ABSTRACT

Boundary Data Smoothness for Solutions of Nonlocal
Boundary Value Problems for nth Order Differential Equations

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Under certain conditions, solutions of the nth order boundary value problem, $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$, $y(a) - \sum_{k=1}^{p} a_k y(\xi_k) = y_1$, $y^{(i-1)}(\gamma) = y_i$, for $2 \le i \le n-1$, and $y(b) - \sum_{l=1}^{q} b_l y(\eta_l) = y_n$, are differentiated with respect to boundary conditions, where $c < a < \xi_1 < \cdots < \xi_p < \gamma < \eta_1 < \cdots < \eta_q < b < d$, $a_1, \ldots, a_p$, $b_1, \ldots, b_q \in \mathbb{R}$, and $y_1, \ldots, y_n \in \mathbb{R}$. The method involves application of Peano's Theorem for initial value problems.

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Boundary Value Problems for nth Order Differential Equations

by

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CHAPTER ONE

Introduction

In this dissertation, we will be concerned with differentiating solutions of certain nonlocal boundary value problems with respect to boundary data for the nth order ordinary differential equation,

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \qquad c < x < d, \tag{1.1}$$

satisfying the boundary conditions

$$\begin{cases} y(a) - \sum_{k=1}^{p} a_k y(\xi_k) &= y_1, \\ y^{(i-1)}(\gamma) &= y_i, \qquad 2 \le i \le n-1, \\ y(b) - \sum_{l=1}^{q} b_l y(\eta_l) &= y_n, \end{cases} \tag{1.2}$$

where $c < a < \xi_1 < \cdots < \xi_p < \gamma < \eta_1 < \cdots < \eta_q < b < d$, $a_1, \ldots, a_p,\ b_1, \ldots, b_q \in \mathbb{R}$, $y_1, \ldots, y_n \in \mathbb{R}$, and where we assume

(i) $f(x, r_1, \ldots, r_n) : (c, d) \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

(ii) $\frac{\partial f}{\partial r_i}(x, r_1, \ldots, r_n) : (c, d) \times \mathbb{R}^n$ is continuous, $\forall\ 1 \le i \le n$, and

(iii) solutions of initial value problems (IVP's) for (1.1) extend to (c, d).

Note that condition (iii) is not a necessary condition, but it allows us to avoid continually making statements in terms of solutions' maximal intervals of existence.

In (1.2), the boundary conditions involving the points a and b are often called *nonlocal* boundary conditions, whereas, we may at times refer to the boundary conditions involving the point γ as *stacked* boundary conditions.

There has been much recent attention on existence of solutions for boundary value problems of type (1.1),(1.2), with primary interest in positive solutions of such problems. The papers by Luca [27] and Su *et al.* [32] are devoted to such questions, and the papers by Henderson and Luca [16–20] and Henderson and Ntouyas [22] deal with positive solutions of coupled systems of boundary value problems like (1.1),(1.2).

Much research also has focused on smoothness with respect to boundary data for solutions of boundary value problems. The paper in 1975 by Spencer [31] seems to be the initial work for such questions. Then in 1984, Henderson [11,12] examined such questions for solutions of (1.1) satisfying right local boundary conditions. Following that, numerous papers by Ehme [6], Ehrke *et al.* [9], Henderson *et al.* [13, 14, 21], Lawrence [26], and Lyons [28] have dealt with smoothness of solutions with respect to boundary data for solutions of (1.1) involving various types of boundary conditions, including conjugate, focal, and nonlocal types. Davis [5] considered analogous questions for Lidstone boundary value problems for (1.1), when n is an even integer, and Ehme *et al.* [7, 8] also addressed analogous questions for solutions of functional differential equations satisfying deviating boundary conditions. For the case of discrete boundary value problems, that is, boundary value problems for finite difference equations, questions of smoothness of solutions with respect to boundary conditions have been dealt with by Benchohra *et al.* [2], Datta [3], Datta and Henderson [4], Henderson and Lee [15], Hopkins *et al.* [23], Janson *et al.* [25], and Lyons [29]. Also, the paper by Baxter *et al.* [1] focused on boundary data smoothness for solutions of boundary value problems for dynamic equations of a time scale.

In Chapter Two, we will characterize the partial derivatives of the solutions of (1.1),(1.2) with respect to the boundary values $y_1, \ldots, y_n$. We will also show that our partial derivatives with respect to $y_1, \ldots, y_n$ form a basis for the solution space of (1.1),(1.2).

In Chapter Three, we will characterize the partial derivatives of the solutions of (1.1),(1.2) with respect to boundary points $a, b, \xi_1, \ldots, \xi_k$, and $\eta_1, \ldots, \eta_l$ from our nonlocal boundary conditions, and with respect to γ from our stacked boundary condition.

In Chapter Four, we will characterize the partial derivatives of the solutions of (1.1),(1.2) with respect to parameters $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$.

For our differentiation with respect to boundary conditions results, given a solution $y(x)$ of (1.1), we will give much attention to the *variational equation* for (1.1) along $y(x)$, which is defined by

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}(x, y(x), y'(x),\ \ldots, y^{(n-1)}(x))z^{(i-1)}. \tag{1.3}$$

Under uniqueness assumptions on solutions of (1.1),(1.2), we will establish analogues of Theorem 1, a result that Hartman [10] attributes to Peano [30] concerning differentiation of solutions of (1.1) with respect to boundary conditions.

Theorem 1 [Peano]: Assume that with respect to (1.1), conditions (i)-(iii) are satisfied. Let $x_0 \in (c,d)$ and $y(x) = y(x, x_0, c_1, \ldots, c_n)$ denote the solution of (1.1) satisfying the initial conditions

$$y^{(i-1)}(x_0) = c_i, \qquad \forall\ 1 \le i \le n. \tag{1.4}$$

Then $\alpha_j := \frac{\partial y}{\partial c_j}$, $\forall\ 1 \le j \le n$, exists on (c,d) and is the solution of the variational equation (1.3) along $y(x)$ satisfying the initial conditions,

$$\alpha_j^{(i-1)}(x_0) = \left\{ \begin{array}{ll} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{array} \right\} = \delta_{ij}, \qquad 1 \le i \le n, \tag{1.5}$$

In addition, our analogues of Theorem 1 depend on the uniqueness of solutions of (1.1),(1.2), a condition we list as an assumption:

(iv) Given $\{a_1, \ldots, a_p\}$, $\{b_1, \ldots, b_q\}$, $c < a < \xi_k < \gamma < \eta_l < b < d$, $1 \le k \le p$, $1 \le l \le q$, if $y(x)$ and $z(x)$ are solutions of (1.1) satisfying

$$\begin{cases} y(a) - \sum_{k=1}^{p} a_k y(\xi_k) & = z(a) - \sum_{k=1}^{p} a_k z(\xi_k), \\ y^{(i-1)}(\gamma) & = z^{(i-1)}(\gamma), \qquad 2 \le i \le n-1, \\ y(b) - \sum_{l=1}^{q} b_l y(\eta_l) & = z(b) - \sum_{l=1}^{q} b_l z(\eta_l), \end{cases}$$

then $y(x) = z(x)$.

The last assumption provides uniqueness of solutions of (1.3) along solutions of $y(x)$ of (1.1).

(v) Given $\{a_1, \ldots, a_p\}$, $\{b_1, \ldots, b_q\}$, $c < a < \xi_k < \gamma < \eta_l < b < d$, $1 \le k \le p$, $1 \le l \le q$, and given any solution $y(x)$ of (1.1), if $u(x)$ is a solution of the variational equation (1.3) along $y(x)$, and

$$\begin{cases} u(a) - \sum_{k=1}^{p} a_k u(\xi_k) & = 0, \\ u^{(i-1)}(\gamma) & = 0, \qquad 2 \le i \le n-1, \\ u(b) - \sum_{l=1}^{q} b_l u(\eta_l) & = 0, \end{cases}$$

then $u(x) = 0$.

CHAPTER TWO

An Analogue for Peano's Theorem for (1.1), (1.2), I

In this chapter, we derive an analogue of Theorem 1 for the boundary value problem (1.1),(1.2). For such a differentiation result, we need continuous dependence of solutions on boundary conditions. We will make use of an application of the Brouwer Invariance of Domain Theorem [24].

Lemma 1: Assume (i)-(iv) are satisfied. Let $u(x)$ be a solution of (1.1) on (c, d), let $a_1, \ldots, a_p,\ b_1, \ldots, b_q \in \mathbb{R}$ and $c < a < \xi_1 < \cdots < \xi_p < \gamma < \eta_1 < \cdots < \eta_q < b < d$ be given, and let $c < r < a$ and $b < s < d$. Then there exists a $\delta > 0$ such that, if

$$\begin{aligned}
|a - \hat{a}| < \delta, \quad |\gamma - \hat{\gamma}| < \delta, \quad |b - \hat{b}| < \delta,& \\
|a_k - \hat{a_k}| < \delta,\ |\xi_k - \hat{\xi_k}| < \delta,& \qquad 1 \le k \le p, \\
|b_l - \hat{b_l}| < \delta,\ |\eta_l - \hat{\eta_l}| < \delta,& \qquad 1 \le l \le q, \\
\left| u(a) - \sum_{k=1}^{p} a_k u(\xi_k) - y_1 \right| < \delta,& \\
\left| u^{(i-1)}(\gamma) - y_i \right| < \delta,& \qquad 2 \le i \le n-1,
\end{aligned}$$

and

$$\left| u(b) - \sum_{l=1}^{q} b_l u(\eta_l) - y_n \right| < \delta,$$

then there exists a unique solution $u_\delta(x)$ of (1.1) such that

$$\begin{cases}
u_\delta(\hat{a}) - \sum_{k=1}^{p} \hat{a_k} u_\delta(\hat{\xi_k}) & = y_1, \\
u_\delta^{(i-1)}(\hat{\gamma}) & = y_i, \qquad 2 \le i \le n-1, \\
u_\delta(\hat{b}) - \sum_{l=1}^{q} \hat{b_l} u_\delta(\hat{\eta_l}) & = y_n,
\end{cases}$$

and for $0 \le j \le n-1$, $\{u_\delta^{(j)}(x)\}$ converges uniformly to $u^{(j)}(x)$, as $\delta \to 0$, on $[r, s]$.

Proof. Fix $x_0 \in (c, d)$. Define the open subset $G \subseteq \mathbb{R}^{2p+2q+n+3}$ by

$$G := \Big\{ (t_1, r_1, \ldots, r_p, \sigma, s_1, \ldots, s_q, t_2, a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_n) \in \mathbb{R}^{2p+2q+n+3} \Big| \\ c < t_1 < r_1 < \cdots < r_p < \sigma < s_1 < \cdots < s_q < t_2 < d, \\ (a_1, \ldots, a_p) \in \mathbb{R}^p, (b_1, \ldots, b_q) \in \mathbb{R}^q, \quad \text{and} \quad (c_1, \ldots, c_n) \in \mathbb{R}^n \Big\}$$

and define a mapping $\kappa : G \to \mathbb{R}^{2p+2q+n+3}$ by

$$\phi(t_1, r_1, \ldots, r_p, \sigma, s_1, \ldots, s_q, t_2, a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_n) := \\ \Big(t_1, r_1, \ldots, r_p, \sigma, s_1, \ldots, s_q, t_2, a_1, \ldots, a_p, b_1, \ldots, b_q, \\ y(t_1) - \sum_{k=1}^{p} a_k y(r_k), y'(\sigma), \ldots, y^{(n-2)}(\sigma), y(t_2) - \sum_{l=1}^{q} b_l y(s_l) \Big),$$

where $y(x)$ is the solution of the initial value problem for (1.1) satisfying $y^{(i-1)}(x_0) = c_i$, $1 \le i \le n$.

By (i) and (ii), solutions of initial value problems for (1.1) are unique, and hence depend continuously upon initial conditions. Therefore, it follows that ϕ is continuous on G. Moreover, from assumption (iv) and uniqueness of solutions of initial value problems for (1.1), it is straightforward that ϕ is one-to-one on G.

By the Brouwer Theorem on Invariance of Domain [24], it follows that $\phi(G)$ is an open subset on $\mathbb{R}^{2p+2q+n+3}$ and that $\phi : G \to \phi(G)$ is a homeomorphism.

The conclusion of the lemma readily follows from the open property of $\phi(G)$ and the continuity of $\phi^{-1} : \phi(G) \to G$. $\square$

For our analogue of Peano's Theorem, we present it in two main theorems.

2.1 - Differentiation with Respect to Boundary Values $(y_1, \ldots, y_n)$

Theorem 2: Assume (i)-(v) are satisfied. Let $u(x)$ be a solution of (1.1) on (c, d), and let $a_1, \ldots, a_p,\ b_1, \ldots, b_q \in \mathbb{R}$ and $c < a < \xi_1 < \cdots < \xi_p < \gamma < \eta_1 < \cdots < \eta_q < b < d$ be given so that

$$u(x) = u(x, a,\ \xi_1, \ldots, \xi_p,\ \gamma,\ \eta_1, \ldots, \eta_q,\ b,\ a_1, \ldots, a_p,\ b_1, \ldots, b_q,\ u_1, \ldots, u_n),$$

where

$$\begin{cases} u(a) - \sum_{k=1}^{p} a_k u(\xi_k) & = u_1, \\ u^{(i-1)}(\gamma) & = u_i, \qquad 2 \le i \le n-1, \\ u(b) - \sum_{l=1}^{q} b_l u(\eta_l) & = u_n. \end{cases} \tag{2.1}$$

Then $D_{u_1} := \frac{\partial u}{\partial u_1}$ exists on (c, d), is a solution of (1.3) along $u(x)$, and satisfies the boundary conditions

$$\begin{cases} D_{u_1}(a) - \sum_{k=1}^{p} a_k D_{u_1}(\xi_k) & = 1, \\ D_{u_1}^{(i-1)}(\gamma) & = 0, \qquad 2 \le i \le n-1, \\ D_{u_1}(b) - \sum_{l=1}^{q} b_l D_{u_1}(\eta_l) & = 0. \end{cases}$$

Then $D_{u_j} := \frac{\partial u}{\partial u_j}$ exists on (c, d), $\forall\ 2 \le j \le n-1$, is a solution of (1.3) along $u(x)$, and satisfies the boundary conditions

$$\begin{cases} D_{u_j}(a) - \sum_{k=1}^{p} a_k D_{u_j}(\xi_k) & = 0, \\ D_{u_j}^{(i-1)}(\gamma) & = \delta_{ij}, \qquad 2 \le i \le n-1, \\ D_{u_j}(b) - \sum_{l=1}^{q} b_l D_{u_j}(\eta_l) & = 0. \end{cases}$$

Then $D_{u_n} := \frac{\partial u}{\partial u_n}$ exists on (c, d), is a solution of (1.3) along $u(x)$, and satisfies the boundary conditions

$$\begin{cases} D_{u_n}(a) - \sum_{k=1}^{p} a_k D_{u_n}(\xi_k) & = 0, \\ D_{u_n}^{(i-1)}(\gamma) & = 0, \qquad 2 \le i \le n-1, \\ D_{u_n}(b) - \sum_{l=1}^{q} b_l D_{u_n}(\eta_l) & = 1. \end{cases}$$

