M-1}$ and for $n > M+1$ there is k_n such that for $m > k_n$, $d(\sigma_{t_m+1}(z_m), z_{m+1}) < 2^{-n}$.

In order to keep track of these points and their interrelations, we define a function $\mathcal{O} : G \to Y$. Set $\mathcal{O}(t_1 \dots t_m) = z_m$ and $\mathcal{O}(e) = z_0$. For notation, let $t_0 = e$. For all other $v \in G$ let n_v be the largest integer such that $t_0 \dots t_{n_v}$ is a prefix of $v = t_0 \dots t_{n_v} v'$. Then let $\mathcal{O}(v) = \sigma_{v'}(z_{n_v})$.

By our construction, $\mathcal{O}$ has the properties that:

1. For $u \in G$, $v \in \Sigma^M$, and $i \in S$, $d(\sigma_i(\mathcal{O}(uv)), \mathcal{O}(uvi)) < 2^{-M-1}$.

2. For every $n \in N$ there is $k_n \in \mathbb{N}$ such that for $|u| > k_n$, $v \in \Sigma^n$, and $i \in R$, $d(\sigma_i(\mathcal{O}(uv)), \mathcal{O}(uvi)) < 2^{-n-1}$.

Define $x : G \to \mathcal{A}$ by $x(u) = \mathcal{O}(u)(e)$. We claim that $x \in X$ and that $\omega_w(x) = A$. By property (1) and Lemma 4.32, every M-block in x is the central M-block for some $\mathcal{O}(v)$. As every $\mathcal{O}(v) \in X$, this implies $x \in X$. Property (2) implies that $d(\sigma(x), \mathcal{O}(v)) < 2^{-n}$ for $|v| > k_n$.

Let $y \in Y$ and $n > \mathbb{N}$. Find k, i, m such that $d(y, x_i^k) < 2^{-n-1}$, $\mathcal{O}(u_1 \dots u_m) = x_i^k$, with $m > k_{n+1}$. Therefore $d(y, \sigma_{u_1 \dots u_m}(x)) < 2^{-n}$, so $y \in \omega_w(x)$.

Now suppose $y \in \omega_w(x)$. For $n \in \mathbb{N}$ find $m > k_{n+1}$ so that $d(y, \sigma_{u_1 \dots u_m}(x)) < 2^{-n-1}$ and $d(\sigma_{u_1 \dots u_m}(x), \mathcal{O}(u_1 \dots u_m)) < 2^{-n-1}$. Therefore $d(y, \mathcal{O}(u_1 \dots u_m)) < 2^{-n}$. As $\mathcal{O}(u_1 \dots u_m) \in Y$ by construction and Y is closed, $y \in Y$.

Therefore $Y = \omega_w(x)$. □

Corollary 4.34. *Let X be a shift of finite type over G. Then $\mathfrak{W}_w = CICT$. If G is a free monoid, then $\mathfrak{W}_w = ICT$.*

It seems at this point we have found the correct notion of internal transitivity to characterize $\mathfrak{W}_w$. The internal transitivity also seems to mimic the trajectory of the orbit of a point used in defining ω_w in that both the chain transitivity and the limit set consider one "linear" direction. However, this is not the case for $\mathfrak{W}_{F_w}$. These limit sets consider multiple directions of a point's orbit; as such, the corresponding internal transitivity should depend on more than just a line connecting points. For this we use block instead of chains.

Definition 4.35. Given a metric space X and $\delta > 0$, a δ-*G-pseudo-orbit* is a function $\mathcal{O} : G \to X$ such that for $u \in G$ and $i \in S$ we have $d(f_i(\mathcal{O}(u)), \mathcal{O}(ui)) < \delta$.

Definition 4.36. Let Y be a closed subset of X. Y is *internally block transitive* ($Y \in IBT$) if for every $\delta > 0$ $x_1, \ldots, x_n \in Y$ there is a δ-G-pseudo-orbit $\mathcal{O}$ of Y containing each x_i.

It seems strange we are calling this property *block* transitive, which implies a finite block, even though we use an infinite pseudo-orbit to define it. We are, however, entirely justified in this notation. For one, we are only concerned with the finte content of the pseudo-orbit; thus, we can restrict it to a finite block. Furthermore, any finite block can easily be extended in a full pseudo-orbit, so long as the underlying space is invariant (which we will prove is the case). The existence of this extension is a crucial property related to the lack of interference between elements of free groups and monoids.

Theorem 4.37. *For any X, IBT is closed.*

Proof. Suppose that $Y \in \overline{IBT}$ and $\epsilon > 0$. By uniform continuity, choose $\frac{\epsilon}{3} > \delta > 0$ so the $d(p, q) < \delta$ implies $d(f_i(p), f_i(q)) < \frac{\epsilon}{3}$. Choose $x_1, \ldots, x_n \in Y$ and $B \in IBT$ with $d_H(Y, B) < \delta$. Find $y_i \in B$ with $d(x_i, y_i) < \delta$ and let $\mathcal{O}$ be a δ-G-pseudo-orbit of B containing $y_1, \ldots, y_n$. We create a pseudo-orbit $\mathcal{O}'$ in A by replacing every y_i in $\mathcal{O}$ with x_i and replacing any other $y \in B$ with $x \in A$ so that $d(x, y) < \delta$. By the choice of δ, it is easy to see that $\mathcal{O}'$ is an ϵ-G-pseudo-orbit containing $x_0, \ldots, x_n$. Therefore $A \in IBT$ and IBT is closed. □

Theorem 4.38. *If $Y \in IBT$, Y is invariant.*

Proof. Let $y \in Y$ and consider $f_i(y)$. For $n \in \mathbb{N}$ there is a $\frac{1}{n}$-G-pseudo-orbit $\mathcal{O}_n$ containing y. Say $\mathcal{O}_n(u_n) = y$ for all n. Then $d(f_i(\mathcal{O}_n(u_n)), \mathcal{O}_n(u_n i)) < \frac{1}{n}$. Hence $d(f_i(y), \mathcal{O}_n(u_n i)) < \frac{1}{n}$. As $\mathcal{O}_n(u_n i) \in Y$ and Y is closed, $f_i(y) \in Y$. □

As in the case for ICT, the existence of these pseudo-orbits does not mean we can concatenate them in any meaningful way. Thus, we must also necessitate extra structure: points at which we can concatenate two blocks together. The following definition will be used for the free group context.

