

ABSTRACT

Knot Equivalence through Braids and Rational Tangles

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A major goal in the study of knot theory is to discover more practical and universal methods that determine knot equivalence. In this paper, we will explore two methods of doing so through the use of rational tangles and braids. We begin by determining a method of converting from knots to rational tangles, then find equivalence between the tangles by means of continued fractions and Prime Fractional Sets, and then convert the tangles back into knots. This leads to the conclusion that two knots have corresponding rational tangles which are equivalent if and only if the original two knots were themselves equivalent. Through a similar course of events, we then do the same with braids through the use of Markovs Moves and come to the conclusion of Markovs Theorem, where two knots are shown to be equivalent if and only if their corresponding braids are determined to be Markov equivalent.

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KNOT EQUIVALENCE THROUGH BRAIDS AND RATIONAL TANGLES

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CHAPTER ONE

Background

The study of knot theory is predominately concerned with the ability to distinguish different knots from each other and, more importantly, be able to tell whether or not two knots are the same. The need for this field of mathematics came with Lord Kelvin's theory of the atom in which it was stated that the chemical properties of elements are related to a system of knotting between atoms [7]. As a response to this, P.G. Tait published the first series of papers devoted to the mathematics of distinguishing knots in 1877 [7].

By 1932, with the German mathematician K. Reidemeister's publishing of the first book on knot theory, *Knotentheorie*, there were means of distinguishing almost any pair of knots from each other, but with complicated calculations that were not ideal for practical use [7]. Reidemeister's techniques become especially difficult when considering the complicated knots of modern knot theory application in the study of DNA recombination [3]. Thus the continued study of knot theory has been dedicated to finding more practical means of distinguishing knots, which has included the use of multiple fields of mathematics such as combinatorics, algebra, geometry, topology, and number theory. During the last century, the field has grown beyond the study of knots, and into tangles and braids as well. In this paper, we will be exploring methods of differentiating between knots through the use of tangles and braids as mediators.

CHAPTER TWO

Introduction to Knots and Tangles

As this is a paper on Knot Theory, we will begin by defining a knot.

Definition 1 *A knot is a simple closed polygonal curve in \mathbb{R}^3 [7].*

So our knotted material is not allowed to go through itself, rather only around itself (simple), is connected at its endpoints (closed), and even though we depict it with two dimensional figures, it is representing an object with length, width, and height (polygonal curve in \mathbb{R}^3). Given this definition of a knot, we can now define a tangle.

Definition 2 *A tangle is a portion of a knot-diagram from which emerges four arcs [5].*

More specifically we will be working with rational tangles, for which we will use Kauffman and Lambropoulou's definition as follows.

Definition 3 *A rational tangle is a proper embedding of two unoriented arcs α_1, α_2 in a 3-ball B^3 such that the embedding must send the endpoints of the arcs to 4 points on the ball's boundary and there exists a homeomorphism of the pairs:*

$$\bar{h} : (B^3, \alpha_1, \alpha_2) \rightarrow (D^2 \times I, \{x, y\} \times I). [6]$$

In other words, a rational tangle is created by taking two unknotted arcs and applying finitely many twists to neighboring endpoints. Additionally, we may also use the three following moves, called the Reidemeister moves.

Definition 4 *Figure 1 shows three operations defined as the Reidemeister moves [7].*

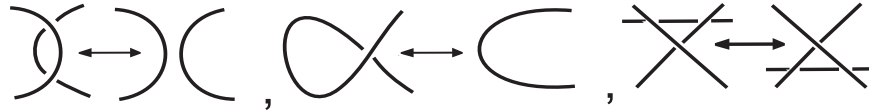


Figure 1: Reidemeister moves

What we are really interested in is how to determine if two tangles or two knots are equivalent, or isotopic.

Definition 5 *Two knots K, N are said to be isotopic, or equivalent, denoted $K \sim N$, if one knot can be transformed into the other through a finite sequence of Reidemeister moves [7].*

Definition 6 *Two rational tangles, S, T , are said to be isotopic, which is denoted $S \sim T$, if and only if any diagrams representing S and T in the plane have identical configurations of their four endpoints on the boundary of the projection disc and they differ by a finite sequence of the well-known Reidemeister moves which take place in the interior of the disc [6].*

Once we have a rational tangle, we must have a means of naming it. Each tangle is composed of smaller horizontal (top row of examples in Figure 2) and vertical tangles (bottom row of examples in Figure 2). We denote horizontal tangles by $[n]$, made of n twists, and vertical tangles by $\frac{1}{[n]}$, made of n twists, where $n \in \mathbb{Z}$. Figure 2 shows

a few examples. Notice that the sign of n depends on the left most (for horizontal twists) or top most (for vertical twists) overcrossing.

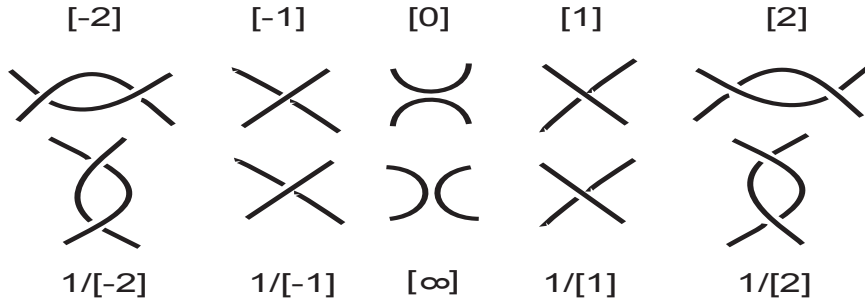


Figure 2: Naming tangles

While building a rational tangle, we can use both addition and multiplication of horizontal and vertical tangles. The product of two rational tangles is defined as connecting the top two endpoints of one tangle to the bottom two endpoints of another. Similarly, the sum of two rational tangles is defined as connecting the two leftmost endpoints of one tangle with the two rightmost points of the other (as shown in Figure 3) [4]. Notice that while the product (sum) of two horizontal (vertical) tangles may be another tangle, it is no longer rational. However, the product (sum) of two vertical (horizontal) tangles will always result in another rational tangle. Furthermore, the product (sum) of two rational tangles will be rational so long as one of the two is a vertical (horizontal) tangle [6].