Proof. Let $1 \le j \le n-1$. For brevity, we denote

$$u(x, a,\ \xi_1, \dots, \xi_p,\ \gamma,\ \eta_1, \dots, \eta_q,\ b,\ a_1, \dots, a_p,\ b_1, \dots, b_q,\ u_1, \dots, u_p)$$

by $u(x, u_j)$. Let $\delta > 0$ be as in Lemma 1, let $0 < |h| < \delta$ be given, and define

$$D_{u_j h}(x) := \frac{1}{h}[u(x, u_j + h) - u(x, u_j)].$$

First, we will look at $D_{u_1 h}(x)$. Note that the boundary conditions for $u(x, u_1 + h)$ are

$$\begin{cases} u(a, u_1 + h) - \sum_{k=1}^{p} a_k u(\xi_k, u_1 + h) & = u_1 + h \\ u^{(i-1)}(\gamma, u_1 + h) & = u_i, \qquad 2 \le i \le n-1 \\ u(b, u_1 + h) - \sum_{l=1}^{q} b_l u(\eta_l, u_1 + h) & = u_n. \end{cases}$$

Now we will look at our boundary conditions satisfied by $D_{u_1 h}(x)$. So for $h \neq 0$,

$$\begin{aligned} D_{u_1 h}(a) - \sum_{k=1}^{p} a_k D_{u_1 h}(\xi_k) &= \frac{1}{h}\left[u(a, u_1 + h) - u(a, u_1)\right] - \sum_{k=1}^{p} \frac{a_k}{h}\left[u(\xi_k, u_1 + h) - u(\xi_k, u_1)\right] \\ &= \frac{1}{h}\left[u(a, u_1 + h) - \sum_{k=1}^{p} a_k u(\xi_k, u_1 + h) - \left(u(a, u_1) - \sum_{k=1}^{p} a_k u(\xi_k, u_1)\right)\right] \\ &= \frac{1}{h}\left[u_1 + h - u_1\right] \\ &= 1. \end{aligned}$$

Now, for $h \neq 0$ and $2 \le i \le n-1$, we have

$$\begin{aligned} D_{u_1 h}^{(i-1)}(\gamma) &= \frac{1}{h}\left[u^{(i-1)}(\gamma, u_1 + h) - u^{(i-1)}(\gamma, u_1)\right] \\ &= \frac{1}{h}\left[u_i - u_i\right] \\ &= 0. \end{aligned}$$

Finally, for $h \neq 0$,

$$\begin{aligned} D_{u_1 h}(b) - \sum_{l=1}^{q} b_l D_{u_1 h}(\eta_l) &= \frac{1}{h}\left[u(b, u_1 + h) - u(b, u_1)\right] - \sum_{l=1}^{q} \frac{b_l}{h}\left[u(\eta_l, u_1 + h) - u(\eta_l, u_1)\right] \\ &= \frac{1}{h}\left[u(b, u_1 + h) - \sum_{l=1}^{q} b_l u(\eta_l, u_1 + h) - \left(u(b, u_1) - \sum_{l=1}^{q} a_l u(\eta_l, u_1)\right)\right] \\ &= \frac{1}{h}\left[u_n - u_n\right] \\ &= 0. \end{aligned}$$

Therefore, our boundary conditions satisfied by $D_{u_1h}(x)$ are

$$\begin{cases} D_{u_1h}(a) - \sum_{k=1}^{p} a_k D_{u_1h}(\xi_k) &= 1, \\ D_{u_1h}^{(i-1)}(\gamma) &= 0, \qquad 2 \le i \le n-1, \\ D_{u_1h}(b) - \sum_{l=1}^{q} b_l D_{u_1h}(\eta_l) &= 0. \end{cases}$$

Next, let

$$\beta_1 := u(\gamma, u_1), \qquad \varepsilon_1(h) := u(\gamma, u_1 + h) - \beta_1,$$

$$\beta_n := u^{(n-1)}(\gamma, u_1), \qquad \varepsilon_n(h) := u^{(n-1)}(\gamma, u_1 + h) - \beta_n.$$

Viewing the solution $u(x, u_1)$ of the boundary value problem (1.1)(2.1) as the solution $y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n)$ of the initial value problem (1.1)(1.4), and denoting $u(x, u_1)$ by $y(x, \gamma, \beta_1, \beta_n)$, we have

$$D_{u_1h}(x) = \frac{1}{h}\left[y\left(x, \gamma, \beta_1 + \varepsilon_1, \beta_n + \varepsilon_n\right) - y(x, \gamma, \beta_1, \beta_n)\right].$$

Then, by utilizing a telescoping sum, we have

$$D_{u_1h}(x) = \frac{1}{h}[y(x, \gamma, \beta_1 + \varepsilon_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n + \varepsilon_n) + y(x, \gamma, \beta_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n)].$$

Applying Theorem 1 and the Mean Value Theorem, we obtain

$$\begin{aligned} D_{u_1h}(x) = \frac{1}{h}\big[&\alpha_1\left(x, y\left(x, \gamma, \beta_1 + \overline{\varepsilon_1}, \beta_n + \varepsilon_n\right)\right)\left(\beta_1 + \varepsilon_1 - \beta_1\right) \\ &+ \alpha_n\left(x, y\left(x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n}\right)\right)\left(\beta_n + \varepsilon_n - \beta_n\right)\big], \end{aligned}$$

where $\beta_1 + \overline{\varepsilon_1}$ is between β_1 and $\beta_1 + \varepsilon_1$, and $\beta_n + \overline{\varepsilon_n}$ is between β_n and $\beta_n + \varepsilon_n$. Thus,

$$D_{u_1h}(x) = \frac{\varepsilon_1}{h}\alpha_1\left(x, y\left(x, \gamma, \beta_1 + \overline{\varepsilon_1}, \beta_n + \varepsilon_n\right)\right) + \frac{\varepsilon_n}{h}\alpha_n\left(x, y\left(x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n}\right)\right).$$

where $\alpha_k(x, y(\cdot))$, $1 \le k \le n$, is the solution of the variational equation (1.3) along $y(\cdot)$ satisfying, in each case,

$$\alpha_k^{(i-1)}(\gamma, y(\cdot)) = \delta_{ik}, \qquad 1 \le i \le n.$$

Note that $y(\cdot)$ denotes different solutions depending on different parameters, $\varepsilon_1, \varepsilon_n, \overline{\varepsilon_1}$, $\overline{\varepsilon_n}$, and $\overline{h}$. Note also that by continuous dependence on both boundary conditions (Lemma 1) and on initial conditions (Kamke's Theorem), $\varepsilon_1 \to 0$ and $\varepsilon_n \to 0$ as $h \to 0$. Thus, it must be the case that $\overline{\varepsilon_1}$, $\overline{\varepsilon_n}$, and $\overline{h}$ are continuous functions of h, and $\overline{\varepsilon_1} \to 0$, $\overline{\varepsilon_n} \to 0$, and $\overline{h} \to 0$ as $h \to 0$. In particular, for each $y(\cdot)$, $\lim\limits_{h\to 0} y(\cdot) = u(x, u_1)$.

Thus, to show $\lim\limits_{h\to 0} D_{u_1 h}(x)$ exists, it suffices to show $\lim\limits_{h\to 0} \frac{\varepsilon_1}{h}$ and $\lim\limits_{h\to 0} \frac{\varepsilon_n}{h}$ exist. Now we can rewrite our boundary conditions above as

$$\begin{aligned}
1 &= D_{u_1 h}(a) - \sum_{k=1}^{p} a_k D_{u_1 h}(\xi_k) \\
&= \frac{\varepsilon_1}{h}\alpha_1(a, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k \left\{\frac{\varepsilon_1}{h}\alpha_1(\xi_k, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(\xi_k, y(\cdot))\right\} \\
&= \frac{\varepsilon_1}{h}\left[\alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_1(\xi_k, y(\cdot))\right] + \frac{\varepsilon_n}{h}\left[\alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_n(\xi_k, y(\cdot))\right],
\end{aligned}$$

and

$$\begin{aligned}
0 &= D_{u_1 h}(b) - \sum_{l=1}^{q} b_l D_{u_1 h}(\eta_l) \\
&= \frac{\varepsilon_1}{h}\alpha_1(b, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l \left\{\frac{\varepsilon_1}{h}\alpha_1(\eta_l, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(\eta_l, y(\cdot))\right\} \\
&= \frac{\varepsilon_1}{h}\left[\alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l\alpha_1(\eta_l, y(\cdot))\right] + \frac{\varepsilon_n}{h}\left[\alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l\alpha_n(\eta_k, y(\cdot))\right].
\end{aligned}$$

Now we will define

$$\begin{aligned}
A_1(h) &:= \alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_1(\xi_k, y(\cdot)), \qquad & A_n(h) &:= \alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_n(\xi_k, y(\cdot)), \\
B_1(h) &:= \alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l\alpha_1(\eta_l, y(\cdot)), \qquad & B_n(h) &:= \alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l\alpha_n(\eta_l, y(\cdot)),
\end{aligned}$$

so that our boundary conditions can be written as

$$\begin{cases}
\frac{\varepsilon_1}{h}A_1(h) + \frac{\varepsilon_n}{h}A_n(h) &= 1, \\
\frac{\varepsilon_1}{h}B_1(h) + \frac{\varepsilon_n}{h}B_n(h) &= 0.
\end{cases}$$

Next, we will show that the coefficients of this system, $\frac{\varepsilon_1}{h}$ and $\frac{\varepsilon_n}{h}$, exist uniquely.

Note that $A_1(h), A_n(h), B_1(h), B_n(h) \in \mathbb{R}$. This system of equations is uniquely solvable for $\frac{\varepsilon_1}{h}$ and $\frac{\varepsilon_n}{h}$ if and only if

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Now we will let $h \to 0$, and noting that $\lim_{h\to 0} y(\cdot) = u(x)$, we define

$$\tilde{A}_1 \coloneqq \alpha_1(a, u(x,u_1)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x,u_1)), \quad \tilde{A}_n \coloneqq \alpha_n(a, u(x,u_1)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x,u_1)),$$
$$\tilde{B}_1 \coloneqq \alpha_1(b, u(x,u_1)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x,u_1)), \quad \tilde{B}_n \coloneqq \alpha_n(b, u(x,u_1)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x,u_1)).$$

We claim that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

To prove our claim, we suppose, to the contrary, that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} = 0.$$

Then there must exist nonzero $\rho_1, \rho_2 \in \mathbb{R}$ such that

$$\begin{cases} \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n &= 0, \\ \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n &= 0. \end{cases}$$

Define

$$w(x) \coloneqq \; \rho_1 \alpha_1(x, u(x,u_1)) + \rho_2 \alpha_n(x, u(x,u_1)).$$

Note that (1.3) is homogeneous and linear, so $w(x)$ is a solution of (1.3) along $u(x)$. By (1.5), our stacked boundary conditions are

$$w^{(i-1)}(\gamma) = \; 0, \qquad 2 \le i \le n-1.$$

Also note that

$$\begin{aligned} w(a) - \sum_{k=1}^{p} a_k w(\xi_k) &= \rho_1 \left[\alpha_1(a, u(x, u_1)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x, u_1)) \right] \\ &\qquad + \rho_2 \left[\alpha_n(a, u(x, u_1)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x, u_1)) \right] \\ &= \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} w(b) - \sum_{l=1}^{q} b_l w(\eta_l) &= \rho_1 \left[\alpha_1(b, u(x, u_1)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x, u_1)) \right] \\ &\qquad + \rho_2 \left[\alpha_n(b, u(x, u_1)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x, u_1)) \right] \\ &= \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n \\ &= 0. \end{aligned}$$

By assumption (v), this implies that $w(x) \equiv 0$, a contradiction to the linear independence of $\alpha_1(x, u(x))$ and $\alpha_n(x, u(x))$. Therefore, we can conclude

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

Since we know our solution, $y(\cdot)$, is continuous with respect to its initial values and $\lim_{h \to 0} y(\cdot) = u(x)$, it is the case that for sufficiently small $h \neq 0$, we have

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Therefore, our coefficients can be obtained by Cramer's rule as

$$\frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} 1 & A_n(h) \\ 0 & B_n(h) \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}}, \qquad \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} A_1(h) & 1 \\ B_1(h) & 0 \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}}.$$

As a result of continuous dependence on initial conditions, we can take the limit as $h \to 0$, and we define

$$E \coloneqq \lim_{h\to 0} \frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} 1 & \tilde{A}_n \\ 0 & \tilde{B}_n \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}, \qquad F \coloneqq \lim_{h\to 0} \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} \tilde{A}_1 & 1 \\ \tilde{B}_1 & 0 \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}.$$

In particular, this allows us to conclude that $\lim\limits_{h\to 0} D_{u_1 h}(x)$ exists, and

$$\lim_{h\to 0} D_{u_1 h}(x) = E\alpha_1(x, u(x, u_1)) + F\alpha_n(x, u(x, u_1)).$$

Let

$$D_{u_1}(x) \coloneqq \lim_{h\to 0} D_{u_1 h}(x),$$

and note by construction of $D_{u_1 h}(x)$,

$$D_{u_1}(x) = \frac{\partial}{\partial u_1} u(x).$$

Furthermore,

$$D_{u_1}(x) = E\alpha_1(x, u(x, u_1)) + F\alpha_n(x, u(x, u_1)),$$

which is a solution of the variational equation (1.3) along $u(x)$. In addition, the boundary conditions satisfied by $D_{u_1}(x)$ are given by

$$\begin{cases} y_1(a) - \sum_{k=1}^{p} a_k y_1(\xi_k) = \lim\limits_{h\to 0} \left[D_{u_1 h}(a) - \sum_{k=1}^{p} a_k D_{u_1 h}(\xi_k) \right] & = 1, \\ y_1^{(i-1)}(\gamma) = \lim\limits_{h\to 0} D_{u_1 h}^{(i-1)}(\gamma) & = 0, \quad 2 \le i \le n-1, \\ y_1(b) - \sum_{l=1}^{q} b_l y_1(\eta_l) = \lim\limits_{h\to 0} \left[D_{u_1 h}(b) - \sum_{l=1}^{q} b_l D_{u_1 h}(\eta_l) \right] & = 0. \end{cases}$$

It can similarly be proven that for $j = n$, our coefficients are defined by

$$G := \frac{\begin{vmatrix} 0 & \tilde{A}_n \\ 1 & \tilde{B}_n \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}, \qquad H := \frac{\begin{vmatrix} \tilde{A}_1 & 0 \\ \tilde{B}_1 & 1 \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}$$

with

$$D_{u_n}(x) = \frac{\partial}{\partial u_n} u(x) = G\alpha_1(x, u(x, u_n)) + H\alpha_n(x, u(x, u_n)).$$

Now we will consider

$$D_{u_j h}(x) = \frac{1}{h}[u(x, u_j + h) - u(x, u_j)], \qquad \forall \quad 2 \le j \le n-1.$$

Note that our boundary conditions of $u(x, u_j + h)$ for $2 \le j \le n-1$ are given by

$$\begin{cases} u(a, u_j + h) - \sum_{k=1}^{p} a_k u(\xi_k, u_j + h) & = u_1 \\ u^{(i-1)}(\gamma, u_j + h) & = u_j + h, \quad 2 \le i = j \le n-1 \\ u^{(i-1)}(\gamma, u_j + h) & = u_i, \qquad 2 \le i \ne j \le n-1 \\ u(b, u_j + h) - \sum_{l=1}^{q} b_l u(\eta_l, u_j + h) & = u_n. \end{cases} \tag{2.2}$$

Now we will look at our boundary conditions for $D_{u_j h}(x)$. So for $h \neq 0$,

$$\begin{aligned} D_{u_j h}(a) - \sum_{k=1}^{p} a_k D_{u_j h}(\xi_k) &= \frac{1}{h}\left[u(a, u_j + h) - u(a, u_j)\right] - \sum_{k=1}^{p} \frac{a_k}{h}\left[u(\xi_k, u_j + h) - u(\xi_k, u_j)\right] \\ &= \frac{1}{h}\left[u(a, u_j + h) - \sum_{k=1}^{p} a_k u(\xi_k, u_j + h) - \left(u(a, u_j) - \sum_{k=1}^{p} a_k u(\xi_k, u_j)\right)\right] \\ &= \frac{1}{h}\left[u_1 - u_1\right] \\ &= 0. \end{aligned}$$