Definition 4.39. If $Y \in IBT$, an element $y \in Y$ is *i,j final* if for every $\delta > 0$, and $x_1, \dots, x_n \in Y$, there is a δ-G-pseudo-orbit $\mathcal{O}$ of Y, indexes u_k such that $\mathcal{O}(u_k) = x_k$ and $u_k \neq u_m$ for $k \neq m$, and indexes u_i, u_j with $\mathcal{O}(u_i) = \mathcal{O}(u_j) = y$ such that u_i, u_j end in $i \neq j$ respectively and u_i, u_j are not prefixes of each other or any u_k.

Definition 4.40. Let Y be a closed subset of X. $Y \in IBT^*$ if and only if $Y \in IBT$ and there exists $y \in Y$ that is i, j-final.

The following is a simplifed analogue for the free monoid case.

Definition 4.41. If $Y \in IBT$, an element $y \in Y$ is *final* if for every $\delta > 0$, $x_1, \dots, x_n \in Y$ there is a δ-G-pseudo-orbit $\mathcal{O}$ of Y and indexes u_k and u_y such that $\mathcal{O}(u_k) = x_k$, $\mathcal{O}(e) = \mathcal{O}(u_y) = y$ with u_y not a prefix of any u_k.

Definition 4.42. Let Y be a closed subset of X. $Y \in IBT^\circ$ if and only if $Y \in IBT$ and there exists $y \in Y$ that is final.

Theorem 4.43. *IBT^* is closed.*

Proof. Let $Y \in \overline{IBT^*}$ and let $\{B_n\}_{n\in\mathbb{N}} \subseteq IBT^*$ converge to Y. In each B_n, there is a point x_n that is i_n, j_n final. Choose i, j so that $i = i_n, j = j_n$ infinitely often. By passing to a subsequence if necessary, we can assume $i = i_n, j = j_n$ for all n. Choose $x \in X$ so $\{x_n\}_{n\in\mathbb{N}}$ converges to x. By a similar technique to Theorem 4.37, it is not hard to see $Y \in IBT^*$ with x being i, j final. □

Theorem 4.44. *IBT° is closed.*

Proof. This follows from the technique used in Theorem 4.37. $\square$

Here, we prove one direction of the characterization of $\mathfrak{W}_{F_w} = IBT^*$.

Lemma 4.45. *Let $w \in W_\infty$, $x \in X$ be given. For a free group action, $\omega_{F_w}(x) \in IBT^*$ with $y \in \omega_w(x)$ i,j-final for some i, j.*

Proof. Let $x_1, \ldots, x_n \in \omega_{F_w}(x)$, $y \in \omega_{F_w}(x)$, and $\delta > 0$ be given. By the uniform continuity of the maps f_i, find $\frac{\delta}{3} > \eta > 0$ such that if $d(x, y) < \eta$ then $d(f_i(x), f_i(y)) < \frac{\delta}{3}$ for $i \in S$. As $y \in \omega_w(x)$ and $\omega_w(x)$ is CICT, let $i = i(y)^{-1}$ and $j = t(y)$, so $i \neq j$.

Now we can find $w_1 \ldots w_{k_i} w_{k_i+1} \in F$ with $i = w_{k_i+1}^{-1}$, $d(y, f_{w_1 \ldots w_{k_i}}(x)) < \eta$ and $k_i > N_\eta$ given in Lemma 4.26. For x_1 we can find $w_1 \ldots w_{k_1} v_{x_1} \in F$ with $k_1 > k_i + 1$ such that $d(x_1, f_{w_1 \ldots w_{k_1} v_{x_1}}(x)) < \eta$ and v_{x_1} does not begin with w_{k_1+1}. By induction we can find $k_{m+1} > k_m$, $v_{x_{m+1}} \in F$ such that $d(x_{m+1}, f_{w_1 \ldots w_{k_{m+1}} v_{x_{m+1}}}(x)) < \eta$ and $v_{x_{m+1}}$ does not begin with $w_{k_{m+1}+1}$. Then we can get $k_j > k_n$ with $w_{k_j} = j$ and $d(y, f_{w_1 \ldots w_{k_j}}(x)$.

Now define $\mathcal{O} : F \to \omega_{F_w}(x)$ with $\mathcal{O}(w_{k_i+2} \ldots w_{k_m} v_{x_m}) = x_m$ and $\mathcal{O}(w_{k_i+1}^{-1}) = \mathcal{O}(w_{k_i+2} \ldots w_{k_j}) = y$. For $u = w_{k_i+1}^{-1} v$, define $\mathcal{O}(u) = f_v(y)$ and for all other u that has not yet been defined, we can choose $z \in \omega_w(x)$ with $d(z, f_{w_1 \ldots w_{k_i+1} u}(x)) < \eta$ and let $\mathcal{O}(u) = z$. By construction $\mathcal{O}$ is an δ-pseudo-orbit.

For the indexes, let $u_i = w_{k_i+1}^{-1}$, $u_j = w_{k_i+2} \ldots w_{k_j}$, and $u_m = w_{k_i+2} \ldots w_{k_m} v_{x_m}$. By construction $\mathcal{O}(u_m) = x_m$ and $u_k \neq u_m$ for $k \neq m$, $\mathcal{O}(u_i) = \mathcal{O}(u_j) = y$, u_i, u_j end in i, j respectively and u_i, u_j are not prefixes of each other or any u_m. $\square$

The same holds for the free monoid context.