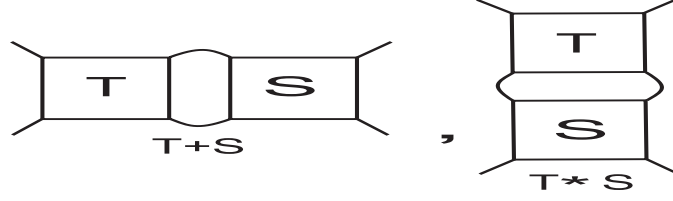


Figure 3: Products and sums of tangles

Definition 7 *A rational tangle is in twist form if and only if it is created by consecutive additions and multiplications by the tangles $[\pm 1]$, starting with one of the trivial tangles $[0]$ or $[\infty]$. [6]*

We can then represent rational tangles of twist form in the following algebraic expressions:

$$[s_k] + (\dots + (\frac{1}{[r_3]} * ([s_1] + (\frac{1}{[r_1]} * [s_0] * \frac{1}{[r_2]})) + [s_2]) * \frac{1}{[r_4]}) + \dots) + [s_{k+1}]$$

or

$$\frac{1}{[r_k]} * (\dots * ([s_3] + (\frac{1}{[r_1]} * ([s_1] + \frac{1}{[r_0]} + [s_2]) * \frac{1}{[r_2]})) + [s_4]) * \dots) * \frac{1}{[r_{k+1}]} [5]$$

depending on whether the tangle begins with a vertical or horizontal twist.

In addition to addition and multiplication, we have another move called a flype.

Definition 8 *A flype is an isotopy of a tangle t of the form $[\pm m] + t$ or $\frac{1}{[\pm m]} * t$. A flype is considered rational if it is applied to a rational tangle [6]. Figure 4 gives an example of a flype for addition and multiplication.*

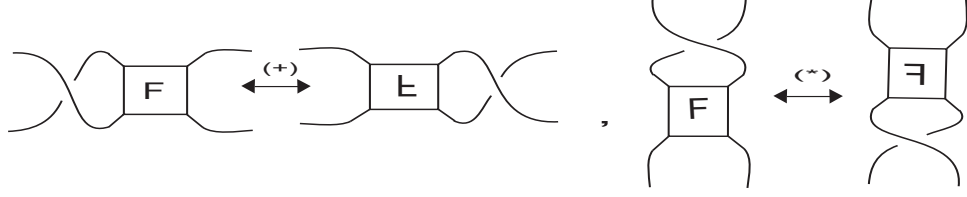


Figure 4: Flype

Definition 9 *A tangle in the form*

$$((([a_n] * \frac{1}{[r_{n-1}]} + [a_{n-2}]) * \dots * \frac{1}{[a_2]}) + [a_1])$$

for $a_1 \in \mathbb{Z}$ and $a_2, \dots, a_n \in \mathbb{Z} - \{0\}$, is referred to as a tangle in standard form [6].

Lemma 1 *Every rational tangle can be written in standard form [6].*

Proof By the definition of rational tangle and the construction of the twist form, every rational tangle can be written in twist form. Notice that by using flypes we can rewrite $[s_1] + \frac{1}{[r_0]} + [s_2]$ as $\frac{1}{[r_0]} + [s_2 + s_1]$ and $\frac{1}{[r_1]} * [s_0] * \frac{1}{[r_2]}$ as $[s_0] * \frac{1}{[r_2 + r_1]}$. Given any twist form tangle, we can start from the outside twists of the tangle, s_0 , and, through the use of these horizontal and vertical flypes, we can make our tangle into the form

$$((((([s_0] * \frac{1}{[r_1 + r_2]}) + [s_1 + s_2]) * \frac{1}{[r_3 + r_4]}) + \dots) + [s_k + s_{k+1}])$$

as desired. ■

Definition 10 *A continued fraction in integer tangles is an algebraic description of a rational tangle as a continued fraction built from the tangles $[a_1], [a_2], \dots, [a_n]$ where $a_1 \in \mathbb{Z}$ and $a_2, \dots, a_n \in \mathbb{Z} - \{0\}$ with all of the numerators equal to one, denoted as*

$$T = [[a_1], [a_2], \dots, [a_{n-1}], [a_n]] := [a_1] + \frac{1}{[a_2] + \dots + \frac{1}{[a_{n-1}] + \frac{1}{[a_n]}}] \cdot [4]$$

Given Lemma 1 and this definition, it is easy to see that each rational tangle can be written as a continued fraction. On the other hand, the following is by no means trivial and was discovered by John Conway in 1970.

Theorem 1 *Two rational tangles are isotopic if and only if they have the same fraction [4]. (An extensive proof of this is given in Kauffman and Lambropoulou's 2004 paper "On the classification of rational tangles" [6].)*

This is truly a remarkable result upon which this paper will heavily rely.