Now, for $2 \le i \le n-1$ where $i \neq j$, we have

$$\begin{aligned} D_{u_j h}^{(i-1)}(\gamma) &= \frac{1}{h}\left[u^{(i-1)}(\gamma, u_j + h) - u^{(i-1)}(\gamma, u_j)\right] \\ &= \frac{1}{h}\left[u_i - u_i\right] \\ &= 0, \end{aligned}$$

and for $i = j$, we have

$$\begin{aligned} D_{u_j h}^{(j-1)}(\gamma) &= \frac{1}{h}\left[u^{(j-1)}(\gamma, u_j + h) - u^{(j-1)}(\gamma, u_j)\right] \\ &= \frac{1}{h}\left[u_j + h - u_j\right] \\ &= 1. \end{aligned}$$

Similar to our first boundary condition, for $h \neq 0$, we have

$$\begin{aligned} D_{u_j h}(b) - \sum_{l=1}^{q} b_l D_{u_j h}(\eta_l) &= \frac{1}{h}\left[u(b, u_j + h) - u(b, u_j)\right] - \sum_{l=1}^{q} \frac{b_l}{h}\left[u(\eta_l, u_j + h) - u(\eta_l, u_j)\right] \\ &= \frac{1}{h}\left[u(b, u_j + h) - \sum_{l=1}^{q} b_l u(\eta_l, u_j + h) - \left(u(b, u_j) - \sum_{l=1}^{q} a_l u(\eta_l, u_j)\right)\right] \\ &= \frac{1}{h}\left[u_n - u_n\right] \\ &= 0. \end{aligned}$$

Therefore, our boundary conditions satisfied by $D_{u_j h}(x)$, for $2 \leq j \leq n-1$, are

$$\begin{cases} D_{u_j h}(a) - \sum_{k=1}^{p} a_k D_{u_j h}(\xi_k) & = 0, \\ D_{u_j h}^{(i-1)}(\gamma) & = \delta_{ij}, \qquad 2 \leq i \leq n-1, \\ D_{u_j h}(b) - \sum_{l=1}^{q} b_l D_{u_j h}(\eta_l) & = 0. \end{cases}$$

Next, let

$$\begin{aligned} \beta_1 &:= u(\gamma, u_j), & \varepsilon_1 &:= u(\gamma, u_j + h) - \beta_1, \\ \beta_n &:= u^{(n-1)}(\gamma, u_j), & \varepsilon_n &:= u^{(n-1)}(\gamma, u_j + h) - \beta_n. \end{aligned}$$

Viewing solutions $u(x, u_j)$ of the boundary value problem (1.1)(2.1) as solutions of initial value problem (1.1)(1.4), and denoting $u(x, u_j) = y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n)$ by $y(x, \gamma, \beta_1, u_j, \beta_n)$, we have

$$D_{u_j h}(x) = \frac{1}{h}\left[y(x, \gamma, \beta_1 + \varepsilon_1, u_j + h, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, u_j, \beta_n)\right].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} D_{u_j h}(x) = \frac{1}{h}\big[&y(x, \gamma, \beta_1 + \varepsilon_1, u_j + h, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, u_j + h, \beta_n + \varepsilon_n) \\ &+ y(x, \gamma, \beta_1, u_j + h, \beta_n + \varepsilon_n) - u(x, \gamma, \beta_1, u_j + h, \beta_n) \end{aligned}$$

$$+ y(x,\gamma,\beta_1,u_j+h,\beta_n) - y(x,\gamma,\beta_1,u_j,\beta_n)\big].$$

Applying Theorem 1 and the Mean Value Theorem, we obtain

$$\begin{aligned} D_{u_jh}(x) = \frac{1}{h}\big[&\alpha_1\left(x,y\left(x,\gamma,\beta_1+\overline{\varepsilon_1},u_j+h,\beta_n+\varepsilon_n\right)\right)\left(\beta_1+\varepsilon_1-\beta_1\right) \\ &+\alpha_n\left(x,y\left(x,\gamma,\beta_1,u_j+h,\beta_n+\overline{\varepsilon_n}\right)\right)\left(\beta_n+\varepsilon_n-\beta_n\right) \\ &+\alpha_j\left(x,y\left(x,\gamma,\beta_1,u_j+\overline{h},\beta_n\right)\right)\left(u_j+h-u_j\right)\big]. \end{aligned}$$

where $u_j+\overline{h}$ is between u_j and u_j+h, $\beta_1+\overline{\varepsilon_1}$ is between β_1 and $\beta_1+\varepsilon_1$, and $\beta_n+\overline{\varepsilon_n}$ is between β_n and $\beta_n+\varepsilon_n$. Thus,

$$\begin{aligned} D_{u_jh}(x) = &\frac{\varepsilon_1}{h}\alpha_1\left(x,y\left(x,\gamma,\beta_1+\overline{\varepsilon_1},u_j+h,\beta_n+\varepsilon_n\right)\right) \\ &+\frac{\varepsilon_n}{h}\alpha_n\left(x,y\left(x,\gamma,\beta_1,u_j+h,\beta_n+\overline{\varepsilon_n}\right)\right) \\ &+\alpha_j\left(x,y\left(x,\gamma,\beta_1,u_j+\overline{h},\beta_n\right)\right), \end{aligned}$$

where $\alpha_k(x,y(\cdot)),\ 1\le k\le n$, is the solution of the variational equation (1.3) along $y(\cdot)$ satisfying, in each case,

$$\alpha_k^{(i-1)}(\gamma)=\delta_{ik},\qquad 1\le i\le n.$$

Note that $y(\cdot)$ denotes different solutions depending on different parameters, $\varepsilon_1,\varepsilon_n,\overline{\varepsilon_1}$, $\overline{\varepsilon_n}$, and $\overline{h}$. Note also that by continuous dependence on both boundary conditions (Lemma 1) and on initial conditions (Kamke's Theorem), $\varepsilon_1\to 0$ and $\varepsilon_n\to 0$ as $h\to 0$. Thus, it must be the case that $\overline{\varepsilon_1}$, $\overline{\varepsilon_n}$, and $\overline{h}$ are continuous functions of h, and $\overline{\varepsilon_1}\to 0$, $\overline{\varepsilon_n}\to 0$, and $\overline{h}\to 0$ as $h\to 0$. In particular, for each $y(\cdot)$, $\lim_{h\to 0} y(\cdot)=u(x,u_j)$.

Thus, to show $\lim_{h\to 0} D_{u_jh}(x)$ exists, it suffices to show $\lim_{h\to 0}\frac{\varepsilon_1}{h}$ and $\lim_{h\to 0}\frac{\varepsilon_n}{h}$ exist. Now, for $2\le j\le n-1$, we can rewrite our boundary conditions above as

$$\begin{aligned} 0 &= D_{u_jh}(a)-\sum_{k=1}^{p} a_k D_{u_jh}(\xi_k) \\ &= \frac{\varepsilon_1}{h}\alpha_1(a,y(\cdot))+\frac{\varepsilon_n}{h}\alpha_n(a,y(\cdot))+\alpha_j(a,y(\cdot)) \end{aligned}$$

$$
\begin{aligned}
&\qquad - \sum_{k=1}^{p} a_k \left\{ \frac{\varepsilon_1}{h} \alpha_1(\xi_k, y(\cdot)) + \frac{\varepsilon_n}{h} \alpha_n(\xi_k, y(\cdot)) + \alpha_j(\xi_k, y(\cdot)) \right\} \\
&= \frac{\varepsilon_1}{h} \left[\alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, y(\cdot)) \right] + \frac{\varepsilon_n}{h} \left[\alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, y(\cdot)) \right] \\
&\qquad - \sum_{k=1}^{p} a_k \alpha_j(\xi_k, y(\cdot)),
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= D_{u_j h}(b) - \sum_{l=1}^{q} b_l D_{u_j h}(\eta_l) \\
&= \frac{\varepsilon_1}{h} \alpha_1(b, y(\cdot)) + \frac{\varepsilon_n}{h} \alpha_n(b, y(\cdot)) + \alpha_j(b, y(\cdot)) \\
&\qquad - \sum_{b=1}^{q} b_l \left\{ \frac{\varepsilon_1}{h} \alpha_1(\eta_l, y(\cdot)) + \frac{\varepsilon_n}{h} \alpha_n(\eta_l, y(\cdot)) + \alpha_j(\eta_l, y(\cdot)) \right\} \\
&= \frac{\varepsilon_1}{h} \left[\alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, y(\cdot)) \right] + \frac{\varepsilon_n}{h} \left[\alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, y(\cdot)) \right] \\
&\qquad - \sum_{l=1}^{q} b_l \alpha_j(\eta_l, y(\cdot)).
\end{aligned}
$$

We will define

$$
\begin{aligned}
A_1(h) &:= \alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, y(\cdot)), & B_1(h) &:= \alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, y(\cdot)), \\
A_n(h) &:= \alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, y(\cdot)), & B_n(h) &:= \alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, y(\cdot)), \\
A_j(h) &:= \sum_{k=1}^{p} a_k \alpha_j(\xi_k, y(\cdot)) & B_j(h) &:= \sum_{l=1}^{q} b_l \alpha_j(\eta_l, y(\cdot)).
\end{aligned}
$$

Note that $A_1(h), A_n(h), B_1(h), B_n(h) \in \mathbb{R}$ and our boundary conditions can be written as

$$
\begin{cases}
\frac{\varepsilon_1}{h} A_1(h) + \frac{\varepsilon_n}{h} A_n(h) &= A_j(h), \\
\frac{\varepsilon_1}{h} B_1(h) + \frac{\varepsilon_n}{h} B_n(h) &= B_j(h).
\end{cases}
$$

This system of equations is uniquely solvable for $\frac{\varepsilon_1}{h}$ and $\frac{\varepsilon_n}{h}$ if and only if

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Now we will let $h \to 0$, and noting that $\lim_{h\to 0} y(\cdot) = u(x)$, we define

$$\tilde{A}_1(h) := \alpha_1(a, u(x)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x)), \qquad \tilde{B}_1(h) := \alpha_1(b, u(x)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x)),$$
$$\tilde{A}_n(h) := \alpha_n(a, u(x)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x)), \qquad \tilde{B}_n(h) := \alpha_n(b, u(x)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x)),$$
$$\tilde{A}_j(h) := \sum_{k=1}^{p} a_k \alpha_j(\xi_k, u(x)), \qquad \tilde{B}_j(h) := \sum_{l=1}^{q} b_l \alpha_j(\eta_l, u(x)).$$

We claim that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

To prove our claim, we suppose, to the contrary, that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} = 0.$$

Then there must exist nonzero $\rho_1, \rho_2 \in \mathbb{R}$ such that

$$\begin{cases} \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n &= 0, \\ \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n &= 0. \end{cases}$$

Define

$$w(x) := \ \rho_1 \alpha_1(x, u(x)) + \rho_2 \alpha_n(x, u(x)).$$

By (1.5), we know that our stacked boundary conditions are

$$w^{(i-1)}(\gamma) = \ 0, \qquad 2 \leq i \leq n-1.$$

Also note that

$$w(a) - \sum_{k=1}^{p} a_k w(\xi_k) = \rho_1 \left[\alpha_1(a, u(x)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x)) \right]$$

$$
\begin{aligned}
&+ \rho_2 \left[\alpha_n(a, u(x)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x)) \right] \\
&= \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
w(b) - \sum_{l=1}^{q} b_l w(\eta_l) &= \rho_1 \left[\alpha_1(b, u(x)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x)) \right] \\
&\qquad + \rho_2 \left[\alpha_n(b, u(x)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x)) \right] \\
&= \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n \\
&= 0.
\end{aligned}
$$

By assumption (v), this implies that $w(x) \equiv 0$, a contradiction to the linear independence of $\alpha_1(x, u(x))$ and $\alpha_n(x, u(x))$. Therefore, we can conclude

$$
\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.
$$

Since we know our solution, $y(\cdot)$, is continuous with respect to its initial values and $\lim_{h \to 0} y(\cdot) = u(x)$, it is the case that for sufficiently small $h \neq 0$, we have

$$
\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.
$$

Therefore, our coefficients can be obtained by Cramer's rule as

$$
\frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} A_j(h) & A_n(h) \\ B_j(h) & B_n(h) \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}} \qquad \text{and} \qquad \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} A_1(h) & A_j(h) \\ B_1(h) & B_j(h) \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}}.
$$

As a result of continuous dependence on initial conditions, we can take the limit as $h \to 0$. We now define

$$S_j \coloneqq \lim_{h\to 0} \frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} \tilde{A}_j & \tilde{A}_n \\ \tilde{B}_j & \tilde{B}_n \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}} \qquad \text{and} \qquad T_j \coloneqq \lim_{h\to 0} \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_j \\ \tilde{B}_1 & \tilde{B}_j \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}.$$

In particular, this allows us to conclude that $\lim_{h\to 0} D_{u_j h}(x)$ exists, and

$$\lim_{h\to 0} D_{u_j h}(x) = S_j \alpha_1 \left(x, u(x, u_j)\right) + T_j \alpha_n \left(x, u(x, u_j)\right) + \alpha_j \left(x, u(x, u_j)\right).$$

Let

$$D_{u_j}(x) \coloneqq \lim_{h\to 0} D_{u_j h}(x),$$

and note by construction of $D_{u_j h}(x)$,

$$D_{u_j}(x) = \frac{\partial}{\partial u_j} u(x).$$

Furthermore,

$$D_{u_j}(x) = S_j \alpha_1 \left(x, u(x, u_j)\right) + T_j \alpha_n \left(x, u(x, u_j)\right) + \alpha_j \left(x, u(x, u_j)\right),$$

which is a solution of the variational equation (1.3) along $u(x)$.

In addition, with $2 \le j \le n-1$, because of the limits of the boundary conditions satisfied by $D_{u_j h}(x)$, we also have

$$\begin{cases} D_{u_j}(a) - \sum_{k=1}^{p} a_k D_{u_j}(\xi_k) = \lim_{h\to 0} \left[D_{u_j h}(a) - \sum_{k=1}^{p} a_k D_{u_j h}(\xi_k)\right] & = 0, \\ D_{u_j}^{(i-1)}(\gamma) = \lim_{h\to 0} D_{u_j h}^{(i-1)}(\gamma) & = \delta_{ij}, \quad 2 \le i \le n-1, \\ D_{u_j}(b) - \sum_{l=1}^{q} b_l D_{u_j}(\eta_l) = \lim_{h\to 0} \left[D_{u_j h}(b) - \sum_{l=1}^{q} b_l D_{u_j h}(\eta_l)\right] & = 0. \end{cases}$$

□

2.2 - A Basis of Solutions for (1.3)

In this section, we show that the partial derivatives obtained in Section 2.1 form a basis for the solution space of (1.3).

Theorem 3: Assume the hypotheses of Theorem 2 are satisfied. Then the family of solutions

$$D_{u_j}(x) \coloneqq \frac{\partial u(x)}{\partial u_j}, \qquad 1 \leq j \leq n,$$

form a basis for the solution space of (1.3) along the solution $u(x)$ of (1.1).