Lemma 4.46. *Let $w \in W_\infty$, $x \in X$ be given. For a free monoid action, $\omega_{F_w}(x) \in IBT^\circ$ with $y \in \omega_w(x)$ final.*

Proof. Let $x_1, \ldots, x_n \in \omega_{F_w}(x)$, $y \in \omega_{F_w}(x)$, and $\delta > 0$ be given. By the uniform continuity of the maps f_i, find $\frac{\delta}{3} > \eta > 0$ such that if $d(x,y) < \eta$ then $d(f_i(x), f_i(y)) < \frac{\delta}{3}$ for $i \in S$.

Now we can find $w_1 \ldots w_k$ such that $d(y, f_{w_1 \ldots w_k}(x)) < \eta$. Find $k_1 > k > N_\eta$ from Lemma 4.26 and $u^1 = u_1 \ldots u_m$ with $u_1 \neq w_{k_1+1}$ and $d(x_1, f_{w_1 \ldots w_{k_1} u^1}(x)) < \eta$. Inductively find $k_i > k_{i-1}$ and $u^k = u_1 \ldots u_m$ with $u_1 \neq w_{k_i+1}$ and $d(x_i, f_{w_1 \ldots w_{k_i} u^k}(x)) < \eta$. Finally, find $k_y > k_n$ and $u^y = u_1 \ldots u_m$ with $u_1 \neq w_{k_n+1}$ and $d(y, f_{w_1 \ldots w_{k_y} u^y}, y) < \eta$. Define $\mathcal{O} : H \to \omega_w(x)$ by $\mathcal{O}(e) = \mathcal{O}(u^y) = y$, $\mathcal{O}(u^k) = x_k$ and for all other u choose $z \in \omega_w(x)$ so that $d(f_{w_1 \ldots w_k u}(x), z) < \eta$. This satisfies the requirements for IBT°. $\square$

Again, we have the machinery in place to prove the full characterization for the case of a shift of finite type over a free group and free monoid.

Theorem 4.47. *Suppose $X \in \mathcal{A}^F$ is an SFT with largest forbidden block size M, and let $Y \subseteq X$ be IBT, invariant, and compact with some $y \in Y$ i,j-final. Then $Y = \omega_{F_w}(\bar{x})$ for some $w \in W$ and $\bar{x} \in X$.*

Proof. For $n > M$, choose $\left\{x_1^n, \ldots, x_{k_n}^n\right\} \subseteq Y$ a 2^{-n} cover of Y. By assumption, we can find a 2^{-n}-pseudo-orbit $\mathcal{O}_n$ with indexes $u_i^n, u_j^n \in \Sigma^{k_n}$ ending in i and j respectively such that $\mathcal{O}_n(u_i^n) = \mathcal{O}_n(u_j^n) = y$ and indexes u_m^n such that $\mathcal{O}_n(u_m^n) = x_m^n$ and u_i^n, u_j^n are not prefixes of u_m^n for all m.

We will inductively construct a function $\mathcal{O} : F \to Y$ and a word $w \in W_\infty$.

First we construct w. Define $w_1 = u_j^{M+1}$ and for $n > 1$, $w_n = w_{n-1}(u_i^{M+n})^{-1} u_j^{M+n}$. By induction we will prove w_n ends with j and begins with w_{n-1}. Note u_j^{M+1} ends with j so w_1 ends with j. Suppose for our inductive step that w_{n-1} ends with j. As u_i^n ends with i and is not a prefix of u_j^n and u_j^n ends with j we have $(u_i^n)^{-1} u_j^n$ beginning with i^{-1} and ending with j. Therefore w_n ends with j. Finally, because $i \neq j$, w_n begins with w_{n-1}. Define $w = \lim_{n \to \infty} w_n$.

For ease in constructing $\mathcal{O}$, define $\mathcal{O}_n'$ to be $\mathcal{O}_n$ restricted to elements of Σ^{k_n} which

do not have u_i^n, u_j^n as a proper prefix. Let $\mathcal{D}_n$ be the domain of $\mathcal{O}'_n$. For $u \in \mathcal{D}_{M+1}$ define $\mathcal{O}(u) = \mathcal{O}'_{M+1}(u)$. Then for $u \in \mathcal{D}_{M+n}$ define $\mathcal{O}(w_{n-1}(u_i^{M+n})^{-1}u) = \mathcal{O}'_{M+n}(u)$. We will show that this step is well-defined. Suppose for some $n < m$ there is v, v' in $\mathcal{D}_n, \mathcal{D}_m$ respectively such that $w_{n-1}(u_i^{M+n})^{-1}v = w_{m-1}(u_i^{M+m})^{-1}v'$. Write $w_{m-1} = w_{n-1}(u_i^{M+n})^{-1}u_j^{M+n} \cdots (u_i^{M+m-1})^{-1}u_j^{M+m-1}$.
Thus $(u_i^{M+n})^{-1}v = (u_i^{M+n})^{-1}u_j^{M+n} \cdots (u_i^{M+m-1})^{-1}u_j^{M+m-1}(u_i^{M+m})^{-1}v'$.
Therefore $v = u_j^{M+n} \cdots (u_i^{M+m-1})^{-1}u_j^{M+m-1}(u_i^{M+m})^{-1}v'$. As v does not contain u_j^{M+n} as a proper prefix, $v = u_j^{M+n}$, $m = n+1$ and $v' = u_i^{M+n+1}$. In this instance, we have $\mathcal{O}(w_{n-1}(u_i^{M+n})^{-1}u_j^{M+n}) = \mathcal{O}'_{M+n}(v) = y$ and also $\mathcal{O}(w_n(u_i^{M+n+1})^{-1}u_i^{M+n+1}) = \mathcal{O}'_{M+n+1}(u_i^{M+n+1}) = y$. Thus $\mathcal{O}$, so far as it has been defined, is well-defined.

To complete the construction of $\mathcal{O}$, for $u \in F$ with $\mathcal{O}(u)$ not already defined, let u' be the largest prefix of u with $\mathcal{O}(u')$ already defined. Such u' always exists as $\mathcal{O}(e)$ is already defined. Then define $\mathcal{O}(u) = \sigma_{u'^{-1}u}(\mathcal{O}(u'))$. Thus we have defined $\mathcal{O} : F \to Y$.