CHAPTER THREE

Rational Tangles and Knot Equivalence

In order to use rational tangles as a method of finding equivalence between two given knots, we must have a way of converting between rational tangles and knots. We begin with the conversion from tangles to knots. Notice that in any given rational tangle, there are 4 free strands. By connecting two of them together and then connecting the remaining two, we have a closed figure.

Definition 11 *A simple closed tangle is a tangle that undergoes the following:*

- 1) *strands may only be attached at their endpoints*
- 2) *two, and only two, strands may be attached at a given point*
- and 3) *all strands must be connected to another strand.*

By connecting three or four at the same point, or connecting two at a point other than their endpoints, the resulting figure would be either not simple or not closed. For the remainder of the paper, if a tangle is referred to as closed, we assume simplicity as well.

Figure 5 shows the three possible ways in which the four strands may be connected.

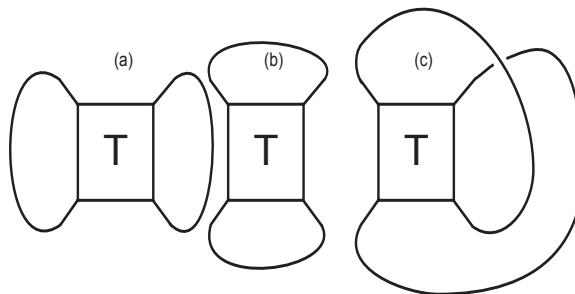


Figure 5: Possible connections for arcs of tangles

Lemma 2 *Every rational tangle can be closed in such a way that it becomes a knot.*

Proof Notice that the right most connection in Figure 5 creates an additional overcrossing, changing the nature of the connected tangle. We will refer to the left and middle connections of Figure 5 as numerator connection and denominator connection respectively. Recall from the definition of rational tangles that these four strands are from 2 unoriented arcs embedded in a 3-ball, B^3 . We will name these two arcs n and m . The following are possibilities for the ordering of m and n in different tangles:

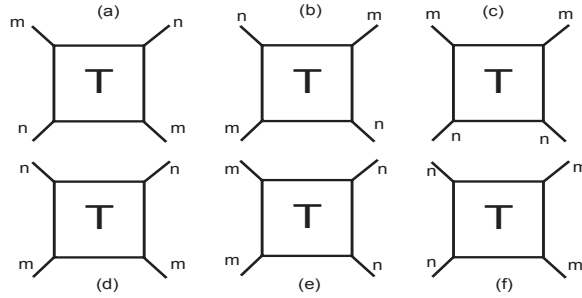


Figure 6: Possible m and n combinations

Notice that a and b are equivalent up to naming. The same is true for c with d and e with f. Since T is isotopic to T^T (the transverse of T), notice that c and d are isotopic to e and f as well. We are now left with 2 possibilities concerning methods of naming the arcs. Let us now consider numerator and denominator connection for each of these, as shown in Figure 7.

Notice that given any time an m connects to an m or an n connects with an n , the arcs will close in on themselves and it will create a link of two knots. On the other hand, if each m connects with an n , it would result in a simple closed figure. Since the connected arcs will inherit the polygonal curve characteristic from the rational tangle, we have a simple closed polygonal curve, which is a knot by definition.

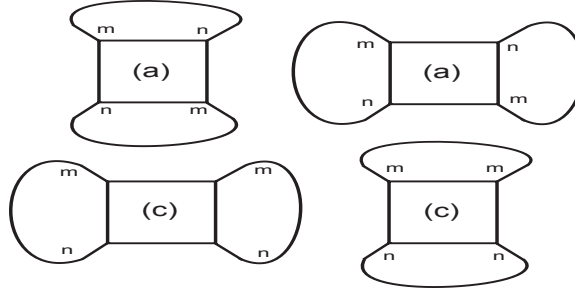


Figure 7: Closure of n and m

This is true with any rational tangle, as, when viewing Figure 7, we see that regardless of how the two arcs are situated, there will always be a means, through either numerator or denominator connection, of creating a closed knot rather than a link.

■

Definition 12 *We call a knot which results from the closure of a rational tangle a rational knot.*

Lemma 3 *Every rational knot can be made into a rational tangle.*

Proof Since by definition, a rational knot is derived from connecting the arcs of a rational tangle, if we were to break the knot at the points where they were originally connected, we would have the rational tangle from which the knot came.

■

We must now explain a method of deciding whether or not a given knot is rational.

Definition 13 *The arcs that border the knot are called external arcs and the remaining arcs are internal arcs. This is best shown through example.*

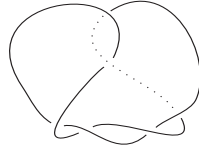


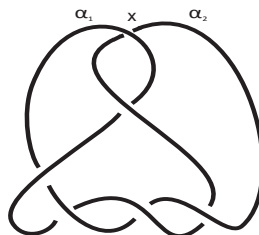
Figure 8: full line = external arc, dotted line = internal arc

The following is the method to be used to test for rationality in a knot:

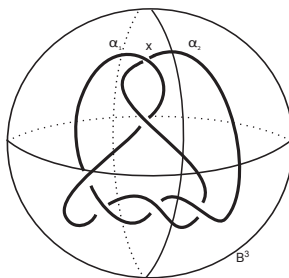
Let S be a knot, and let α_1 and α_2 be two external arcs of S that share a common crossing, x . Let B^3 be a 3-ball such that it completely contains S . Now, pull α_1 and α_2 out of B^3 without creating additional crossings. This creates a tangle within B^3 by definition, which we will call S' . Now check to see if S' is rational according to the definition given in Chapter 1. If so, S is a rational knot.

This method is used in the following example.