Proof. The solution space of (1.3) along the solution $u(x)$ is an n-dimensional vector space. Hence, it suffices to show that the family $\{D_{u_j}(x)\}_{j=1}^{n}$ is a linearly independent family on (c, d). Suppose there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$c_1 D_{u_1}(x) + \cdots + c_n D_{u_n}(x) = 0, \qquad c < x < d.$$

Let

$$w(x) \coloneqq c_1 D_{u_1}(x) + \cdots + c_n D_{u_n}(x).$$

Then, since $w(x) = 0$,

$$\begin{aligned}
0 &= w(a) - \sum_{k=1}^{p} a_k w(\xi_k) \\
&= c_1 \left(D_{u_1}(a) - \sum_{k=1}^{p} a_k D_{u_1}(\xi_k) \right) + \sum_{j=2}^{n} c_j \left(D_{u_j}(a) - \sum_{k=1}^{n} a_k D_{u_j}(\xi_k) \right) \\
&= c_1 \cdot 1 + \sum_{j=2}^{n} c_j \cdot 0 \\
&= c_1.
\end{aligned}$$

Similarly, $c_n = 0$. So

$$w(x) = \sum_{j=2}^{n-1} c_j D_{u_j}(x).$$

Next, let $2 \leq j_0 \leq n-1$ be arbitrary but fixed. Again, by Theorem 2,

$$\begin{aligned} 0 &= w^{(j_0-1)}(\gamma) \\ &= \sum_{\substack{j=2 \\ j \neq j_0}}^{n-1} c_j D_{u_j}^{(j_0-1)}(\gamma) + c_{j_0} D_{u_{j_0}}^{(j_0-1)}(\gamma) \\ &= \sum_{\substack{j=2 \\ j \neq j_0}}^{n-1} c_j \cdot 0 + c_{j_0} \cdot 1 \\ &= c_{j_0}. \end{aligned}$$

It follows that $c_j = 0$ for $1 \leq j \leq n$, and we conclude that the family $\left\{D_{u_j}(x)\right\}_{j=1}^{n}$ is linearly independent, hence the conclusion of the theorem. $\square$

CHAPTER THREE

An Analogue for Peano's Theorem for (1.1), (1.2), II

In Chapter three, we will characterize partial derivatives of the solutions of (1.1),(1.2) with respect to boundary points $a, b, \xi_1, \ldots, \xi_k$, and $\eta_1, \ldots, \eta_l$ from our nonlocal boundary conditions, and with respect to γ from our stacked boundary condition.

3.1 - Differentiation with Respect to Boundary Points (a and b)

Theorem 4: Assume (i)-(v) are satisfied. Let $u(x)$ be a solution of (1.1) on (c, d), let $a_1, \ldots, a_p,\ b_1, \ldots, b_q \in \mathbb{R}$ and $c < a < \xi_1 < \cdots < \xi_p < \gamma < \eta_1 < \cdots < \eta_q < b < d$ be given so that

$$u(x) = u(x, a,\ \xi_1, \ldots, \xi_p,\ \gamma,\ \eta_1, \ldots, \eta_q,\ b,\ a_1, \ldots, a_p,\ b_1, \ldots, b_q,\ u_1, \ldots, u_n),$$

where

$$\begin{cases} u(a) - \sum_{k=1}^{p} a_k u(\xi_k) &= u_1, \\ u^{(i-1)}(\gamma) &= u_i, \qquad 2 \le i \le n-1, \\ u(b) - \sum_{l=1}^{q} b_l u(\eta_l) &= u_n. \end{cases} \tag{2.1}$$

Then $D_a := \frac{\partial u}{\partial a}$ exists on (c, d), is a solution of (1.3) along $u(x)$, and satisfies

$$\begin{cases} D_a(a) - \sum_{k=1}^{p} a_k D_a(\xi_k) &= -u'(a), \\ D_a^{(i-1)}(\gamma) &= 0, \qquad 2 \le i \le n-1, \\ D_a(b) - \sum_{l=1}^{q} b_l D_a(\eta_l) &= 0. \end{cases}$$

Then $D_b := \frac{\partial u}{\partial b}$ exists on (c, d), is a solution of (1.3) along $u(x)$, and satisfies

$$\begin{cases} D_b(a) - \sum_{k=1}^{p} a_k D_b(\xi_k) &= 0, \\ D_b^{(i-1)}(\gamma) &= 0, \qquad 2 \le i \le n-1, \\ D_b(b) - \sum_{l=1}^{q} b_l D_b(\eta_l) &= -u'(b). \end{cases}$$

Proof. For brevity, we will denote

$$u(x, a,\ \xi_1, \dots, \xi_p,\ \gamma,\ \eta_1, \dots, \eta_q,\ b,\ a_1, \dots, a_p,\ b_1, \dots, b_q,\ u_1, \dots, u_p)$$

by $u(x, a)$. Let $\delta > 0$ be as in Lemma 1, $0 < |h| < \delta$ be given, and define

$$D_{ah}(x) := \frac{1}{h}[u(x, a+h) - u(x, a)].$$

Note that our boundary conditions for $u(x, a+h)$ read

$$\begin{cases} u(a+h, a+h) - \sum_{k=1}^{p} a_k u(\xi_k, a+h) & = u_1, \\ u^{(i-1)}(\gamma, a+h) & = u_i, \qquad\qquad 2 \le i \le n-1, \\ u(b, a+h) - \sum_{l=1}^{q} b_l u(\eta_l, a+h) & = u_n. \end{cases}$$

Now we will look at our boundary conditions satisfied by D_{ah}. So for $h \neq 0$,

$$\begin{aligned} D_{ah}(a) - \sum_{k=1}^{p} a_k D_{ah}(\xi_k) &= \frac{1}{h}\left[u(a, a+h) - u(a, a)\right] - \sum_{k=1}^{p} \frac{a_k}{h}\left[u(\xi_k, a+h) - u(\xi_k, a)\right] \\ &= \frac{1}{h}\Bigg[u(a, a+h) - \sum_{k=1}^{p} a_k u(\xi_k, a+h) \\ &\qquad - \left(u(a, a) - \sum_{k=1}^{p} a_k u(\xi_k, a)\right)\Bigg] \\ &= \frac{1}{h}\left[u(a, a+h) - \sum_{k=1}^{p} a_k u(\xi_k, a+h) - u_1\right] \\ &= \frac{1}{h}\Big[u(a, a+h) - \{u(a+h, a+h) - u_1\} - u_1\Big] \\ &= -\frac{1}{h}\left[u(a+h, a+h) - u(a, a+h)\right]. \end{aligned}$$

We will define

$$\Delta_a(h) := -\frac{1}{h}\left[u(a+h, a+h) - u(a, a+h)\right].$$

Now, for $h \neq 0$ and $2 \le i \le n-1$,

$$\begin{aligned} D_{ah}^{(i-1)}(\gamma) &= \frac{1}{h}\left[u^{(i-1)}(\gamma, a+h) - u^{(i-1)}(\gamma, a)\right] \\ &= \frac{1}{h}\left[u_i - u_i\right] \\ &= 0. \end{aligned}$$

Finally, for $h \neq 0$,

$$\begin{aligned} D_{ah}(b) - \sum_{l=1}^{q} b_l D_{ah}(\eta_l) &= \frac{1}{h}\left[u(b, a+h) - u(b,a)\right] - \sum_{l=1}^{q} \frac{b_l}{h}\left[u(\eta_l, a+h) - u(\eta_l, a)\right] \\ &= \frac{1}{h}\Bigg[u(b, a+h) - \sum_{l=1}^{q} b_l u(\eta_l, a+h) \\ &\qquad\qquad - \left(u(b,a) - \sum_{l=1}^{q} a_l u(\eta_l, a)\right)\Bigg] \\ &= \frac{1}{h}\left[u_n - u_n\right] \\ &= 0. \end{aligned}$$

Therefore, our boundary conditions satisfied by $D_{ah}(x)$ are

$$\begin{cases} D_{ah}(a) - \sum_{k=1}^{p} a_k D_{ah}(\xi_k) &= \Delta_a(h), \\ D_{ah}^{(i-1)}(\gamma) &= 0, \qquad\qquad 2 \leq i \leq n-1, \\ D_{ah}(b) - \sum_{l=1}^{q} b_l D_{ah}(\eta_l) &= 0. \end{cases}$$

Next, let

$$\begin{aligned} \beta_1 &\coloneqq u(\gamma, a), & \varepsilon_1 &\coloneqq u(\gamma, a+h) - \beta_1, \\ \beta_n &\coloneqq u^{(n-1)}(\gamma, a), & \varepsilon_n &\coloneqq u^{(n-1)}(\gamma, a+h) - \beta_n. \end{aligned}$$

Viewing the solution $u(x,a)$ of the boundary value problem (1.1)(2.1) as the solution $y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n)$ of the initial value problem (1.1)(1.4), and denoting $u(x,a)$ by $y(x, \gamma, \beta_1, \beta_n)$, we have

$$D_{ah}(x) = \frac{1}{h}\left[y(x, \gamma, \beta_1 + \varepsilon_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n)\right].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} D_{ah}(x) = \frac{1}{h}[&y(x, \gamma, \beta_1 + \varepsilon_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n + \varepsilon_n) \\ &+ y(x, \gamma, \beta_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n)]. \end{aligned}$$

Applying Theorem 1 and the Mean Value Theorem, we obtain

$$D_{ah}(x) = \frac{1}{h}[\alpha_1 (x, y (x, \gamma, \beta_1 + \overline{\varepsilon_1}, \beta_n + \varepsilon_n)) (\beta_1 + \varepsilon_1 - \beta_1) \\ + \alpha_n (x, y (x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n})) (\beta_n + \varepsilon_n - \beta_n)],$$

where $\beta_1 + \overline{\varepsilon_1}$ is between β_1 and $\beta_1 + \varepsilon_1$ and $\beta_n + \overline{\varepsilon_n}$ is between β_n and $\beta_n + \varepsilon_n$. Thus,

$$D_{ah}(x) = \frac{\varepsilon_1}{h}\alpha_1 (x, y (x, \gamma, \beta_1 + \overline{\varepsilon_1}, \beta_n + \varepsilon_n)) + \frac{\varepsilon_n}{h}\alpha_n (x, y (x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n})),$$

where $\alpha_k(x, y(\cdot)),\ 1 \le k \le n$, is the solution of the variational equation (1.3) along $y(\cdot)$ satisfying, in each case,

$$\alpha_k^{(i-1)}(\gamma, y(\cdot)) = \delta_{ik}, \quad 1 \le i \le n,$$

Note that $y(\cdot)$ denotes different solutions depending on different parameters, $\varepsilon_1, \varepsilon_n, \overline{\varepsilon_1}$ and $\overline{\varepsilon_n}$. Note also that by continuous dependence on both boundary conditions (i.e. Lemma 1) and on initial conditions (Kamke's Theorem), $\varepsilon_1 \to 0$ and $\varepsilon_n \to 0$ as $h \to 0$. Thus, it must be the case that $\overline{\varepsilon_1}$, $\overline{\varepsilon_n}$, and $\overline{h}$ are continuous functions of h, and $\overline{\varepsilon_1} \to 0$, $\overline{\varepsilon_n} \to 0$, and $\overline{h} \to 0$ as $h \to 0$. In particular, for each $y(\cdot)$, $\lim_{h\to 0} y(\cdot) = u(x, a)$.

Thus, to show $\lim_{h\to 0} D_{ah}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\varepsilon_1}{h}$ and $\lim_{h\to 0} \frac{\varepsilon_n}{h}$ exist. Now we can rewrite our boundary conditions above as

$$\begin{aligned}\Delta_a(h) &= D_{ah}(a) - \sum_{k=1}^{p} a_k D_{ah}(\xi_k) \\ &= \frac{\varepsilon_1}{h}\alpha_1(a, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k \left\{\frac{\varepsilon_1}{h}\alpha_1(\xi_k, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(\xi_k, y(\cdot))\right\} \\ &= \frac{\varepsilon_1}{h}\left[\alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_1(\xi_k, y(\cdot))\right] + \frac{\varepsilon_n}{h}\left[\alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_n(\xi_k, y(\cdot))\right],\end{aligned}$$

and

$$\begin{aligned}0 &= D_{ah}(b) - \sum_{l=1}^{q} b_l D_{ah}(\eta_l) \\ &= \frac{\varepsilon_1}{h}\alpha_1(a, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(a, y(\cdot)) - \sum_{l=1}^{q} b_l \left\{\frac{\varepsilon_1}{h}\alpha_1(\eta_l, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(\eta_l, y(\cdot))\right\}\end{aligned}$$

$$= \frac{\varepsilon_1}{h}\left[\alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, y(\cdot))\right] + \frac{\varepsilon_n}{h}\left[\alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_k, y(\cdot))\right].$$

We will define

$$A_1(h) \coloneqq \alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, y(\cdot)), \qquad A_n(h) \coloneqq \alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, y(\cdot)),$$

$$B_1(h) \coloneqq \alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, y(\cdot)), \qquad B_n(h) \coloneqq \alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, y(\cdot)).$$

Note that $A_1(h), A_n(h), B_1(h), B_n(h), \Delta_a(h) \in \mathbb{R}$, and our boundary conditions can be written as

$$\begin{cases} \frac{\varepsilon_1}{h} A_1(h) + \frac{\varepsilon_n}{h} A_n(h) & = \Delta_a(h), \\ \frac{\varepsilon_1}{h} B_1(h) + \frac{\varepsilon_n}{h} B_n(h) & = \ 0. \end{cases}$$

This system of equations is uniquely solvable for $\frac{\varepsilon_1}{h}$ and $\frac{\varepsilon_n}{h}$ if and only if

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Now we will let $h \to 0$, and noting that $\lim_{h \to 0} y(\cdot) = u(x, a)$, we define

$$\tilde{A}_1 \coloneqq \alpha_1(a, u(x,a)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x,a)), \quad \tilde{A}_n \coloneqq \alpha_n(a, u(x,a)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x,a)),$$

$$\tilde{B}_1 \coloneqq \alpha_1(b, u(x,a)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x,a)), \qquad \tilde{B}_n \coloneqq \alpha_n(b, u(x,a)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x,a)).$$

We claim that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

To prove our claim, we suppose, to the contrary, that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} = 0.$$

Then there must exist nonzero $\rho_1, \rho_2 \in \mathbb{R}$ such that

$$\begin{cases} \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n &= 0, \\ \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n &= 0. \end{cases}$$

Define

$$w(x) \coloneqq \ \rho_1 \alpha_1(x, u(x,a)) + \rho_2 \alpha_n(x, u(x,a)).$$

Note that (1.3) is homogeneous and linear, so $w(x)$ is a solution of (1.3) along $u(x)$. By (1.5), our stacked boundary conditions are

$$w^{(i-1)}(\gamma) = \ 0, \quad 2 \le i \le n-1.$$

Also note that

$$\begin{aligned} w(a) - \sum_{k=1}^{p} a_k w(\xi_k) &= \rho_1 \left[\alpha_1(a, u(x,a)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x,a) \right] \\ &\qquad + \rho_2 \left[\alpha_n(a, u(x,a)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x,a) \right] \\ &= \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} w(b) - \sum_{l=1}^{q} b_l w(\eta_l) &= \rho_1 \left[\alpha_1(b, u(x,a)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x,a) \right] \\ &\qquad + \rho_2 \left[\alpha_n(b, u(x,a)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x,a) \right] \\ &= \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n \\ &= 0. \end{aligned}$$

By assumption (v), this implies that $w(x) \equiv 0$, a contradiction to the linear independence of $\alpha_1(x, u(x,a))$ and $\alpha_n(x, u(x,a))$. Therefore, we can conclude

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

Since we know our solution $y(\cdot)$ is continuous with respect to its initial values and $\lim_{h\to 0} y(\cdot) = u(x,a)$, it is the case that for sufficiently small $h \neq 0$, we have