From the construction, $\mathcal{O}$ has the properties that:

1. For $u \in F$, $v \in \Sigma^M$, and $i \in S$, $d(\sigma_i(\mathcal{O}(uv)), \mathcal{O}(uvi)) < 2^{-M-1}$.

2. For $n \in N$ there is $k_n \in \mathbb{N}$ such that for $|u| > k_n$, $v \in \Sigma^n$, and $i \in S$, $d(\sigma_i(\mathcal{O}(uv)), \mathcal{O}(uvi)) < 2^{-n-1}$.

Just as in Theorem 4.33, by defining $\bar{x}$ by $\bar{x}(v) = \mathcal{O}(v)(e)$ these properties imply that $\bar{x} \in X$ and $d(\sigma(\bar{x}), \mathcal{O}(v)) < 2^{-n}$ for $|v| > k_n$.

Finally, we show that $Y = \omega_{F_w}(\bar{x})$. First $\omega_{F_w}(\bar{x}) \subseteq Y$ as every N-block in $\bar{x}$ is an N-block of an element of A. Now let $z \in Y$ and $n \in \mathbb{N}$. We must find a $u \in F$ with $u_{|n} = w_{|n}$ and $d(\sigma_u(\bar{x}), z) < \frac{1}{n}$. Note $|w_n| \geq n$. Find x_m^{M+n+1} such that $d(x_m^{M+n}, z) < 2^{-(M+n+1)}$. Let $u = w_n(u_i^{M+n})^{-1}u_m^{M+n+1}$. Thus $u_{|n} = w_{|n}$ and $d(\sigma_u(\bar{x}), x_m^{M+n+1}) < 2^{-(M+n+1)}$. Therefore $d(\sigma_u(\bar{x}), z) < 2^{-(M+n+1)} < \frac{1}{n}$. Thus $Y = \omega_{F_w}(\bar{x})$. □

Note that this construction depends only upon the properties of $A \in IBT^*$. Therefore, as long as one replaces the shift maps σ with a general function f, the construction works in all other IBT^* sets with F actions.

Corollary 4.48. *Let X be a shift of finite type over a free group G. Then $\mathfrak{W}_{F_w} = IBT^*$.*

Using a slightly varied construction, we can get a similar result for free monoid actions.

Theorem 4.49. *Suppose $X \in \mathcal{A}^H$ is an SFT with largest forbidden block size M, and let $Y \subseteq X$ be IBT, invariant, and compact with some $y \in Y$ final. Then $Y = \omega_{F_w}(\bar{x})$ for some $w \in W_\infty$ and $\bar{x} \in X$.*

Proof. For $n > M$, choose $\{x_1^n, \dots, x_{k_n}^n\} \subseteq Y$ a 2^{-n} cover of Y. By assumption, we can find a 2^{-n}-pseudo-orbit $\mathcal{O}_n$ with indexes u_y^n, u_m^n such that $\mathcal{O}(e) = \mathcal{O}(u_y^n) = y$, $\mathcal{O}_n(u_m^n) = x_m^n$ and u_y^n is not a prefix of u_m^n for all m.

Define $w = u_y^{M+1} u_y^{M+2} \dots$. For notation, define $u_y^0 = e$. For $u \in H$, find the maximal m such that $u = u_y^0 \dots u_y^m u'$ and define $\mathcal{O}(u) = \mathcal{O}_{m+1}(u')$. Letting $\bar{x}(v) = \mathcal{O}(v)(e)$, the same reasoning as above gives $\omega_{F_w}(\bar{x}) = Y$. $\square$

Again, this construction also works for any $A \in IBT^\circ$, so long as one replaces the maps σ with f.

Corollary 4.50. *Let X be an SFT over a free monoid H. Then $\mathfrak{W}_{F_w} = IBT^\circ$.*

Shadowing in Group Actions

We now will develop the machinery to prove the characterizations $\mathfrak{W}_w = CICT$ and $\mathfrak{W}_{F_w} = IBT^*$ for more general contexts. As in the case of $\mathbb{Z}$ actions, these contexts depend on a notion of *shadowing*.

Definition 4.51. For $\epsilon > 0$, a function $\mathcal{O} : G \to X$ is *ϵ-shadowed by $x \in X$* if $d(f_u(x), \mathcal{O}(u)) < \epsilon$ for all $u \in G$.

Definition 4.52. A G action on a compact metric space X has the *G-shadowing property* if for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-G-pseudo-orbit is ϵ-shadowed by a point in X.

Lemma 4.53. *If X is not an SFT, then for every $n \in \mathbb{N}$ there is $m > n$ such that there is a forbidden m-block of X such that every sub-block is not forbidden.*

Proof. We will prove the contrapositive. Write $X = X_{\mathcal{F}}$ for some set of forbidden blocks $\mathcal{F}$. Let m be the largest integer such that there is an m-block $B \in \mathcal{F}$ such that every sub-block of B is not in $\mathcal{F}$. Let $\mathcal{F}' = \{B \in \mathcal{F} : B \text{ is a k-block for } k \leq m\}$. Note that $\mathcal{F}'$ is finite as there only finitely many k-blocks for $k \leq m$. We claim that $X_{\mathcal{F}} = X_{\mathcal{F}'}$. As $\mathcal{F}' \subseteq \mathcal{F}$, $X_{\mathcal{F}'} \supseteq X_{\mathcal{F}}$. Now suppose $x \notin X_{\mathcal{F}}$. Thus x contains a forbidden k-block B in $\mathcal{F}$. If $k \leq m$, $B \in \mathcal{F}'$ so $x \notin X_{\mathcal{F}'}$. If $k > m$, then B contains a forbidden l-block for $l < k$. By induction, B contains a forbidden l-block for $l \leq m$. In either case, x contains a block forbidden in $\mathcal{F}'$ so $x \notin X_{\mathcal{F}'}$. As $X = X_{\mathcal{F}} = X_{\mathcal{F}'}$, X is an SFT. $\square$