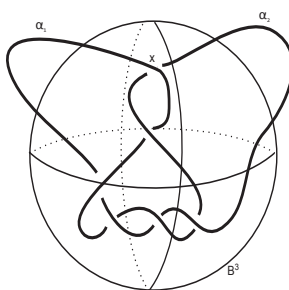
Starting with the 6_1 knot, we choose our external arcs α_1 and α_2 with the common crossing x .



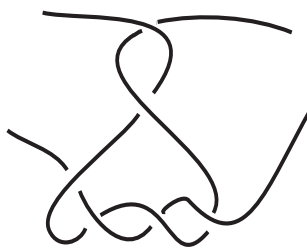
We set it in a 3-ball



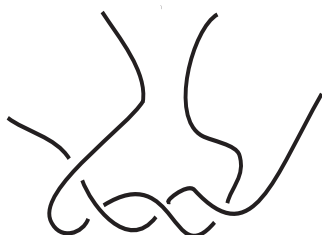
and then pull out, or extend, α_1 and α_2 out of the sphere,



giving us a tangle.



We now check to see if it is rational. First untwist the top strands twice.



Then untwist the right strands four times,



giving us the trivial tangle. Hence the tangle is rational by definition and therefore the original knot, the 6_1 knot, is a rational knot. Through this method, it can be seen that the first 24 knots are rational, except for 8_5 [5]. We will refer to this method as the *Rational Check Method*.

We now have a method to convert from a rational knot to a rational tangle and vice-versa! By applying this method to each pair of external arcs connected by a common crossing, some interesting results emerge. Notice that in the example with the 6_1 knot, the resulting rational tangle was $[-2, 4] = \frac{-7}{4}$. If the rational check method is then applied to the other pairs of α_1 and α_2 choices, the $[-1, -3, -2] = \frac{-9}{5}$ tangle results.

Definition 14 *Given a knot K , we call the set of fractions that correspond to tangles which when closed form K , the fraction set of the knot K .*

Examine this fraction set. In the previous example it was shown that the 6_1 knot could be formed from the connection of the $[-2, 4] = \frac{-7}{4}$ tangle. But if one were to twist the left two strands three times, it would create the $[3, -2, 4] = \frac{17}{7}$ tangle. Applying numerator closure and simplifying a bit, it would again give way to the 6_1

knot. We could then have an arbitrary number of twists with those two left-most strands, and with numerator closure still have the 6_1 knot. So for $n \in \mathbb{Z}$, the tangle $[n, -2, 4]$ is in the fraction set of 6_1 , giving way to an infinite set of fractions which, when closed, will produce the 6_1 knot.

In order to navigate around these infinite sets, we will only look at the fractions which, when connected, are indivisible, or, will not simplify through untwisting.

Definition 15 *We call the corresponding fraction of the tangles which, when connected, do not simplify through untwisting, prime fractions.*

Definition 16 *The set of prime fractions resulting from the rational tangles of a given knot is the prime fraction set of a knot. The notation for this set is $P(K_n)$, where K_n is the name of the basic knot associated with the set.*

In continuation of the example for the rational check method, notice that $P(6_2) = \{\frac{-7}{4}, \frac{-9}{5}\}$. We are now equipped with the tools needed to prove the main theorem.

Theorem 2 *Two knots are equivalent if and only if their corresponding rational tangle fractions are of the same prime fraction set.*

Proof First we show that if two knots are equivalent, then their corresponding rational tangles are of the same prime fraction set. Suppose that the two fractions are not of the same set. Then when the tangles corresponding to the two fractions are reconnected, they will be two different basic knots. Therefore there are no sequence of Reidemeister moves which will transform the first knot into the second, and they are

not equivalent. Thus if two knots are equivalent, then their corresponding rational tangles are of the same prime fraction set.

We now show that given two fractions within the same prime fraction set, the closure of their corresponding tangles will result in equivalent knots. Let $t, s \in P(K_n)$, for some basic knot K_n , and T, S be the two tangles represented by those fractions. Now call T', S' the knots created by the closure of these two tangles. Then by the definition of the prime fractional set of K_n , $T' \sim K_n$ and $S' \sim K_n$ so $T' \sim K_n \sim S'$ thus $T' \sim S'$ as desired.

■

CHAPTER FOUR

Introduction to Braids

We must now address the issue of our inability to use rational tangles on knots that are not rational. The problem was that two arcs were not enough to allow us to transform all knots into rational tangles. So what would happen if we were to add additional arcs? To examine this new territory, we must leave the realm of tangles and enter the world of braids.

Definition 17 *An n -braid is a collection of n strings in (x,y,t) -space, which are disjoint and monotone in the t direction [2].*

To further get a picture of what this is, we begin by producing a cube bounded by $[0, 1]$ in the x direction, $[-\frac{1}{2}, \frac{1}{2}]$ in the y direction, and $[0, 1]$ in the t direction [3]. Next, denote the set of n arcs as $\{a_i\}_{i=1}^n$. Set one of the endpoints of a_i on the bottom of the cube at $(\frac{i}{n+1}, 0, 0)$ and then the other endpoint of the arc on the top of the cube at $(\frac{i}{n+1}, 0, 1)$. Do this for each $1 \leq i \leq n$. This creates the trivial braid; an example of which is displayed by Figure 9.