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Our coefficients can be obtained by Cramer's rule as

$$\frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} \Delta_a(h) & A_n(h) \\ 0 & B_n(h) \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}} \qquad \text{and} \qquad \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} A_1(h) & \Delta_a(h) \\ B_1(h) & 0 \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}}.$$

As a result of continuous dependence on initial conditions, we can take the limit as $h \to 0$. Notice that

$$\lim_{h\to 0} \Delta_a(h) = -\frac{\partial u(x,a)}{\partial x}\bigg|_{x=a} = -u'(a).$$

We will define

$$K := \lim_{h\to 0} \frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} -u'(a) & \tilde{A}_n \\ 0 & \tilde{B}_n \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}} \qquad \text{and} \qquad L := \lim_{h\to 0} \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} \tilde{A}_1 & -u'(a) \\ \tilde{B}_1 & 0 \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}.$$

In particular, this allows us to conclude that $\lim_{h\to 0} D_{ah}(x)$ exists, and

$$\lim_{h\to 0} D_{ah}(x) = K\alpha_1(x, u(x,a)) + L\alpha_n(x, u(x,a)).$$

Let

$$D_a(x) := \lim_{h\to 0} D_{ah}(x),$$

and note by construction of $D_{ah}(x)$,

$$D_a(x) = \frac{\partial}{\partial a} u(x).$$

Furthermore,

$$D_a(x) = K\alpha_1(x, u(x,a)) + L\alpha_n(x, u(x,a)),$$

which is a solution of the variational equation (1.3) along $u(x)$. In addition, the boundary conditions satisfied by $D_a(x)$ are given by

$$\begin{cases} D_a(a) - \sum_{k=1}^{p} a_k D_a(\xi_k) = \lim\limits_{h\to 0} \left[D_{ah}(a) - \sum_{k=1}^{p} a_k D_{ah}(\xi_k)\right] & = -u'(a), \\ D_a^{(i-1)}(\gamma) = \lim\limits_{h\to 0} D_{ah}^{(i-1)}(\gamma) & = 0, \qquad 2 \le i \le n-1, \\ D_a(b) - \sum_{l=1}^{q} b_l D_a(\eta_l) = \lim\limits_{h\to 0} \left[D_{ah}(b) - \sum_{l=1}^{q} b_l D_{ah}(\eta_l)\right] & = 0. \end{cases}$$

In complete analogy, it can be similarly shown that $D_b(x)$ exists on (c,d) and satisfies the boundary conditions

$$\begin{cases} D_b(a) - \sum_{k=1}^{p} a_k D_b(\xi_k) = \lim\limits_{h\to 0} \left[D_{bh}(a) - \sum_{k=1}^{p} a_k D_{bh}(\xi_k)\right] & = 0, \\ D_b^{(i-1)}(\gamma) = \lim\limits_{h\to 0} D_{bh}^{(i-1)}(\gamma) & = 0, \qquad 2 \le i \le n-1, \\ D_b(b) - \sum_{l=1}^{q} b_l D_b(\eta_l) = \lim\limits_{h\to 0} \left[D_{bh}(b) - \sum_{l=1}^{q} b_l D_{bh}(\eta_l)\right] & = -u'(b), \end{cases}$$

where $D_{bh}(x) = \frac{1}{h}[u(x, b+h) - u(x,b)]$. □

3.2 - Differentiation with Respect to Boundary Point (γ)

Theorem 5: Assume (i)-(v) are satisfied. Let $u(x)$ be a solution of (1.1) on (c,d), let $a_1, \ldots, a_p,\ b_1, \ldots, b_q \in \mathbb{R}$ and $c < a < \xi_1 < \cdots < \xi_p < \gamma < \eta_1 < \cdots < \eta_q < b < d$ be given so that

$$u(x) = u(x, a,\ \xi_1, \ldots, \xi_p,\ \gamma,\ \eta_1, \ldots, \eta_q,\ b,\ a_1, \ldots, a_p,\ b_1, \ldots, b_q,\ u_1, \ldots, u_n),$$

where

$$\begin{cases} u(a) - \sum_{k=1}^{p} a_k u(\xi_k) & = u_1, \\ u^{(i-1)}(\gamma) & = u_i, \qquad 2 \le i \le n-1, \\ u(b) - \sum_{l=1}^{q} b_l u(\eta_l) & = u_n. \end{cases} \tag{2.1}$$

Then $D_\gamma(x) := \frac{\partial}{\partial\gamma}u(x)$ exists on (c, d), is a solution of (1.3) along $u(x)$, and satisfies the respective boundary conditions,

$$\begin{cases} D_\gamma(a) - \sum_{k=1}^{p} a_k D_\gamma(\xi_k) &= 0, \\ D_\gamma^{(i-1)}(\gamma) &= -u^{(i)}(\gamma), \qquad 2 \le i \le n-1, \\ D_\gamma(b) - \sum_{l=1}^{q} b_l D_\gamma(\eta_l) &= 0. \end{cases}$$

Proof. For brevity, we will denote

$$u(x, a,\ \xi_1, \dots, \xi_p,\ \gamma,\ \eta_1, \dots, \eta_q,\ b,\ a_1, \dots, a_p,\ b_1, \dots, b_q,\ u_1, \dots, u_p)$$

by $u(x, \gamma)$. Let $\delta > 0$ be as in Lemma 1, $0 < |h| < \delta$ be given, and define

$$D_{\gamma h}(x) := \frac{1}{h}[u(x, \gamma + h) - u(x, \gamma)].$$

Note that the boundary conditions for $u(x, \gamma + h)$ are

$$\begin{cases} u(a, \gamma+h) - \sum_{k=1}^{p} a_k u(\xi_k, \gamma+h) &= u_1, \\ u^{(i-1)}(\gamma+h, \gamma+h) &= u_i, \qquad 2 \le i \le n-1, \\ u(b, \gamma+h) - \sum_{l=1}^{q} b_l u(\eta_l, \gamma+h) &= u_n. \end{cases}$$

Now we will look at our boundary conditions satisfied by $D_{\gamma h}$. So for $h \neq 0$,

$$\begin{aligned} D_{\gamma h}(a) - \sum_{k=1}^{p} a_k D_{\gamma h}(\xi_k) &= \frac{1}{h}[u(a, \gamma+h) - u(a, \gamma)] - \sum_{k=1}^{p} a_k \frac{1}{h}[u(\xi_k, \gamma+h) - u(\xi_k, \gamma)] \\ &= \frac{1}{h}[u(a, \gamma+h) - \sum_{k=1}^{p} a_k u(\xi_k, \gamma+h)] - \frac{1}{h}[u(a, \gamma) - \sum_{k=1}^{p} a_k u(\xi_k, \gamma)] \\ &= \frac{1}{h}[u_1 - u_1] \\ &= 0. \end{aligned}$$

Now for $h \neq 0$ and $2 \le i \le n-1$,

$$\begin{aligned} D_{\gamma h}^{(i-1)}(\gamma) &= \frac{1}{h}[u^{(i-1)}(\gamma, \gamma+h) - u^{(i-1)}(\gamma, \gamma)], \\ &= \frac{1}{h}[u^{(i-1)}(\gamma, \gamma+h) - u^{(i-1)}(\gamma+h, \gamma+h) + u^{(i-1)}(\gamma+h, \gamma+h) - u^{(i-1)}(\gamma, \gamma)], \\ &= \frac{1}{h}[u^{(i-1)}(\gamma, \gamma+h) - u^{(i-1)}(\gamma+h, \gamma+h) + u_i - u_i], \end{aligned}$$

$$= -\frac{1}{h}[u^{(i-1)}(\gamma+h,\gamma+h) - u^{(i-1)}(\gamma,\gamma+h)].$$

We now define, for $2 \le i \le n-1$,

$$\Delta_i(h) \coloneqq -\frac{1}{h}[u^{(i-1)}(\gamma+h,\gamma+h) - u^{(i-1)}(\gamma,\gamma+h)],$$

Finally, for $h \neq 0$,

$$\begin{aligned}
D_{\gamma h}(b) - \sum_{l=1}^{q} b_l D_{\gamma h}(\eta_l) &= \frac{1}{h}[u(b,\gamma+h) - u(b,\gamma)] - \sum_{l=1}^{q} b_l \frac{1}{h}[u(\xi_k,\gamma+h) - u(\xi_k,\gamma)] \\
&= \frac{1}{h}[u(b,\gamma+h) - \sum_{l=1}^{q} b_l u(\xi_k,\gamma+h)] - \frac{1}{h}[u(b,\gamma) - \sum_{l=1}^{q} b_l u(\xi_k,\gamma)] \\
&= \frac{1}{h}[u_n - u_n] \\
&= 0.
\end{aligned}$$

Therefore, our boundary conditions satisfied by $D_{\gamma h}(x)$ are

$$\begin{cases}
D_{\gamma h}(a) - \sum_{k=1}^{p} a_k D_{\gamma h}(\xi_k) & = 0, \\
D_{\gamma h}^{(i-1)}(\gamma) & = \Delta_i(h), \qquad 2 \le i \le n-1, \\
D_{\gamma h}(b) - \sum_{l=1}^{q} b_l D_{\gamma h}(\eta_l) & = 0.
\end{cases}$$

Next, let

$$\begin{aligned}
\beta_1 &\coloneqq u(\gamma,\gamma), \\
\beta_n &\coloneqq u^{(n-1)}(\gamma,\gamma), \\
\varepsilon_1 &\coloneqq u(\gamma,\gamma+h) - \beta_1, \\
&\vdots \\
\varepsilon_i &\coloneqq u^{(i-1)}(\gamma,\gamma+h) - u_i, \qquad 2 \le i \le n-1 \\
&\vdots \\
\varepsilon_n &\coloneqq u^{(n-1)}(\gamma,\gamma+h) - \beta_n.
\end{aligned} \tag{3.1}$$

Viewing the solution $u(x,\gamma)$ of the boundary value problem (1.1)(2.1) as the solution $y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n)$ of the initial value problem (1.1)(1.4), we have

$$D_{\gamma h}(x) = \frac{1}{h}[y(x, \gamma, \beta_1 + \varepsilon_1, \varepsilon_2 + u_2, \ldots, \varepsilon_{n-1} + u_{n-1}, \beta_n + \varepsilon_n) \\ - y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n)].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} D_{\gamma h}(x) = \frac{1}{h}[&y(x, \gamma, \beta_1 + \varepsilon_1, \varepsilon_2 + u_2, \ldots, \varepsilon_{n-1} + u_{n-1}, \beta_n + \varepsilon_n) \\ &- y(x, \gamma, \beta_1, u_2 + \varepsilon_2, \ldots, \varepsilon_{n-1} + u_{n-1}, \beta_n + \varepsilon_n) \\ &+ y(x, \gamma, \beta_1, u_2 + \varepsilon_2, \ldots, \varepsilon_{n-1} + u_{n-1}, \beta_n + \varepsilon_n) \\ &- y(x, \gamma, \beta_1, u_2, u_3 + \varepsilon_3, \ldots, u_{n-1} + \varepsilon_{n-1}, \beta_n + \varepsilon_n) \\ &+ y(x, \gamma, \beta_1, u_2, u_3 + \varepsilon_3, \ldots, u_{n-1} + \varepsilon_{n-1}, \beta_n + \varepsilon_n) \\ &\vdots \\ &- y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n + \varepsilon_n) \\ &+ y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n + \varepsilon_n) \\ &- y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n)]. \end{aligned}$$

Applying Theorem 1 and the Mean Value Theorem, we obtain

$$\begin{aligned} D_{\gamma h}(x) = \frac{1}{h}\Big[&\alpha_1 \left(x, y(x, \gamma, \beta_1 + \overline{\varepsilon_1}, \varepsilon_2 + u_2, \ldots, \varepsilon_{n-1} + u_{n-1}, \beta_n + \varepsilon_n)\right) (\beta_1 + \varepsilon_1 - \beta_1) \\ &+ \alpha_2 \left(x, y(x, \gamma, \beta_1, u_2 + \overline{\varepsilon_2}, \ldots, \varepsilon_{n-1} + u_{n-1}, \beta_n + \varepsilon_n)\right) (u_2 + \varepsilon_2 - u_2) \\ &\vdots \\ &+ \alpha_{n-1} \left(x, y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1} + \overline{\varepsilon_{n-1}}, \beta_n + \varepsilon_n)\right) (u_{n-1} + \varepsilon_{n-1} - u_{n-1}) \\ &+ \alpha_n \left(x, y\left(x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n}\right)\right) (u_n + \varepsilon_n - u_n)\Big], \end{aligned}$$

where $\beta_1 + \overline{\varepsilon_1}$ is between β_1 and $\beta_1 + \varepsilon_1$, $\beta_n + \overline{\varepsilon_n}$ is between β_n and $\beta_n + \varepsilon_n$, and $u_i + \overline{\varepsilon_i}$ is between u_i and $u_i + \varepsilon_i$ for $2 \leq i \leq n - 1$. Thus,

$$
\begin{aligned}
D_{\gamma h}(x) = {} & \frac{\varepsilon_1}{h}\alpha_1\left(x, y(x,\gamma,\beta_1+\overline{\varepsilon_1}, u_2+\varepsilon_2,\ldots,u_{n-1}+\varepsilon_{n-1},\beta_n+\varepsilon_n)\right) \\
& + \frac{\varepsilon_2}{h}\alpha_2\left(x, y(x,\gamma,\beta_1, u_2+\overline{\varepsilon_2},\ldots,u_{n-1}+\varepsilon_{n-1},\beta_n+\varepsilon_n)\right) \\
& \vdots \\
& + \frac{\varepsilon_{n-1}}{h}\alpha_{n-1}\left(x, y(x,\gamma,\beta_1, u_2,\ldots,u_{n-1}+\overline{\varepsilon_{n-1}},\beta_n+\varepsilon_n)\right) \\
& + \frac{\varepsilon_n}{h}\alpha_n\left(x, y\left(x,\gamma,\beta_1,\beta_n+\overline{\varepsilon_n}\right)\right), \\
= {} & \sum_{i=1}^{n}\frac{\varepsilon_i}{h}\alpha_i(x,y(\cdot))
\end{aligned}
$$

where $\alpha_k(x, y(\cdot)),\ 1 \le k \le n$, is the solution of the variational equation (1.3) along $y(\cdot)$ satisfying, in each case,

$$\alpha_k^{(i-1)}(\gamma, y(\cdot)) = \delta_{ik}, \quad 1 \le i \le n,$$

Note that $y(\cdot)$ denotes different solutions depending on different parameters, $\varepsilon_1,\ldots,\varepsilon_n,\overline{\varepsilon_1},\ldots,\overline{\varepsilon_n}$. Note also by continuous dependence on both boundary conditions (Lemma 1) and on initial conditions (Kamke's Theorem), $\varepsilon_i \to 0$ as $h \to 0$, $\forall\ 1 \le i \le n$. Thus, it must be the case that, $\forall\ 1 \le i \le n$, $\overline{\varepsilon_i}$ are continuous functions of h and $\overline{\varepsilon_i} \to 0$ as $h \to 0$. In particular, for each $y(\cdot)$, $\lim\limits_{h\to 0} y(\cdot) = u(x,\gamma)$.