Lemma 4.54. *Suppose $\mathcal{O}$ is a 2^{-m}-pseudo-orbit. Then for $u \in F$ and $v \in \Sigma^{m-1}$, $\mathcal{O}(u)(v) = \mathcal{O}(uv)(e)$.*

Proof. This result follows from the same argument as Lemma 4.32. $\square$

Theorem 4.55. *A shift space X is an SFT if and only if X has the shadowing property.*

Proof. Suppose X is an M-step SFT and let $\epsilon > 0$ be given. Choose $k > M$ so $2^{-k} < \epsilon$. We claim every 2^{-k-1}-pseudo-orbit can be ϵ-shadowed. Let $\mathcal{O}$ be such a pseudo-orbit. Construct $x \in \mathcal{A}^F$ by $x(u) = \mathcal{O}(u)(e)$. Let $u \in F$ and $v \in \Sigma^k$. By definition, $\sigma_u(x)(v) = x(uv) = \mathcal{O}(uv)(1)$. By the previous lemma, the right-hand term equals $\mathcal{O}(u)(v)$. Thus $\sigma_u(x)_{|\Sigma^k} = \mathcal{O}(u)_{|\Sigma^k}$ so $d(\sigma_u(x), \mathcal{O}(u)) < 2^{-k}$. This implies x ϵ-shadows $\mathcal{O}$. Furthermore, $x \in X$ as every M-block in x is an M-block in an element of X. Thus x contains no forbidden blocks.

Now suppose X is not an SFT. Let $\epsilon = 2^{-1}$ and suppose X has the shadowing property. Thus there is a $\delta > 0$ such that every δ-pseudo-orbit can be ϵ-shadowed. Choose m with $2^{-m} < \delta$. By a previous lemma, there is a $k > m+2$ such that X has a forbidden k-block B with all sub-blocks of B not forbidden. For $i \in S$ let B_i be the (k-1)-block of B centered at i. As these are not forbidden, there exists an $x_i \in X$ such that $x_{i|\Sigma^{k-1}} = B_i$. Let $B_1 = B_{|\Sigma^{k-1}}$ and $x_1 \in X$ such that $x_{1|\Sigma^{k-1}} = B_1$. For $u \in F$, define $\mathcal{O}(iu) = \sigma_u(x_i)$ and $\mathcal{O}(e) = x_1$. For $i \in S$, $d(\sigma_{i^{-1}}(x_i), x_1) < 2^{-k+2}$ as B_i and B_0 overlap on a (k-2)-block. For $i, j \in S$ and $u \in F$ with $iuj \neq 1$, $\mathcal{O}(iuj) = \sigma_{uj}(x_i) = \sigma_j(\sigma_u(x_i)) = \sigma_j(\mathcal{O}(iu))$. Therefore $\mathcal{O}$ is a δ-pseudo-orbit. Suppose that $x \in X$ ϵ-shadows $\mathcal{O}$. Then for $u \in F$, $d(\sigma_u(x), \mathcal{O}(u)) < \epsilon$. Particularly, this implies $x(u) = \mathcal{O}(u)(e)$. We claim that x contains B. Clearly, $x(e) = B(e)$. For $i \in S$ and $u \in \Sigma^{k-1}$, $x(iu) = \sigma_{iu}(x)(e) = \mathcal{O}(iu)(e) = \sigma_u(x_i)(e) = x_i(u) = \mathcal{O}(i)(u) = B_i(u) = B(iu)$. Therefore our claim is correct, hence $\mathcal{O}$ cannot be ϵ-shadowed, contradicting the assumption of X having the shadowing property. $\square$

Theorem 4.56. *For an G action on X with G-shadowing, then $CICT = \overline{\mathfrak{W}_w}$.*

Proof. Let $Y \in CICT$. For $n \in \mathbb{N}$, find δ_n such that any δ_n-pseudo-orbit can $\frac{1}{n}$ shadowed. Define $\mathcal{O}$ as follows. Fix $k_0 > \frac{1}{\delta_n}$. For $k > k_0$ let $\left\{x_i^k\right\}_{i=0}^{n_k} \subseteq Y$ be sequence that $\frac{1}{k}$ covers Y. As Y is CICT, there is a $\frac{1}{k}$-chain from x_i^k to x_{i+1}^k indexed by u_i where u_i begins with $i(x_i^k)$ and ends with $t(x_{i+1}^k)$. By concatenating these chains, we can get a $\frac{1}{k}$-chain $\left\{y_0^k, \ldots, y_{n_k}^k\right\}$ from x_0^k to $x_{n_k}^k$ indexed by $w_k = v_1^k \ldots v_{m_k}^k$ such that $v_1^k = i(x_0^k)$, $v_{m_k}^k = t(x_{n_k}^k)$, and for every i there is an n such that $x_i^k = y_n^k$. We also have a $\frac{1}{k+1}$-chain $\left\{z_0^k, \ldots, z_{l_k}^k\right\}$ from $x_{n_k}^k$ to x_0^{k+1} indexed by $w_k' = v_1' \ldots v_{l_k}'$ with $v_1' = i(x_{n_k}^k)$ and $v_{l_k}' = i(x_0^{k+1})$.

Concatenating $w = w_{k_0} w_{k_0}' w_{k_0+1} w_{k_0+1}' \ldots$ and

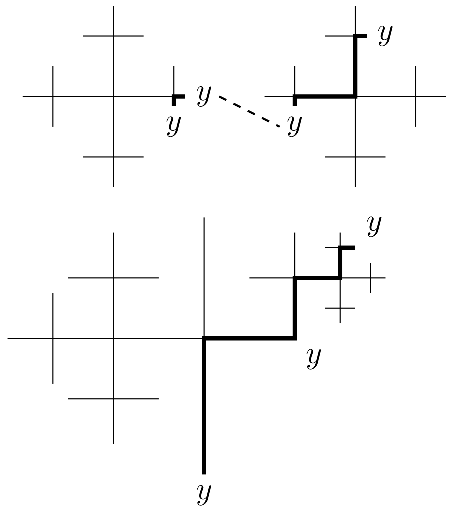

Figure 4.2: One step of the construction.