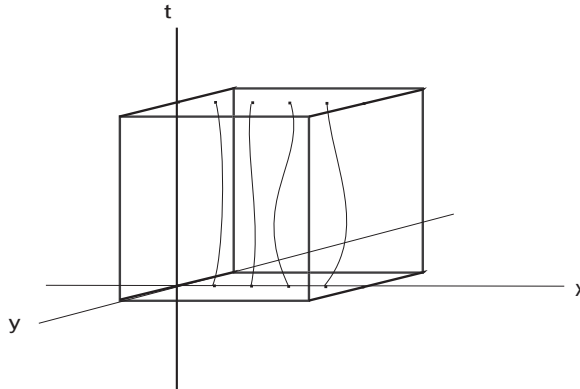


Figure 9: The trivial 4-Braid

The braids then deviate from this trivial braid when for some $1 \leq i, j \leq n$, arc a_i begins at point $(\frac{i}{n+1}, 0, 0)$ but ends at $(\frac{j}{n+1}, 0, 1)$, where $j \neq i$. An example of this is given in Figure 10.

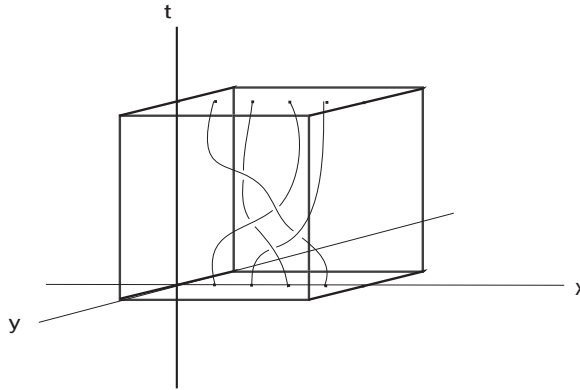


Figure 10: A non-trivial 4-Braid

Recall the three Reidemeister moves. We used these in order to define equivalence between knots and tangles. Due to the monotonic increasing nature of the arcs of the braids and the fact that more than just two arcs are being used, the three moves need to be altered.

Definition 18 *Two braids are said to be braid equivalent if and only if one can be transformed into the other through a finite use of the three moves in Figure 11 [2]:*

The set of braids with n arcs are denoted B_n . We now want to define braid multiplication, which is essentially done by stacking their respective cubes on top of each other and then altering the t values to shift the two cubes back into the unit cube [2]. Let α and β be two braids in B_n . Let the x and y values remain the same for each of the coordinates in the braids, but alter the t coordinates using the following two transformations. For the t coordinates in α , we use the transformation $T(x, y, t) = (x, y, \frac{t}{2})$

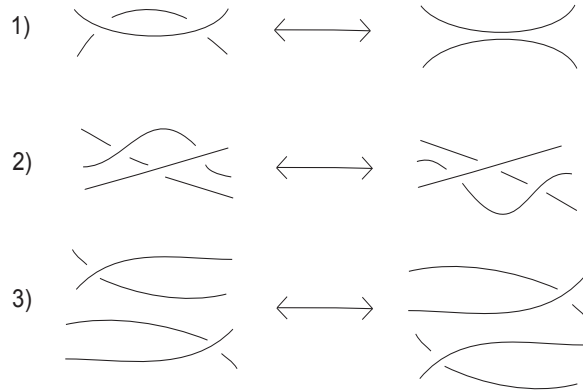


Figure 11: Equivalence moves for braids

and for the t coordinates in β , we use the transformation $T(x, y, t) = (x, y, \frac{t+1}{2})$ [2].

For an example, let the above trivial 4-braid be α and the above nontrivial 4-braid be β . The following shows the stacking of their respective cubes and then the coordinate shift, resulting in the product $\alpha\beta$ or $\alpha * \beta$.

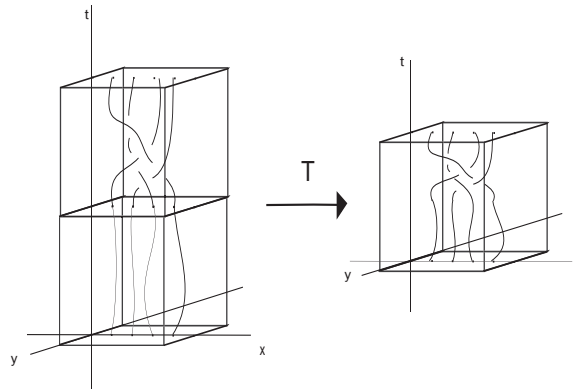


Figure 12: Braid multiplication

At this point, it is easy to see that braid multiplication is associative up to braid equivalence. The order in which we multiply only effects how condensed the product braid is. For $\alpha, \beta, \gamma \in B_n$, notice that $(\alpha * \beta) * \gamma$ will result in a braid with α and β more condensed than γ , and the braid resulting from $\alpha * (\beta * \gamma)$ will be such that β

and γ are more condensed than α . Through our altered Reidemeister moves, we can easily stretch and condense the strands in the t direction to show equivalence.

For the remainder of the paper, we will draw the braids without the cube and axes out of simplicity.

Next we will provide a manner of naming a given braid. Begin by taking the (y, t) plane projection of the braid and orientating it such that the t axis is horizontal. Then label the arcs $1, 2, \dots, n$ starting from the bottom strand. Next, with the use of the 3rd equivalence move from Figure 11, move the crossings such that no two crossings occupy the same coordinate in the y value. We then split the braid up into a product of braids such that each braid consists of one of the crossings of the original braid. The original braid is then described by a sequence of permutations of the arcs.

There are two possibilities for each crossing: either the strand from the top comes over the bottom strand, or the bottom comes over the top strand. These are shown in Figure 13 by (a) and (b) respectively.