Thus, to show $\lim\limits_{h\to 0} D_{\gamma h}(x)$ exists, it suffices to show $\lim\limits_{h\to 0} \frac{\varepsilon_i}{h}$ exists for $1 \le i \le n$. Note from (3.1), that for $2 \le i \le n-1$,

$$\frac{\varepsilon_i}{h} = \Delta_i(h).$$

To find $\frac{\varepsilon_1}{h}$ and $\frac{\varepsilon_n}{h}$, we begin by rewriting our boundary conditions above as

$$
\begin{aligned}
0 &= D_{\gamma h}(a) - \sum_{k=1}^{p} a_k D_{\gamma h}(\xi_k) \\
&= \sum_{i=1}^{n}\frac{\varepsilon_i}{h}\alpha_i(a,y(\cdot)) - \sum_{k=1}^{p} a_k\left\{\sum_{i=1}^{n}\frac{\varepsilon_i}{h}\alpha_i(\xi_k,y(\cdot))\right\} \\
&= \sum_{i=1}^{n}\frac{\varepsilon_i}{h}\left[\alpha_i(a,y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_i(\xi_k,y(\cdot))\right]
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= D_{\gamma h}(b) - \sum_{l=1}^{q} b_l D_{\gamma h}(\eta_l) \\
&= \sum_{i=1}^{n} \frac{\varepsilon_i}{h}\alpha_i(b, y(\cdot)) - \sum_{l=1}^{q} b_l \left\{ \sum_{i=1}^{n} \frac{\varepsilon_i}{h}\alpha_i(\eta_l, y(\cdot)) \right\} \\
&= \sum_{i=1}^{n} \frac{\varepsilon_i}{h} \left[\alpha_i(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_i(\eta_l, y(\cdot)) \right].
\end{aligned}
$$

We will define, for $1 \leq i \leq n$,

$$
\begin{aligned}
A_i(h) &\coloneqq \alpha_i(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_i(\xi_k, y(\cdot)), \\
B_i(h) &\coloneqq \alpha_i(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_i(\eta_l, y(\cdot)).
\end{aligned}
$$

Note that $A_i(h), B_i(h), \Delta_i(h) \in \mathbb{R}$ for $1 \leq i \leq n$, and our boundary conditions can be written as

$$
\begin{cases}
\frac{\varepsilon_1}{h} A_1(h) + \frac{\varepsilon_n}{h} A_n(h) & = -\sum_{i=2}^{n-1} \Delta_i(h) A_i(h), \\
\frac{\varepsilon_1}{h} B_1(h) + \frac{\varepsilon_n}{h} B_n(h) & = -\sum_{i=2}^{n-1} \Delta_i(h) B_i(h).
\end{cases}
$$

This system of equations is uniquely solvable for $\frac{\varepsilon_1}{h}$ and $\frac{\varepsilon_n}{h}$ if and only if

$$
\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.
$$

Noting that $\lim_{h \to 0} y(\cdot) = u(x, \gamma)$, we now define

$$
\begin{aligned}
\tilde{A}_i &\coloneqq \alpha_i(a, u(x, \gamma)) - \sum_{k=1}^{p} a_k \alpha_i(\xi_k, u(x, \gamma)), \\
\tilde{B}_i &\coloneqq \alpha_i(b, u(x, \gamma)) - \sum_{l=1}^{q} b_l \alpha_i(\eta_l, u(x, \gamma)).
\end{aligned}
$$

We claim that

$$
\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0
$$

To prove our claim, we suppose, to the contrary, that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} = 0.$$

Then there must exist nonzero $\rho_1, \rho_2 \in \mathbb{R}$ such that

$$\begin{cases} \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n & = 0, \\ \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n & = 0. \end{cases}$$

Define

$$w(x) \coloneqq \ \rho_1 \alpha_1(x, u(x,\gamma)) + \rho_n \alpha_n(x, u(x,\gamma)).$$

Note that (1.3) is homogeneous and linear, so $w(x)$ is a solution of (1.3) along $u(x)$. By (1.5), our stacked boundary conditions are

$$w^{(i-1)}(\gamma) = \ 0, \quad 2 \leq i \leq n-1.$$

Also note that

$$\begin{aligned} w(a) - \sum_{k=1}^{p} a_k w(\xi_k) &= \frac{\varepsilon_1}{h} \alpha_1(a, u(x,\gamma)) + \rho_1 \alpha_n(a, u(x,\gamma)) \\ &\qquad - \rho_1 \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x,\gamma) - \rho_1 \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x,\gamma) \\ &= \rho_1 \tilde{A}_1 + \rho_1 \tilde{A}_n \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} w(b) - \sum_{l=1}^{q} b_l w(\eta_l) &= \rho_1 \alpha_1(b, u(x,\gamma)) + \rho_1 \alpha_n(b, u(x,\gamma)) \\ &\qquad - \rho_1 \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x,\gamma) - \rho_1 \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x,\gamma) \\ &= \rho_1 \tilde{B}_1 + \rho_1 \tilde{B}_n \\ &= 0. \end{aligned}$$

By assumption (v), this implies that $w(x) \equiv 0$, a contradiction to the linear independence of $\alpha_1(x, u(x,\gamma))$ and $\alpha_n(x, u(x,\gamma))$. Therefore, we can conclude

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

Since we know our solution $y(\cdot)$ is continuous with respect to its initial values and $\lim_{h\to 0} y(\cdot) = u(x,\gamma)$, we have, for $h \neq 0$ and sufficiently small,

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Our coefficients can be obtained by Cramer's rule as

$$\frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} -\sum_{i=2}^{n-1} \Delta_i(h) A_i(h) & A_n(h) \\ -\sum_{i=2}^{n-1} \Delta_i(h) B_i(h) & B_n(h) \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}}$$

and

$$\frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} A_1(h) & -\sum_{i=2}^{n-1} \Delta_i(h) A_i(h) \\ B_1(h) & -\sum_{i=2}^{n-1} \Delta_i(h) B_i(h) \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}}.$$

As a result of continuous dependence on initial conditions, we can take the limit as $h \to 0$. Notice that

$$\lim_{h\to 0} \Delta_i(h) = \lim_{h\to 0} -\frac{u^{(i-1)}(\gamma+h, \gamma+h) - u^{(i-1)}(\gamma, \gamma+h)}{h},$$
$$= -u^{(i)}(\gamma).$$

We will define

$$M := \lim_{h\to 0} \frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} -\sum_{i=2}^{n-1} u^{(i)}(\gamma) A_i(h) & \tilde{A}_n \\ -\sum_{i=2}^{n-1} u^{(i)}(\gamma) B_i(h) & \tilde{B}_n \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}$$

and

$$N := \lim_{h\to 0}\frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} \tilde{A}_1 & -\sum_{i=2}^{n-1} u^{(i)}(\gamma)A_i(h) \\ \tilde{B}_1 & -\sum_{i=2}^{n-1} u^{(i)}(\gamma)B_i(h) \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}.$$

In particular, this allows us to conclude that $\lim_{h\to 0} D_{\gamma h}(x)$ exists, and

$$\lim_{h\to 0} D_{\gamma h}(x) = M\alpha_1(x, u(x,\gamma)) - \sum_{i=2}^{n-1} u^{(i)}(\gamma)\alpha_i(x, u(x,\gamma)) + N\alpha_n(x, u(x,\gamma))$$

Let

$$D_\gamma(x) := \lim_{h\to 0} D_{\gamma h}(x),$$

and note by construction of $D_{\gamma h}(x)$,

$$D_\gamma(x) = \frac{\partial}{\partial\gamma}u(x).$$

Furthermore,

$$\begin{cases} D_\gamma(a) - \sum_{k=1}^{p} a_k D_\gamma(\xi_k) = \lim_{h\to 0}\left[D_{\gamma h}(a) - \sum_{k=1}^{p} a_k D_{\gamma h}(\xi_k)\right] & = 0, \\ D_\gamma^{(i-1)}(\gamma) = \lim_{h\to 0} D_{\gamma h}^{(i-1)}(\gamma) & = -u^{(i)}(\gamma), \;\; 2 \le i \le n-1, \\ D_\gamma(b) - \sum_{l=1}^{q} b_l D_\gamma(\eta_l) = \lim_{h\to 0}\left[D_{\gamma h}(b) - \sum_{l=1}^{q} b_l D_{\gamma h}(\eta_l)\right] & = 0, \end{cases}$$

□

3.3 - Differentiation with Respect to Nonlocal Boundary Points (ξ_k and η_l)

Theorem 6: Assume (i)-(v) are satisfied. Let $u(x)$ be a solution of (1.1) on (c,d), let $a_1,\ldots,a_p,\ b_1,\ldots,b_q \in \mathbb{R}$ and $c < a < \xi_1 < \cdots < \xi_p < \gamma < \eta_1 < \cdots < \eta_q < b < d$ be given so that

$$u(x) = u(x, a,\ \xi_1,\ldots,\xi_p,\ \gamma,\ \eta_1,\ldots,\eta_q,\ b,\ a_1,\ldots,a_p,\ b_1,\ldots,b_q,\ u_1,\ldots,u_n),$$

where

$$\begin{cases} u(a) - \sum_{k=1}^{p} a_k u(\xi_k) & = u_1, \\ u^{(i-1)}(\gamma) & = u_i, \qquad 2 \le i \le n-1, \\ u(b) - \sum_{l=1}^{q} b_l u(\eta_l) & = u_n. \end{cases} \tag{2.1}$$

Then $D_{\xi_m}(x) := \frac{\partial}{\partial \xi_m} u(x)$ exists on (c,d), $\forall\ 1 \le m \le p$, is a solution of (1.3) along $u(x)$, and satisfies the respective boundary conditions,

$$\begin{cases} D_{\xi_m}(a) - \sum_{k=1}^{p} a_k D_{\xi_m}(\xi_k) & = a_m u'(\xi_m), \\ D_{\xi_m}^{(i-1)}(\gamma) & = 0, \qquad 2 \le i \le n-1, \\ D_{\xi_m}(b) - \sum_{l=1}^{q} b_l D_{\xi_m}(\eta_l) & = 0. \end{cases}$$

Then $D_{\eta_m}(x) := \frac{\partial}{\partial \eta_m} u(x)$ exists on (c,d), $\forall\ l \le m \le q$, is a solution of (1.3) along $u(x)$, and satisfies the respective boundary conditions,

$$\begin{cases} D_{\eta_m}(a) - \sum_{k=1}^{p} a_k D_{\eta_m}(\xi_k) & = 0, \\ D_{\eta_m}^{(i-1)}(\gamma) & = 0, \qquad 2 \le i \le n-1, \\ D_{\eta_m}(b) - \sum_{l=1}^{q} b_l D_{\eta_m}(\eta_l) & = b_m u'(\eta_m). \end{cases}$$

Proof. For brevity, we will denote

$$u(x, a,\ \xi_1, \ldots, \xi_p,\ \gamma,\ \eta_1, \ldots, \eta_q,\ b,\ a_1, \ldots, a_p,\ b_1, \ldots, b_q,\ u_1, \ldots, u_p)$$

by $u(x, \xi_m)$. Let $\delta > 0$ be as in Lemma 1, $0 < |h| < \delta$ be given, and define

$$D_{\xi_m h}(x) := \frac{1}{h}[u(x, \xi_m + h) - u(x, \xi_m)].$$

Note that the boundary conditions for $u(x, \xi_m + h)$ read

$$\begin{cases} u(a, \xi_m + h) - \sum_{\substack{k=1 \\ k \ne m}}^{p} a_k u(\xi_k, \xi_m + h) - a_m u(\xi_m + h, \xi_m + h) & = u_1, \\ u^{(i-1)}(\gamma) & = u_i,\ 2 \le i \le n-1, \\ u(b) - \sum_{l=1}^{q} b_l u(\eta_l) & = u_n. \end{cases}$$

Now we will look at our boundary conditions satisfied by $D_{\xi_m h}$. So for $h \neq 0$,

$$\begin{aligned}
D_{\xi_m h}(a) - \sum_{k=1}^{p} a_k D_{\xi_m h}(\xi_k) &= \frac{1}{h}[u(a, \xi_m + h) - u(a, \xi_m)] \\
&\quad - \sum_{k=1}^{p} a_k \frac{1}{h}[u(\xi_k, \xi_m + h) - u(\xi_k, \xi_m)], \\
&= \frac{1}{h}[u(a, \xi_m + h) - \sum_{k=1}^{p} a_k u(\xi_k, \xi_m + h)] \\
&\quad - \frac{1}{h}[u(a, \xi_m) - \sum_{k=1}^{p} a_k u(\xi_k, \xi_m)], \\
&= \frac{1}{h}[u(a, \xi_m + h) - \sum_{k=1}^{p} a_k u(\xi_k, \xi_m + h) - u_1], \\
&= \frac{1}{h}[u(a, \xi_m + h) - \sum_{k=1}^{p} a_k u(\xi_k, \xi_m + h) + a_m u(\xi_m, \xi_m + h) \\
&\quad - a_m u(\xi_m, \xi_m + h) - u_1] \\
&= \frac{1}{h}[u(a, \xi_m + h) - \sum_{\substack{k=1 \\ k \neq m}}^{p} a_k u(\xi_k, \xi_m + h) \\
&\quad - a_m u(\xi_m, \xi_m + h) - u_1], \\
&= \frac{1}{h}[u(a, \xi_m + h) - \sum_{\substack{k=1 \\ k \neq m}}^{p} a_k u(\xi_k, \xi_m + h) - a_m u(\xi_m + h, \xi_m + h) \\
&\quad + a_m u(\xi_m + h, \xi_m + h) - a_m u(\xi_m, \xi_m + h) - u_1], \\
&= \frac{1}{h}[u_1 + a_m u(\xi_m + h, \xi_m + h) - a_m u(\xi_m, \xi_m + h) - u_1], \\
&= \frac{1}{h}[a_m u(\xi_m + h, \xi_m + h) - a_m u(\xi_m, \xi_m + h)].
\end{aligned}$$

We will define

$$\Delta_{\xi_m}(h) \coloneqq \frac{1}{h}[a_m u(\xi_m + h, \xi_m + h) - a_m u(\xi_m, \xi_m + h)].$$

Now for $h \neq 0$ and $2 \leq i \leq n-1$,

$$\begin{aligned}
D_{\xi_m h}^{(i-1)}(\gamma) &= \frac{1}{h}[u^{(i-1)}(\gamma, \xi_m + h) - u^{(i-1)}(\gamma, \xi_m)] \\
&= \frac{1}{h}[u_i - u_i]
\end{aligned}$$

$$= 0.$$

Finally, for $h \neq 0$,

$$\begin{aligned} D_{\xi_m h}(b) - \sum_{l=1}^{q} b_l D_{\xi_m h}(\eta_l) &= \frac{1}{h}[u(b, \xi_m + h) - u(b, \xi_m)] \\ &\quad - \sum_{l=1}^{q} b_l \frac{1}{h}[u(\xi_k, \xi_m + h) - u(\xi_k, \xi_m)] \\ &= \frac{1}{h}[u(b, \xi_m + h) - \sum_{l=1}^{q} b_l u(\xi_k, \xi_m + h)] \\ &\quad - \frac{1}{h}[u(b, \xi_m) - \sum_{l=1}^{q} b_l u(\xi_k, \xi_m)] \\ &= \frac{1}{h}[u_n - u_n] \\ &= 0. \end{aligned}$$

Therefore, our boundary conditions satisfied by $D_{\xi_m h}(x)$ are

$$\begin{cases} D_{\xi_m h}(a) - \sum_{k=1}^{p} a_k D_{\xi_m h}(\xi_k) & = \Delta_{\xi_m}(h), \\ D_{\xi_m h}^{(i-1)}(\gamma) & = 0, \qquad 2 \leq i \leq n-1, \\ D_{\xi_m h}(b) - \sum_{l=1}^{q} b_l D_{\xi_m h}(\eta_l) & = 0. \end{cases}$$

Next, let

$$\begin{aligned} \beta_1 &:= u(\gamma, \xi_m), & \varepsilon_1 &:= u(\gamma, \xi_m + h) - \beta_1, \\ \beta_n &:= u^{(n-1)}(\gamma, \xi_m), & \varepsilon_n &:= u^{(n-1)}(\gamma, \xi_m + h) - \beta_n. \end{aligned}$$