$\left\{y_0^{k_0}, \ldots, y_{n_{k_0}}^{k_0}, z_1^{k_0}, \ldots, z_{l_{k_0}}^{k_0}, y_1^{k_0+1}, \ldots, y_{n_{k_0+1}}^{k_0+1}, \ldots\right\}$ yields a sequence $\{z_0, z_1, \ldots\} \subseteq Y$ and $w = t_1 t_2 \cdots \in W_\infty$ such that for all $i \in \mathbb{N}$ $d(f_{t_{i+1}}(z_i), z_{i+1}) < \delta_n$ and for $n > \frac{1}{\delta_n}$ there is k_n such that for $m > k_n$, $d(f_{t_m+1}(z_m), z_{m+1}) < \frac{1}{n}$.

To construct the psuedo orbit, set $\mathcal{O}(t_1 \ldots t_m) = z_m$ and $\mathcal{O}(e) = z_0$. For notation, let $t_0 = 1$. For all other $v \in F$, let n_v be the largest integer such that $t_0 \ldots t_{n_v}$ is a prefix of $v = t_0 \ldots t_{n_v} v'$. Then let $\mathcal{O}(v) = f_{v'}(z_m)$.

Let x be a point that shadows this pseudo-orbit. We show that $d_H(\omega_w(x), Y) < \frac{1}{n}$. For any element of $y \in \omega_w(x)$, $d(y, Y) < \frac{1}{n}$ by the nature of the construction. If $a \in Y$, then there is a prefix v_m of w_n with $d(a, (O)(v_m)) < \frac{1}{m}$. By definition, $\{f_{v_m}(x)\}$ converges to a point $z \in \omega_w(x)$. Thus $d(a, z) < \frac{1}{n}$. Therefore $d_H(\omega_w(x_n), Y) < \frac{1}{n}$. As n was arbitrary, $Y \in \overline{\mathfrak{W}_w}$ and $CICT \subseteq \overline{\mathfrak{W}_w}$. We have already shown that $\mathfrak{W}_w \subseteq CICT$ and $CICT$ is closed; thus $\overline{\mathfrak{W}_w} \subseteq CICT$. $\square$

With a similar construction, we arrive at an analagous result for IBT sets.

Theorem 4.57. *For an F action on X with F-shadowing, $IBT^* = \overline{\mathfrak{W}_{F_w}}$.*

Proof. Let $Y \in IBT^*$. For $k \in \mathbb{N}$, find δ_k such that any δ_k-pseudo-orbit can $\frac{1}{k}$ shadowed. Define $\mathcal{O}$ as follows. Fix $k_0 > \frac{1}{\delta_k}$. For $k \geq k_0$ choose $\left\{x_1^n, \ldots, x_{k_n}^n\right\} \subseteq Y$

a $\frac{1}{n}$ cover of Y. By assumption, we can find a $\frac{1}{n}$-pseudo-orbit $\mathcal{O}_n$ with indexes $u_i^n, u_j^n \in \Sigma^{k_n}$ ending in i and j respectively such that $\mathcal{O}_n(u_i^n) = \mathcal{O}_n(u_j^n) = y$ and indexes u_m^n such that $\mathcal{O}_n(u_m^n) = x_m^n$ and u_i^n, u_j^n are not prefixes of u_m^n for all m.

We will inductively construct $\mathcal{O} : F \to Y$ and a word $w \in W_\infty$.

First we construct w. Define $w_0 = u_j^{k_0}$ and for $n > 0$, $w_n = w_{n-1}(u_i^{k_0+n})^{-1}u_j^{k_0+n}$. By induction we will prove w_n ends with j and begins with w_{n-1}. Note $u_j^{k_0}$ ends with j so w_0 ends with j. Suppose for our inductive step that w_{n-1} ends with j. As u_i^n ends with i and is not a prefix of u_j^n and u_j^n ends with j we have $(u_i^n)^{-1}u_j^n$ beginning with i^{-1} and ending with j. Therefore w_n ends with j. Finally, because $i \neq j$, w_n begins with w_{n-1}. Define $w = \lim_{n\to\infty} w_n$.

For ease in constructing $\mathcal{O}$, define $\mathcal{O}'_n$ to be $\mathcal{O}_n$ restricted to elements of Σ^{k_n} which do not have u_i^n, u_j^n as a proper prefix. Let $\mathcal{D}_n$ be the domain of $\mathcal{O}'_n$. For $u \in \mathcal{D}_{k_0}$ define $\mathcal{O}(u) = \mathcal{O}'_M(u)$. Then for $u \in \mathcal{D}_{k-0+n}$ define $\mathcal{O}^{k_0}(w_{n-1}(u_i^{k_0+n})^{-1}u) = \mathcal{O}'_{k_0+n}(u)$. We will show that this step is well-defined. Suppose for some $n < m$ there is v, v' in $\mathcal{D}_n, \mathcal{D}_m$ respectively such that $w_{n-1}(u_i^{k_0+n})^{-1}v = w_{m-1}(u_i^{k_0+m})^{-1}v'$. Write $w_{m-1} = w_{n-1}(u_i^{k_0+n})^{-1}u_j^{k_0+n}\cdots(u_i^{k_0+m-1})^{-1}u_j^{k_0+m-1}$.
Thus $(u_i^{k_0+n})^{-1}v = (u_i^{k_0+n})^{-1}u_j^{k_0+n}\cdots(u_i^{k_0+m-1})^{-1}u_j^{k_0+m-1}(u_i^{k_0+m})^{-1}v'$.
Therefore $v = u_j^{k_0+n}\cdots(u_i^{k_0+m-1})^{-1}u_j^{k_0+m-1}(u_i^{k_0+m})^{-1}v'$. As v does not contain $u_j^{k_0+n}$ as a proper prefix, $v = u_j^{k_0+n}$, $m = n+1$ and $v' = u_i^{k_0+n+1}$. In this instance, we have that $\mathcal{O}(w_{n-1}(u_i^{k_0+n})^{-1}u_j^{k_0+n}) = \mathcal{O}'_{k_0+n}(v) = y$ and $\mathcal{O}(w_n(u_i^{k_0+n+1})^{-1}u_i^{k_0+n+1}) = \mathcal{O}'_{k_0+n+1}(u_i^{k_0+n+1}) = y$. Thus $\mathcal{O}$, so far as it has been defined is well-defined.