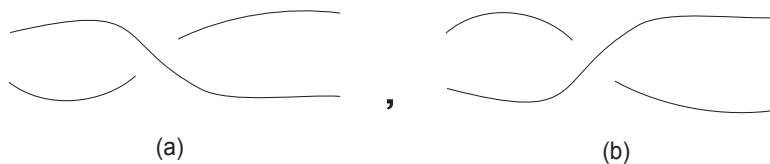
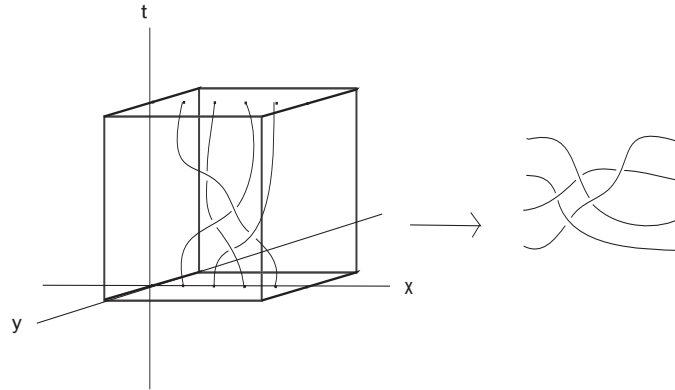


Figure 13: Braid crossings

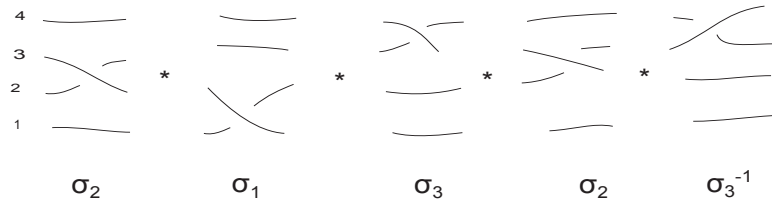
We will call the bottom strand the i th strand, and the top strand the $i + 1$ strand, where $0 \leq i \leq n - 1$. When case (a) in Figure 13 appears in a braid, we call the permutation of strands σ_i , and when case (b) in Figure 13 appears in a braid, we call the permutation of strands σ_i^{-1} . We then multiply these permutations to get the

name of the braid [2]. We will use the braid from above as an example.

First, we orientate the braid as previously described.



Then we name the strands and split the braid into a product of braids with one crossing in each braid and name each crossing accordingly.



Then multiply the braids back together, giving us the name $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1}$.

Lemma 4 *The set of braids B_n is a group under braid multiplication.*

Proof We have already shown associativity, and for the identity, we will use the trivial braid. So it remains to be shown that every n -braid has an inverse.

Recall the first of the altered Reidemeister moves given in Figure 11. Notice that it is the product of two braids, $\sigma_1\sigma_1^{-1}$. When multiplied together, they are equivalent to the trivial braid, or more specifically, the identity. Notice that this will then work for any braid. Let $\mu = \sigma_i^{\pm 1}\sigma_j^{\pm 1}\sigma_k^{\pm 1}\dots\sigma_m^{\pm 1} \in B_n$ where $1 \leq i, j, k, m \leq n - 1$. Then we

can create the braid $\nu = \sigma_m^{\mp 1} \dots \sigma_k^{\mp 1} \sigma_j^{\mp 1} \sigma_i^{\mp 1} \in B_n$ such that $\mu\nu = 1 = Identity$. Thus B_n is a group. ■

Let us use the $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1}$ from above as an example. We will create its inverse by multiplying braids to the end of it until we obtain 1, or the trivial knot. In doing so, we see that $\beta^{-1} = \sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}$ is the inverse for $\beta = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1}$. Notice that $\beta * \beta^{-1} = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1} * \sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1} = \sigma_2\sigma_1\sigma_3\sigma_2(\sigma_3^{-1} * \sigma_3)\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1} = \sigma_2\sigma_1\sigma_3(\sigma_2 * \sigma_2^{-1})\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1} = \sigma_2\sigma_1(\sigma_3 * \sigma_3^{-1})\sigma_1^{-1}\sigma_2^{-1} = \sigma(2\sigma_1 * \sigma_1^{-1})\sigma_2^{-1} = \sigma_2 * \sigma_2^{-1} = 1$, the identity. This process is shown in Figure 14.

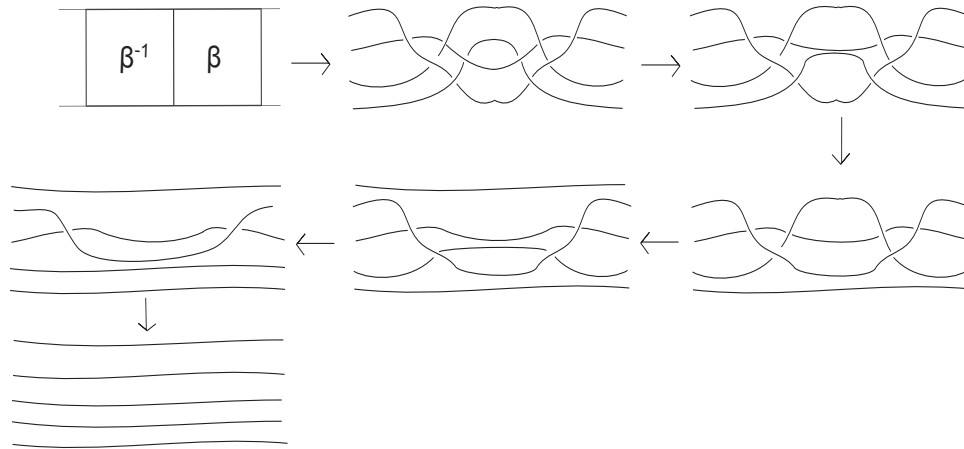


Figure 14: The multiplication of β and β^{-1}

With these tools, we are now ready to look at a method of finding equivalence between knots through the use of braids.