Viewing the solution $u(x, \xi_m)$ of the boundary value problem (1.1)(2.1) as the solution $y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n)$ of the initial value problem (1.1)(1.4), and denoting $u(x, \xi_m)$ by $y(x, \gamma, \beta_1, \beta_n)$, we have

$$D_{\xi_m h}(x) = \frac{1}{h}\left[y(x, \gamma, \beta_1 + \varepsilon_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n)\right].$$

Then, by utilizing a telescoping sum, we have

$$D_{\xi_m h}(x) = \frac{1}{h}[y(x, \gamma, \beta_1 + \varepsilon_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n + \varepsilon_n)$$

$$+ y(x, \gamma, \beta_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n)].$$

Applying Theorem 1 and the Mean Value Theorem, we obtain

$$D_{\xi_m h}(x) = \frac{1}{h}[\alpha_1 (x, y (x, \gamma, \beta_1 + \overline{\varepsilon_1}, \beta_n + \varepsilon_n)) (\beta_1 + \varepsilon_1 - \beta_1)$$
$$+ \alpha_n (x, y (x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n})) (\beta_n + \varepsilon_n - \beta_n)].$$

where $\beta_1 + \overline{\varepsilon_1}$ is between β_1 and $\beta_1 + \varepsilon_1$ and $\beta_n + \overline{\varepsilon_n}$ is between β_n and $\beta_n + \varepsilon_n$. Thus,

$$D_{\xi_m h}(x) = \frac{\varepsilon_1}{h}\alpha_1 (x, y (x, \gamma, \beta_1 + \overline{\varepsilon_1}, \beta_n + \varepsilon_n)) + \frac{\varepsilon_n}{h}\alpha_n (x, y (x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n})),$$

where $\alpha_k(x, y(\cdot)),\ 1 \leq k \leq n$, is the solution of the variational equation (1.3) along $y(\cdot)$ satisfying, in each case,

$$\alpha_k^{(i-1)}(\gamma, y(\cdot)) = \delta_{ik}, \quad 1 \leq i \leq n.$$

Note that $y(\cdot)$ denotes different solutions depending on different parameters, $\varepsilon_1, \varepsilon_n, \overline{\varepsilon_1}$, and $\overline{\varepsilon_n}$. Note also that by continuous dependence on both boundary conditions (Lemma 1) and on initial conditions (Kamke's Theorem), $\varepsilon_1 \to 0$ and $\varepsilon_n \to 0$ as $h \to 0$. Thus, it must be the case that $\overline{\varepsilon_1}$ and $\overline{\varepsilon_n}$ are continuous functions of h and $\overline{\varepsilon_1} \to 0$ and $\overline{\varepsilon_n} \to 0$ as $h \to 0$. In particular, for each $y(\cdot)$, $\lim_{h\to 0} y(\cdot) = u(x, \xi_m)$.

Thus, to show $\lim_{h\to 0} D_{\xi_m h}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\varepsilon_1}{h}$ and $\lim_{h\to 0} \frac{\varepsilon_n}{h}$ exist. Now we can rewrite our boundary conditions above as

$$\begin{aligned}
\Delta_{\xi_m}(h) &= D_{\xi_m h}(a) - \sum_{k=1}^{p} a_k D_{\xi_m h}(\xi_k) \\
&= \frac{\varepsilon_1}{h}\alpha_1(a, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k \left\{\frac{\varepsilon_1}{h}\alpha_1(\xi_k, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(\xi_k, y(\cdot))\right\} \\
&= \frac{\varepsilon_1}{h}\left[\alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_1(\xi_k, y(\cdot))\right] + \frac{\varepsilon_n}{h}\left[\alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_n(\xi_k, y(\cdot))\right],
\end{aligned}$$

and

$$0 = D_{\xi_m h}(b) - \sum_{l=1}^{q} b_l D_{\xi_m h}(\eta_l)$$

$$= \frac{\varepsilon_1}{h}\alpha_1(a, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(a, y(\cdot)) - \sum_{l=1}^{q} b_l \left\{ \frac{\varepsilon_1}{h}\alpha_1(\eta_l, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(\eta_l, y(\cdot)) \right\}$$

$$= \frac{\varepsilon_1}{h}\left[\alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l\alpha_1(\eta_l, y(\cdot))\right] + \frac{\varepsilon_n}{h}\left[\alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l\alpha_n(\eta_k, y(\cdot))\right].$$

We will define

$$A_1(h) \coloneqq \alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_1(\xi_k, y(\cdot)), \qquad A_n(h) \coloneqq \alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k\alpha_n(\xi_k, y(\cdot)),$$

$$B_1(h) \coloneqq \alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l\alpha_1(\eta_l, y(\cdot)), \qquad B_n(h) \coloneqq \alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l\alpha_n(\eta_l, y(\cdot)).$$

Note that $A_1(h), A_n(h), B_1(h), B_n(h), \Delta_{\xi_m}(h) \in \mathbb{R}$, and our boundary conditions can be written as

$$\begin{cases} \frac{\varepsilon_1}{h}A_1(h) + \frac{\varepsilon_n}{h}A_n(h) & = \Delta_{\xi_m}(h), \\ \frac{\varepsilon_1}{h}B_1(h) + \frac{\varepsilon_n}{h}B_n(h) & = \ 0. \end{cases}$$

This system of equations is uniquely solvable for $\frac{\varepsilon_1}{h}$ and $\frac{\varepsilon_n}{h}$ if and only if

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Now we will let $h \to 0$, and noting that $\lim_{h\to 0} y(\cdot) = u(x, \xi_m)$, we define

$$\tilde{A}_1 \coloneqq \alpha_1(a, u(x, \xi_m)) - \sum_{k=1}^{p} a_k\alpha_1(\xi_k, u(x, \xi_m)), \qquad \tilde{A}_n \coloneqq \alpha_n(a, u(x, \xi_m)) - \sum_{k=1}^{p} a_k\alpha_n(\xi_k, u(x, \xi_m)),$$

$$\tilde{B}_1 \coloneqq \alpha_1(b, u(x, \xi_m)) - \sum_{l=1}^{q} b_l\alpha_1(\eta_l, u(x, \xi_m)), \qquad \tilde{B}_n \coloneqq \alpha_n(b, u(x, \xi_m)) - \sum_{l=1}^{q} b_l\alpha_n(\eta_l, u(x, \xi_m)).$$

We claim that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

To prove our claim, we suppose, to the contrary, that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} = 0.$$

Then there must exist nonzero $\rho_1, \rho_2 \in \mathbb{R}$ such that

$$\begin{cases} \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n &= 0, \\ \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n &= 0. \end{cases}$$

Define

$$w(x) \coloneqq \rho_1 \alpha_1(x, u(x, \xi_m)) + \rho_2 \alpha_n(x, u(x, \xi_m)).$$

Note that (1.3) is homogeneous and linear, so $w(x)$ is a solution of (1.3) along $u(x)$. By (1.5), our stacked boundary conditions are

$$w^{(i-1)}(\gamma) = 0, \quad 2 \le i \le n-1.$$

Also note that

$$\begin{aligned} w(a) - \sum_{k=1}^{p} a_k w(\xi_k) &= \rho_1 \left[\alpha_1(a, u(x, \xi_m)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x, \xi_m) \right] \\ &\quad + \rho_2 \left[\alpha_n(a, u(x, \xi_m)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x, \xi_m) \right] \\ &= \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} w(b) - \sum_{l=1}^{q} b_l w(\eta_l) &= \rho_1 \left[\alpha_1(b, u(x, \xi_m)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x, \xi_m) \right] \\ &\quad + \rho_2 \left[\alpha_n(b, u(x, \xi_m)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x, \xi_m) \right] \\ &= \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n \\ &= 0. \end{aligned}$$

By assumption (v), this implies that $w(x) \equiv 0$, a contradiction to the linear independence of $\alpha_1(x, u(x, \xi_m))$ and $\alpha_n(x, u(x, \xi_m))$. Therefore, we can conclude

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

Since we know our solution $y(\cdot)$ is continuous with respect to its initial values and $\lim_{h\to 0} y(\cdot) = u(x, \xi_m)$, it is the case that for sufficiently small $h \neq 0$, we have

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Our coefficients can be obtained by Cramer's rule as

$$\frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} \Delta_{\xi_m}(h) & A_n(h) \\ 0 & B_n(h) \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}} \qquad \text{and} \qquad \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} A_1(h) & \Delta_{\xi_m}(h) \\ B_1(h) & 0 \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}}.$$

As a result of continuous dependence on initial conditions, we can take the limit as $h \to 0$. Notice that

$$\lim_{h\to 0} \Delta_{\xi_m}(h) = a_m u'(\xi_m).$$

We will define

$$P \coloneqq \lim_{h\to 0} \frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} a_m u'(\xi_m) & \tilde{A}_n \\ 0 & \tilde{B}_n \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}} \qquad \text{and} \qquad Q \coloneqq \lim_{h\to 0} \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} \tilde{A}_1 & a_m u'(\xi_m) \\ \tilde{B}_1 & 0 \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}.$$

In particular, this allows us to conclude that $\lim_{h\to 0} D_{\xi_m h}(x)$ exists, and

$$\lim_{h\to 0} D_{\xi_m h}(x) = P\alpha_1(x, u(x, \xi_m)) + Q\alpha_n(x, u(x, \xi_m)).$$

Let

$$D_{\xi_m}(x) \coloneqq \lim_{h\to 0} D_{\xi_m h}(x),$$

and note by construction of $D_{\xi_m h}(x)$,

$$D_{\xi_m}(x) = \frac{\partial}{\partial \xi_m} u(x).$$

Furthermore,

$$D_{\xi_m}(x) = P\alpha_1(x, u(x, \xi_m)) + Q\alpha_n(x, u(x, \xi_m)),$$

which is a solution of the variational equation (1.3) along $u(x)$. In addition, the boundary conditions satisfied by $D_{\xi_m}(x)$, for $1 \leq m \leq p$, are given by

$$\begin{cases} D_{\xi_m}(a) - \sum_{k=1}^{p} a_k D_{\xi_m}(\xi_k) = \lim_{h \to 0} [D_{\xi_m h}(a) - \sum_{k=1}^{p} a_k D_{\xi_m h}(\xi_k)] & = a_m u'(\xi_m), \\ D_{\xi_m}^{(i-1)}(\gamma) = \lim_{h \to 0} D_{\xi_m h}^{(i-1)}(\gamma) & = 0, \ \ 2 \leq i \leq n-1, \\ D_{\xi_m}(b) - \sum_{l=1}^{q} b_l D_{\xi_m}(\eta_l) = \lim_{h \to 0} [D_{\xi_m h}(b) - \sum_{l=1}^{q} b_l D_{\xi_m h}(\eta_l)] & = 0. \end{cases}$$

In complete analogy, it can be similarly shown that $D_{\eta_m}(x)$ exists on (c, d) and satisfies the boundary conditions

$$\begin{cases} D_{\eta_m}(a) - \sum_{k=1}^{p} a_k D_{\eta_m}(\xi_k) = \lim_{h \to 0} \left[D_{\eta_m h}(a) - \sum_{k=1}^{p} a_k D_{\eta_m h}(\xi_k)\right] & = 0, \\ D_{\eta_m}^{(i-1)}(\gamma) = \lim_{h \to 0} D_{\eta_m h}^{(i-1)}(\gamma) & = 0, \ \ 2 \leq i \leq n-1, \\ D_{\eta_m}(b) - \sum_{l=1}^{q} b_l D_{\eta_m}(\eta_l) = \lim_{h \to 0} \left[D_{\eta_m h}(b) - \sum_{l=1}^{q} b_l D_{\eta_m h}(\eta_l)\right] & = b_m u'(\eta_m). \end{cases}$$

□

CHAPTER FOUR

Differentiation of Solutions of (1.1),(1.2) with Respect to Parameters

In Chapter four, we will characterize partial derivatives of the solutions of (1.1),(1.2) with respect to parameters $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$.

4.1 - Differentiation with Respect to Parameters (a_k and b_l)

Theorem 7: Assume (i)-(v) are satisfied. Let $u(x)$ be a solution of (1.1) on (c, d), let $a_1, \ldots, a_p,\ b_1, \ldots, b_q \in \mathbb{R}$ and $c < a < \xi_1 < \cdots < \xi_p < \gamma < \eta_1 < \cdots < \eta_q < b < d$ be given so that

$$u(x) = u(x, a,\ \xi_1, \ldots, \xi_p,\ \gamma,\ \eta_1, \ldots, \eta_q,\ b,\ a_1, \ldots, a_p,\ b_1, \ldots, b_q,\ u_1, \ldots, u_n),$$

where

$$\begin{cases} u(a) - \sum_{k=1}^{p} a_k u(\xi_k) &= u_1, \\ u^{(i-1)}(\gamma) &= u_i, \qquad 2 \le i \le n-1, \\ u(b) - \sum_{l=1}^{q} b_l u(\eta_l) &= u_n. \end{cases} \tag{2.1}$$

Then $D_{a_m}(x) := \frac{\partial}{\partial a_m} u(x)$ exists on (c, d), $\forall\ 1 \le m \le p$, is a solution of (1.3) along $u(x)$, and satisfies the boundary conditions

$$\begin{cases} D_{a_m}(a) - \sum_{k=1}^{p} a_k D_{a_m}(\xi_k) &= u(\xi_m), \\ D_{a_m}^{(i-1)}(\gamma) &= 0, \qquad 2 \le i \le n-1, \\ D_{a_m}(b) - \sum_{l=1}^{q} b_l D_{a_m}(\eta_l) &= 0. \end{cases}$$

Then $D_{b_m}(x) := \frac{\partial}{\partial b_m} u(x)$ exists on (c, d), $\forall\ l \leq m \leq q$, is a solution of (1.3) along $u(x)$, and satisfies the boundary conditions

$$\begin{cases} D_{b_m}(a) - \sum_{k=1}^{p} a_k D_{b_m}(\xi_k) & = 0, \\ D_{b_m}^{(i-1)}(\gamma) & = 0, \qquad 2 \leq i \leq n-1, \\ D_{b_m}(b) - \sum_{l=1}^{q} b_l D_{b_m}(\eta_l) & = u(\eta_m). \end{cases}$$

Proof. For brevity, we will denote

$$u(x, a,\ \xi_1, \ldots, \xi_p,\ \gamma,\ \eta_1, \ldots, \eta_q,\ b,\ a_1, \ldots, a_p,\ b_1, \ldots, b_q,\ u_1, \ldots, u_p)$$

by $u(x, a_m)$. Let $\delta > 0$ be as in Lemma 1, $0 < |h| < \delta$ be given, and define

$$D_{a_m h}(x) := \frac{1}{h}[u(x, a_m + h) - u(x, a_m)].$$

Note that the boundary conditions for $u(x, a_m + h)$ read

$$\begin{cases} u(a, a_m + h) - \sum_{k=1}^{p} a_k u(\xi_k, a_m + h) - h u(\xi_m, a_m + h) & = u_1, \\ u^{(i-1)}(\gamma, a_m + h) & = u_i,\ \ 2 \leq i \leq n-1, \\ u(b, a_m + h) - \sum_{l=1}^{q} b_l u(\eta_l, a_m + h) & = u_n. \end{cases}$$