To complete the construction of $\mathcal{O}$, for $u \in F$ with $\mathcal{O}(u)$ not already defined, let u' be the largest prefix of u with $\mathcal{O}(u')$ already defined. Such u' always exists as $\mathcal{O}(e)$ is already defined. Then define $\mathcal{O}(u) = \sigma_{u'^{-1}u}(\mathcal{O}(u'))$. Thus we have defined $\mathcal{O} : F \to Y$.

From the construction, we can see that $\mathcal{O}$ is a δ_k-pseudo-orbit. Let $x \in X$ $\frac{1}{k}$ shadow $\mathcal{O}$. By an analagous argument to 4.56, $d_H(\omega_{F_w}(x), Y) < \frac{1}{k}$. Thus $Y \in IBT^*$. $\square$

Theorem 4.58. *For an H action on X with H-shadowing, $IBT^\circ = \overline{\mathfrak{W}_{F_w}}$.*

Proof. Let $Y \in IBT^\circ$. For $k \in \mathbb{N}$, find δ_k such that any δ_k-pseudo-orbit can $\frac{1}{k}$-shadowed. Define $\mathcal{O}$ as follows.

Fix $k_0 > \frac{1}{\delta_k}$. For $n \geq k_0$ choose $\{x_1^n, \ldots, x_{k_n}^n\} \subseteq Y$ a $\frac{1}{n}$ cover of Y. By assumption, we can find a $\frac{1}{n}$-pseudo-orbit $\mathcal{O}_n$ with indexes u_y^n, u_m^n such that $\mathcal{O}_n(e) = \mathcal{O}_n(u_y^n) = y$, $\mathcal{O}_n(u_m^n) = x_m^n$ and u_y^n is not a prefix of u_m^n for all m.

Define $w = u_y^{k_0} u_y^{k_0} \ldots$. For notation, define $u_y^{k_0-1} = e$. For $u \in H$, find the maximal m such that $u = u_y^{k_0-1} \ldots u_y^m u'$ and define $\mathcal{O}(u) = \mathcal{O}_{m+1}(u')$. Clearly, $\mathcal{O}$ is a δ_k-H-pseudo-orbit.

By the same argument above, choosing x to $\frac{1}{k}$-shadow $\mathcal{O}$ implies $d_H(\omega_{F_w}(x), Y) < \frac{1}{k}$. Thus $Y \in IBT^\circ$. $\square$

In the previous results, we were able to get the desired characterization of $\mathfrak{W}_w$ and $\mathfrak{W}_{F_w}$ up to the closure of the internal transitive sets. In order to remove these closures from the characterization, it is necessary to have a strong form of shadowing.

Definition 4.59. An *asymptotic F-pseudo-orbit* is a function $\mathcal{O} : F \to X$ such that for every $\delta > 0$ there is an integer n such that for $|w| > n$ and $u \in S$, $d(f_u(\mathcal{O}(w)), \mathcal{O}(wu)) < \delta$.

Definition 4.60. A function $\mathcal{O} : F \to X$ is *asymptotically shadowed* if there is an $x \in X$ such that for every $\epsilon > 0$ there is an integer n such that for $|w| > n$ $d(f_w(x), \mathcal{O}(w)) < \epsilon$.

It is not the case that in a shift of finite type every asymptotic pseudo-orbit can be asymptotically shadowed. Consider for instance the shift of finite type of $\{0,1\}^F$ given by forbidding any 0 adjacent to a 1. This shift of finite type has two elements: x_0 and x_1, the constant maps of 0 and 1 respectively. We can construct an asymptotic pseudo-orbit by first choosing $j \in S$, then defining $\mathcal{O}(ju) = x_0$ for $u \in F$, $\mathcal{O}(iu) = x_1$ for $i \neq j \in S$ and $u \in F$, and $\mathcal{O}(1) = x_1$. Suppose that some y in the subshift asymptotically shadows $\mathcal{O}$. Then y must contain both 0 and 1, meaning it contains a 0 adjacent to a 1, so y is not in the subshift. This contradicts $\mathcal{O}$ being asymptotically shadowed.

Theorem 4.61. *If X is an m-step SFT, every asymptotic, 2^{-m-1}-pseudo-orbit can be asymptotically shadowed.*

Proof. Let $\mathcal{O}$ be such a pseudo-orbit. Construct x by $x(u) = \mathcal{O}(u)(1)$. By the previous theorem, $x \in X$.

For $k > m+1$ find l_k such that for $|u| > l_k$, $d(\sigma_i(\mathcal{O}(u)), \mathcal{O}(ui)) < 2^{-k}$ for all $i \in S$. We claim that for $|u| > l_{k+1} + k$, $d(\sigma_u(x), \mathcal{O}(u)) < 2^{-k}$. Notice by the choice of u that $\mathcal{O}_{|u\Sigma^k}$ is a finite portion of a 2^{-k-1} pseudo-orbit. Hence by a previous lemma, $\mathcal{O}(u)(v) = \mathcal{O}(uv)(e)$ for $v \in \Sigma^k$. Thus $\sigma_u(x)(v) = x(uv) = \mathcal{O}(uv)(1) = \mathcal{O}(u)(v)$. Hence $d(\sigma_u(x), \mathcal{O}(u)) < 2^{-k}$, so our claim is correct. Therefore x asymptotically shadows $\mathcal{O}$. $\square$

Definition 4.62. A G action on a compact metric spaces has the *weak* G-asymptotic shadowing property if there exists $\delta > 0$ such that every asymptotic G-pseudo-orbit which is also a δ-G-pseudo-orbit is asymptotically shadowed.