CHAPTER FIVE

Braids and Knot Equivalence

We will approach using braids as a method of finding equivalence between two given knots in a way similar to our approach with rational tangles by first finding a way of converting between braids and knots. We begin with the conversion from braids to knots.

We will use a similar method for closing braids as we had for closing tangles, but, we will be more exact in our choices of strands to be connected. Starting with an n -braid, orient it in the same way as was done in Chapter 3 for naming braids. Then, number the strands on both the left and right from the bottom to the top with 1 through n . Next, connect the strands on the right with the strands on the left with the same number in such a way that no additional crossings are created. This results in a knot or a link [1].

Definition 19 *The use of the above method on a braid β illustrates what is known as braid closure and the resulting braid is the closed braid, denoted $\tilde{\beta}$.*

This differs from the rational tangle closure in that we are not guaranteed a knot. We will find later that this is of little trouble to us, as we will never use braids that close into links. Figure 15 is an example of the closure of a 4-braid. With little effort, we see that this closure of the $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1}$ braid is actually the trivial knot.

We must now show how to go from a knot to a braid. We will do so by using the following method.

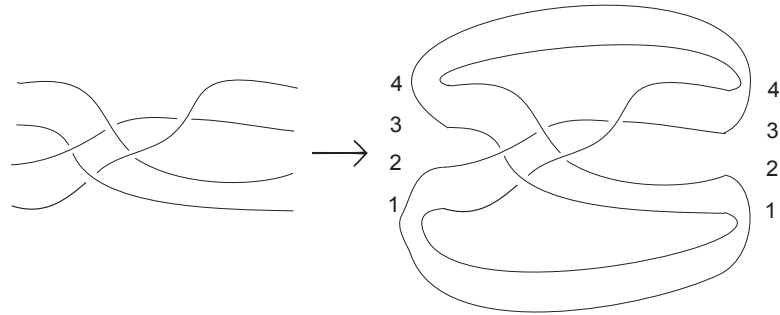


Figure 15: Braid Closure of a 4-braid

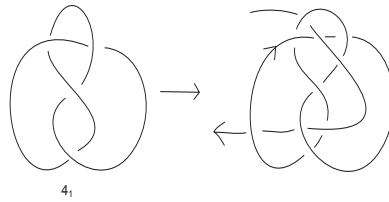
Begin by creating an axis through the knot according to the following rules:

- 1) the axis is a line that intersects every region of the knot projected onto the plane at least once,
- 2) it begins and ends in the infinite regions,
- 3) we assign the under and overpasses of the knot over and under the axis alternately by following the orientation of the knot.

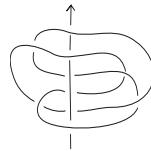
Now, perform a coordinate transformation in such a way that the axis becomes straight. At this point, we must check to make sure that our resulting figure will be orientated in such a way that it is monotonously increasing. We do so by looking for any time that a $[0]$ appears. This $[0]$ is of the same form as with the rational tangles. A figure with this characteristic will, once opened, have arcs moving in the vertical direction, against the monotone increasing orientation. In the cases where $[0]$ occurs, pull the top string to the bottom of the figure and the bottom arc to the top of the figure. This will create more crossings, but will allow the resulting figure to be a braid. Then cut the knot at the points where it is overcrosses the axis and then lay the ends out on either side. This results in a braid.

Definition 20 *The above method of converting a knot into a braid is called the Alexander Method [1].*

We will show this in an example with the 4_1 knot. Start with the knot and draw the axis through each of its regions, starting in the infinite region and create alternating over and undercrossings in the intersections of the knot and the axis.



We then straighten the axis,



cut the knot at the overcrossings with the axis,



and then pull back the cut strands



which, through some simplification, gives us a braid:



which is actually the $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$ braid.

Lemma 5 *Every knot can be made into a braid through the use of the Alexander Method.*

Proof This is true by the construction of our conversion from knot to braid. We already had a polygonally curved figure, as it was inherited from the knot, and we assured its monotone increasing nature through the eradication of the [0] occurrences. Thus resulting in a braid. ■

Corollary 3 *Let A be a knot and α be a braid resulting from the use of the Alexander Method on A . Then the braid closure of α , $\tilde{\alpha}$, is knot equivalent to A .*

Proof Up until the breaking of the arcs in the Alexander Method, we only used transformations which were equivalent to the Reidemeister moves. Notice then that the breaking of the knot as defined by the Alexander Method has the broken strands laid over the rest of the figure, at which point they are fixed. When we close the braid, we use the method described in order to reconnect the strands in the same manner they were broken, as to return to the original knot. ■

We now have methods which allow us to convert any given knot into a braid, and any given braid into a knot or link. In our attempt to come closer to being able to find equivalence between two knots through the use of their corresponding braids, we

will first look at two moves which, when applied to braids, do not effect the knot that they close into.

Definition 21 *The following two moves are called the Markov moves and they are denoted, M_1 and M_2 respectively.*

The First Markov Move: (M_1)

Let $\beta \in B_n$. Then $M_1(\beta) := g\beta g^{-1}$ and likewise, we define M_1^{-1} such that $M_1^{-1}(\beta) := g^{-1}\beta g$, where g is any n -braid [3].