Now we will look at our boundary conditions satisfied by $D_{a_m h}$. So for $h \neq 0$,

$$\begin{aligned} D_{a_m h}(a) - \sum_{k=1}^{p} a_k D_{a_m h}(\xi_k) &= \frac{1}{h}[u(a, a_m + h) - u(a, a_m)] \\ &\qquad - \sum_{k=1}^{p} a_k \frac{1}{h}[u(\xi_k, a_m + h) - u(\xi_k, a_m)] \\ &= \frac{1}{h}\Bigg[u(a, a_m + h) - \sum_{k=1}^{p} a_k u(\xi_k, a_m + h) \\ &\qquad - u(a, a_m) - \sum_{k=1}^{p} a_k u(\xi_k, a_m)\Bigg] \\ &= \frac{1}{h}\left[u(a, a_m + h) - \sum_{k=1}^{p} a_k u(\xi_k, a_m + h) - u_1\right] \\ &= \frac{1}{h}\Bigg[u(a, a_m + h) - \sum_{k=1}^{p} a_k u(\xi_k, a_m + h) \end{aligned}$$

$$
\left. - hu(\xi_m, a_m + h) + hu(\xi_m, a_m + h) - u_1 \right]
$$

$$
\begin{aligned}
&= \frac{1}{h}[u_1 + hu(\xi_m, a_m + h) - u_1] \\
&= u(\xi_m, a_m + h).
\end{aligned}
$$

We will define

$$
\Delta_{a_m}(h) = u(\xi_m, a_m + h)
$$

Now for $h \neq 0$ and $2 \leq i \leq n-1$,

$$
\begin{aligned}
D_{a_m h}^{(i-1)}(\gamma) &= \frac{1}{h}[u^{(i-1)}(\gamma, a_m + h) - u^{(i-1)}(\gamma, a_m)] \\
&= \frac{1}{h}[u_i - u_i] \\
&= 0.
\end{aligned}
$$

Finally, for $h \neq 0$,

$$
\begin{aligned}
D_{a_m h}(b) - \sum_{l=1}^{q} b_l D_{a_m}(\eta_l) &= \frac{1}{h}[u(b, a_m + h) - u(b, a_m)] \\
&\quad - \sum_{l=1}^{q} b_l \frac{1}{h}[u(\xi_k, a_m + h) - u(\xi_k, a_m)] \\
&= \frac{1}{h}[u(b, a_m + h) - \sum_{l=1}^{q} b_l u(\xi_k, a_m + h)] \\
&\quad - \frac{1}{h}[u(b, a_m) - \sum_{l=1}^{q} b_l u(\xi_k, a_m)] \\
&= \frac{1}{h}[u_n - u_n] \\
&= 0.
\end{aligned}
$$

Therefore, our boundary conditions satisfied by $D_{a_m h}(x)$, for $h \neq 0$, are

$$
\begin{cases}
D_{a_m h}(a) - \sum_{k=1}^{p} a_k D_{a_m h}(\xi_k) & = \Delta_{a_m}(h), \\
D_{a_m h}^{(i-1)}(\gamma) & = 0, \qquad 2 \leq i \leq n-1, \\
D_{a_m h}(b) - \sum_{l=1}^{q} b_l D_{a_m h}(\eta_l) & = 0.
\end{cases}
$$

Next, define

$$\beta_1 \coloneqq u(\gamma, a_m), \qquad\qquad \varepsilon_1 \coloneqq u(\gamma, a_m + h) - \beta_1,$$

$$\beta_n \coloneqq u^{(n-1)}(\gamma, a_m), \qquad\qquad \varepsilon_n \coloneqq u^{(n-1)}(\gamma, a_m + h) - \beta_n.$$

Viewing the solution $u(x, a_m)$ of the boundary value problem (1.1)(2.1) as the solution $y(x, \gamma, \beta_1, u_2, \ldots, u_{n-1}, \beta_n)$ of the initial value problem (1.1)(1.4), and denoting $u(x, a_m)$ by $y(x, \gamma, \beta_1, \beta_n)$, we have

$$D_{a_m h}(x) = \frac{1}{h}\left[y(x, \gamma, \beta_1 + \varepsilon_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n)\right].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} D_{a_m h}(x) = \frac{1}{h}[&y(x, \gamma, \beta_1 + \varepsilon_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n + \varepsilon_n) \\ &+ y(x, \gamma, \beta_1, \beta_n + \varepsilon_n) - y(x, \gamma, \beta_1, \beta_n)]. \end{aligned}$$

Applying Theorem 1 and the Mean Value Theorem, we obtain

$$\begin{aligned} D_{a_m h}(x) = \frac{1}{h}[&\alpha_1\left(x, y\left(x, \gamma, \beta_1 + \overline{\varepsilon_1}, \beta_n + \varepsilon_n\right)\right)\left(\beta_1 + \varepsilon_1 - \beta_1\right) \\ &+ \alpha_n\left(x, y\left(x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n}\right)\right)\left(\beta_n + \varepsilon_n - \beta_n\right)]. \end{aligned}$$

where $\beta_1 + \overline{\varepsilon_1}$ is between β_1 and $\beta_1 + \varepsilon_1$ and $\beta_n + \overline{\varepsilon_n}$ is between β_n and $\beta_n + \varepsilon_n$. Thus,

$$D_{a_m h}(x) = \frac{\varepsilon_1}{h}\alpha_1\left(x, y\left(x, \gamma, \beta_1 + \overline{\varepsilon_1}, \beta_n + \varepsilon_n\right)\right) + \frac{\varepsilon_n}{h}\alpha_n\left(x, y\left(x, \gamma, \beta_1, \beta_n + \overline{\varepsilon_n}\right)\right),$$

where $\alpha_k(x, y(\cdot)),\ 1 \le k \le n$, is the solution of the variational equation (1.3) along $y(\cdot)$ satisfying, in each case,

$$\alpha_k^{(i-1)}(\gamma, y(\cdot)) = \delta_{ik}, \quad 1 \le i \le n.$$

Note that $y(\cdot)$ denotes different solutions depending on different parameters, $\varepsilon_1, \varepsilon_n, \overline{\varepsilon_1}$, and $\overline{\varepsilon_n}$. Note also that by continuous dependence on both boundary conditions (Lemma 1) and on initial conditions (Kamke's Theorem), $\varepsilon_1 \to 0$ and $\varepsilon_n \to 0$ as $h \to 0$. Thus, it must be the case that $\overline{\varepsilon_1}$ and $\overline{\varepsilon_n}$ are continuous functions of h and $\overline{\varepsilon_1} \to 0$ and $\overline{\varepsilon_n} \to 0$ as $h \to 0$. In particular, for each $y(\cdot)$, $\lim\limits_{h \to 0} y(\cdot) = u(x, a_m)$.

Thus, to show $\lim\limits_{h\to 0} D_{a_m h}(x)$ exists, it suffices to show $\lim\limits_{h\to 0} \frac{\varepsilon_1}{h}$ and $\lim\limits_{h\to 0} \frac{\varepsilon_n}{h}$ exist. Now we can rewrite our boundary conditions above as

$$\begin{aligned}
\Delta_{a_m}(h) &= D_{a_m h}(a) - \sum_{k=1}^{p} a_k D_{a_m h}(\xi_k) \\
&= \frac{\varepsilon_1}{h}\alpha_1(a, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(a, y(\cdot)) \\
&\qquad - \sum_{k=1}^{p} a_k \left\{ \frac{\varepsilon_1}{h}\alpha_1(\xi_k, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(\xi_k, y(\cdot)) \right\} \\
&= \frac{\varepsilon_1}{h}\left[\alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, y(\cdot)) \right] \\
&\qquad + \frac{\varepsilon_n}{h}\left[\alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, y(\cdot)) \right]
\end{aligned}$$

and

$$\begin{aligned}
0 &= D_{a_m h}(b) - \sum_{l=1}^{q} b_l D_{a_m h}(\eta_l) \\
&= \frac{\varepsilon_1}{h}\alpha_1(a, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(a, y(\cdot)) \\
&\qquad - \sum_{l=1}^{q} b_l \left\{ \frac{\varepsilon_1}{h}\alpha_1(\eta_l, y(\cdot)) + \frac{\varepsilon_n}{h}\alpha_n(\eta_l, y(\cdot)) \right\} \\
&= \frac{\varepsilon_1}{h}\left[\alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, y(\cdot)) \right] \\
&\qquad + \frac{\varepsilon_n}{h}\left[\alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_k, y(\cdot)) \right].
\end{aligned}$$

We will define

$$\begin{aligned}
A_1(h) &\coloneqq \alpha_1(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, y(\cdot)), \qquad & A_n(h) &\coloneqq \alpha_n(a, y(\cdot)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, y(\cdot)), \\
B_1(h) &\coloneqq \alpha_1(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, y(\cdot)), \qquad & B_n(h) &\coloneqq \alpha_n(b, y(\cdot)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, y(\cdot)).
\end{aligned}$$

Note that $A_1(h), A_n(h), B_1(h), B_n(h), \Delta_{a_m}(h) \in \mathbb{R}$, and our boundary conditions can be written as

$$\begin{cases} \frac{\varepsilon_1}{h}A_1(h) + \frac{\varepsilon_n}{h}A_n(h) & = \Delta_{a_m}(h), \\ \frac{\varepsilon_1}{h}B_1(h) + \frac{\varepsilon_n}{h}B_n(h) & = \ 0. \end{cases}$$

This system of equations is uniquely solvable for $\frac{\varepsilon_1}{h}$ and $\frac{\varepsilon_n}{h}$ if and only if

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Now we will let $h \to 0$, and noting that $\lim_{h\to 0} y(\cdot) = u(x, a_m)$, we define

$$\tilde{A}_1 \coloneqq \alpha_1(a, u(x, a_m)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x, a_m)), \quad \tilde{A}_n \coloneqq \alpha_n(a, u(x, a_m)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x, a_m)),$$

$$\tilde{B}_1 \coloneqq \alpha_1(b, u(x, a_m)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x, a_m)), \quad \tilde{B}_n \coloneqq \alpha_n(b, u(x, a_m)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x, a_m)).$$

We claim that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

To prove our claim, we suppose, to the contrary, that

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} = 0.$$

Then there must exist nonzero $\rho_1, \rho_2 \in \mathbb{R}$ such that

$$\begin{cases} \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n & = 0, \\ \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n & = 0. \end{cases}$$

Define

$$w(x) \coloneqq \ \rho_1 \alpha_1(x, u(x, a_m)) + \rho_2 \alpha_n(x, u(x, a_m)).$$

Note that (1.3) is homogeneous and linear, so $w(x)$ is a solution of (1.3) along $u(x)$.

By (1.5), our stacked boundary conditions are

$$w^{(i-1)}(\gamma) = \ 0, \quad 2 \leq i \leq n-1.$$

Also note that

$$\begin{aligned}
w(a) - \sum_{k=1}^{p} a_k w(\xi_k) &= \rho_1 \left[\alpha_1(a, u(x, a_m)) - \sum_{k=1}^{p} a_k \alpha_1(\xi_k, u(x, a_m)\right] \\
&\quad + \rho_2 \left[\alpha_n(a, u(x, a_m)) - \sum_{k=1}^{p} a_k \alpha_n(\xi_k, u(x, a_m)\right] \\
&= \rho_1 \tilde{A}_1 + \rho_2 \tilde{A}_n \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
w(b) - \sum_{l=1}^{q} b_l w(\eta_l) &= r_1 \left[\alpha_1(b, u(x, a_m)) - \sum_{l=1}^{q} b_l \alpha_1(\eta_l, u(x, a_m)\right] \\
&\quad + r_2 \left[\alpha_n(b, u(x, a_m)) - \sum_{l=1}^{q} b_l \alpha_n(\eta_l, u(x, a_m)\right] \\
&= \rho_1 \tilde{B}_1 + \rho_2 \tilde{B}_n \\
&= 0.
\end{aligned}$$

By assumption (v), this implies that $w(x) \equiv 0$, a contradiction to the linear independence of $\alpha_1(x, u(x, a_m))$ and $\alpha_n(x, u(x, a_m))$. Therefore, we can conclude

$$\det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{bmatrix} \neq 0.$$

Since we know our solution $y(\cdot)$ is continuous with respect to its initial values and $\lim_{h \to 0} y(\cdot) = u(x, a_m)$, it is the case that for sufficiently small $h \neq 0$, we have

$$\det \begin{bmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{bmatrix} \neq 0.$$

Our coefficients can be obtained by Cramer's rule as

$$\frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} u(\xi_m, a_m + h) & A_n(h) \\ 0 & B_n(h) \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}} \quad \text{and} \quad \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} A_1(h) & u(\xi_m, a_m + h) \\ B_1(h) & 0 \end{vmatrix}}{\begin{vmatrix} A_1(h) & A_n(h) \\ B_1(h) & B_n(h) \end{vmatrix}}.$$

As a result of continuous dependence on initial conditions, we can take the limit as $h \to 0$. We will define

$$V := \lim_{h \to 0} \frac{\varepsilon_1}{h} = \frac{\begin{vmatrix} u(\xi_m) & \tilde{A}_n \\ 0 & \tilde{B}_n \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}} \quad \text{and} \quad W := \lim_{h \to 0} \frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} \tilde{A}_1 & u(\xi_m) \\ \tilde{B}_1 & 0 \end{vmatrix}}{\begin{vmatrix} \tilde{A}_1 & \tilde{A}_n \\ \tilde{B}_1 & \tilde{B}_n \end{vmatrix}}.$$

In particular, this allows us to conclude that $\lim_{h \to 0} D_{a_m h}(x)$ exists, and

$$\lim_{h \to 0} D_{a_m h}(x) = V\alpha_1(x, u(x, a_m)) + W\alpha_n(x, u(x, a_m)).$$

Let

$$D_{a_m}(x) := \lim_{h \to 0} D_{a_m h}(x),$$

and note by construction of $D_{a_m h}(x)$,

$$D_{a_m}(x) = \frac{\partial}{\partial a_m} u(x).$$

Furthermore,

$$D_{a_m}(x) = V\alpha_1(x, u(x, a_m)) + W\alpha_n(x, u(x, a_m)),$$

which is a solution of the variational equation (1.3) along $u(x)$. In addition, the boundary conditions satisfied by $D_{a_m}(x)$, for $1 \leq m \leq p$, are given by

$$\begin{cases} D_{a_m}(a) - \sum_{k=1}^{p} a_k D_{a_m}(\xi_k) = \lim_{h \to 0} \left[D_{a_m h}(a) - \sum_{k=1}^{p} a_k D_{a_m h}(\xi_k) \right] & = u(\xi_m), \\ D_{a_m}^{(i-1)}(\gamma) = \lim_{h \to 0} D_{a_m h}^{(i-1)}(\gamma) & = 0, \ \ 2 \leq i \leq n-1, \\ D_{a_m}(b) - \sum_{l=1}^{q} b_l D_{a_m}(\eta_l) = \lim_{h \to 0} \left[D_{a_m h}(b) - \sum_{l=1}^{q} b_l D_{a_m h}(\eta_l) \right] & = 0. \end{cases}$$

In complete analogy, it can be similarly shown that $D_{b_m}(x)$ exists on (c,d) and satisfies the boundary conditions

$$\begin{cases} D_{b_m}(a) - \sum_{k=1}^{p} a_k D_{b_m}(\xi_k) = \lim_{h\to 0} \left[D_{b_m h}(a) - \sum_{k=1}^{p} a_k D_{b_m h}(\xi_k)\right] & = 0, \\ D_{b_m}^{(i-1)}(\gamma) = \lim_{h\to 0} D_{b_m h}^{(i-1)}(\gamma) & = 0, \ \ 2 \le i \le n-1, \\ D_{b_m}(b) - \sum_{l=1}^{q} b_l D_{b_m}(\eta_l) = \lim_{h\to 0} \left[D_{b_m h}(b) - \sum_{l=1}^{q} b_l D_{b_m h}(\eta_l)\right] & = u(\eta_m). \end{cases}$$

□

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