With this definition, we are finally able to give the full characterizations we were after.

Theorem 4.63. *For an G action on X with the weak G-asymptotic shadowing property, $\mathfrak{W}_w(X) = CICT$.*

Proof. It remains to show that $CICT \subseteq \mathfrak{W}_w(X)$. Let $A \in CICT$ and find $\delta > 0$ that witnesses the asymptotic shadowing property. Choose $k > \frac{1}{\delta}$ and let $\mathcal{O}$ and w be as defined in Theorem 4.56. By construction, $\mathcal{O}$ is an asymptotic δ-F-pseudo-orbit. If $x \in X$ asymptotically shadows $\mathcal{O}$, it is not difficult to see that $A = \omega_w(x)$. $\square$

We can also use the constructions given in Theorems 4.57 and 4.58 to obtain the following results.

Theorem 4.64. *For a G action on X with the weak G-asymptotic shadowing property, $\mathfrak{W}_{F_w}(X) = IBT^*$.*

Theorem 4.65. *For an H action on X with the weak H-asymptotic shadowing property, $\mathfrak{W}_{F_w}(X) = IBT^\circ$.*

Conclusion

In this chapter, we have defined certain limit sets for free group and monoid actions, defined types of internal transitivites in these actions, and showed these definitions are equivalent under certain types of these actions. In this way, we generalized many of the notions of $\mathbb{Z}$ actions to this setting.

However, there remains work to be done still. For one, we considered only four types of limit sets. There are certainly other interesting sets to consider that also can be characterized by a kind of internal transitivity. In this way, the work on characterizing limit sets is in no way comprehensive. Also, while the definition of $CICT$ does seem to be the correct definition of internal transitivity to chracterize $\mathfrak{W}_w$, this may not be the case for IBT^* or IBT°. While these conditions are certainly necessary and sufficient, they are not elegant with the many conditions. As formulated in the following questions, it is not known how they may be simplified.

Problem 4.66. For an F action, if Y is IBT, compact, and invariant, does there necessarily exist some $y \in Y$ which is i, j-final? For an H action, if Y is IBT, compact, and invariant, does there necessarily exist some $y \in Y$ which is final?

Thus, the work done in this chapter can be improved and simplified. Finally, this thesis only considers the interaction of limit sets and chain transitivity in the specific context of the four limit sets defined for free group and monoid actions. The general case for any limit set in any group or monoid action is still an open question, and it would be interesting to consider how the structure of the family of subsets used to define the limit set relates to the structure of the internal transitivity used to characterize the limit set, if such a characterization exists. The results of this thesis seem to suggest there is some relation.

[1] Jung-Chao Ban and Chih-Hung Chang. "Tree-shifts: irreducibility, mixing, and the chaos of tree-shifts". In: *Trans. Amer. Math. Soc.* 369.12 (2017), pp. 8389–8407. ISSN: 0002-9947. DOI: 10.1090/tran/6906. URL: https://doi.org/10.1090/tran/6906.

[2] June Barrow-Green. *Poincaré and the three body problem*. Vol. 11. History of Mathematics. American Mathematical Society, Providence, RI; London Mathematical Society, London, 1997, pp. xvi+272. ISBN: 0-8218-0367-0.

[3] Andrew D. Barwell et al. "Characterizations of ω-limit sets in topologically hyperbolic systems". In: *Discrete Contin. Dyn. Syst.* 33.5 (2013), pp. 1819–1833. ISSN: 1078-0947. DOI: 10.3934/dcds.2013.33.1819. URL: https://doi.org/10.3934/dcds.2013.33.1819.

[4] Andrew Barwell et al. "A characterization of ω-limit sets in shift spaces". In: *Ergodic Theory Dynam. Systems* 30.1 (2010), pp. 21–31. ISSN: 0143-3857. DOI: 10.1017/S0143385708001089. URL: https://doi.org/10.1017/S0143385708001089.

[5] Douglas Lind and Brian Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995, pp. xvi+495. ISBN: 0-521-55124-2; 0-521-55900-6. DOI: 10.1017/CBO9780511626302. URL: https://doi.org/10.1017/CBO9780511626302.

[6] Jonathan Meddaugh and Brian E. Raines. "Shadowing and internal chain transitivity". In: *Fund. Math.* 222.3 (2013), pp. 279–287. ISSN: 0016-2736. DOI: 10.4064/fm222-3-4. URL: https://doi.org/10.4064/fm222-3-4.

[7] Jonathan Meddaugh and Brian E. Raines. "The structure of limit sets for $\mathbb{Z}^d$ actions". In: *Discrete Contin. Dyn. Syst.* 34.11 (2014), pp. 4765–4780. ISSN: 1078-0947. DOI: 10.3934/dcds.2014.34.4765. URL: https://doi.org/10.3934/dcds.2014.34.4765.

[8] Josiney A. Souza. "On limit behavior of semigroup actions on noncompact spaces". In: *Proc. Amer. Math. Soc.* 140.11 (2012), pp. 3959–3972. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-2012-11248-1. URL: https://doi.org/10.1090/S0002-9939-2012-11248-1.

[9] Susan G. Williams, ed. *Symbolic dynamics and its applications*. Vol. 60. Proceedings of Symposia in Applied Mathematics. Lectures of the American Mathematical Society Short Course held in San Diego, CA, January 4–5, 2002, AMS Short Course Lecture Notes. American Mathematical Society, Providence, RI, 2004, pp. viii+156. ISBN: 0-8218-3157-7. DOI: 10.1090/psapm/060. URL: https://doi.org/10.1090/psapm/060.