The Second Markov Move: (M_2)

Let $\beta \in B_n$. Then define $M_2 : B_n \rightarrow B_{n+1}$ such that $M_2(\beta) := \beta\sigma_n^{\pm 1}$. We then define $M_2^{-1} : B_{n+1} \rightarrow B_n$ such that $M_2^{-1}(\beta\sigma_n^{\pm 1}) := \beta$ [3].

Definition 22 *Two braids, $\beta \in B_n$ and $b \in B_m$ are said to be Markov equivalent, denoted $\beta \sim_M b$ if and only if there exist a finite sequence of Markov moves that take β to b [3].*

Theorem 4 (Markov's Theorem) *Given braids $\beta \in B_n$ and $\delta \in B_m$, $\tilde{\beta} \sim \tilde{\delta}$ if and only if $\beta \sim_M \delta$ [3].*

Proof Kurpita has a comprehensive proof of this in his book *A Study of Braids*, but here we will show that $\beta \sim_M \delta$ implies $\tilde{\beta} \sim \tilde{\delta}$. To do so, we will show that the application of the Markov moves do not alter the closure of the braid. We begin with M_1 , and then M_1^{-1} follows similarly. Let $\beta \in B_n$ and α be another n -braid. Then $M_1(\beta) = \alpha\beta\alpha^{-1}$. If we take the closure of this transformed braid, we see that, through the use of the braid Reidemeister moves, the α^{-1} braid can “travel” around the closure and reconnect back on the bottom, attaching itself to the α braid as shown

in Figure 16. This results in the identity multiplied with the β braid closure, which is equivalent to the β braid closure [3]. Thus the application of the M_1 move does not change the closure of the braid and so the closure of the original braid is equivalent to the closure of the braid after the use of the M_1 move.

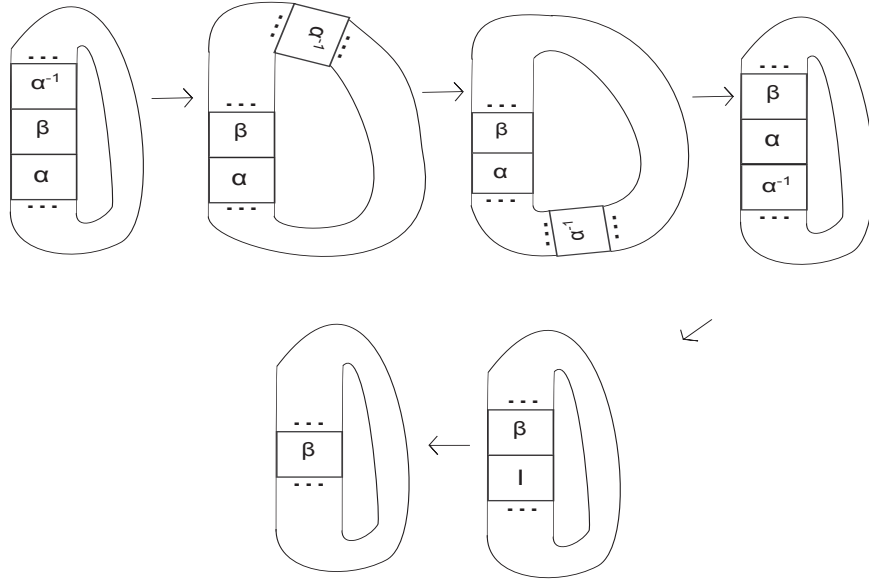


Figure 16: The traveling path of α^{-1} in the closure of M_1

We now show that the braid closure is not effected by the M_2 move. We will show the $\beta\sigma_n^{-1}$ case, from which the $\beta\sigma$ case follows. Notice, by referencing Figure 17, after the closure of the $M_2(\beta)$ braid, the last crossing created by the σ_n^{-1} is able to be unlooped and we are left with the closure of the β braid [3]. Thus the application of the M_2 move does not change the closure of the braid and so the closure of the original braid is equivalent to the closure of the braid after the use of the M_2 move.

Therefore, the closure of a braid is equivalent to the closure of the same braid with a finite number of the Markov moves applied to it.

■

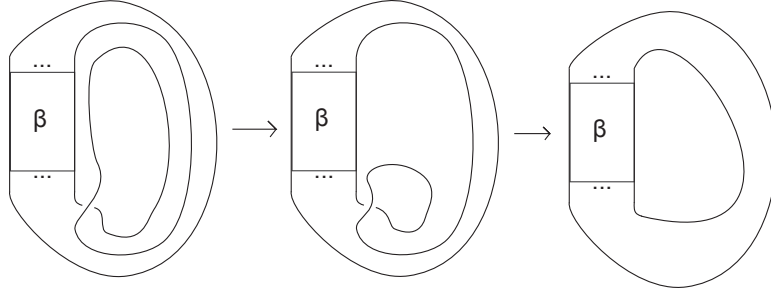


Figure 17: The the simplification of the closure of M_2

So given two knots A and B , we can convert them to their corresponding braids α and β . If, through a finite sequence of Markov moves, we can transform α into β , then their corresponding braid closures $\tilde{\alpha}$, $\tilde{\beta}$ are equivalent. Therefore, $A \sim \tilde{\alpha} \sim \tilde{\beta} \sim B$, resulting in $A \sim B$ as desired.

There is currently no algorithm to ultimately know if two braids are Markov equivalent [1]. That being said, there may be cases where two braids are Markov equivalent, but we are unable to detect it. Even with this downside, Markov's Theorem is still very powerful as it can be applied to all knots, not just those that are rational.

CHAPTER SIX

Conclusion

As stated in Chapter One, one of the main goals of the study of knots is to discover more practical methods of determining knot equivalence. In the method given through the use of rational tangles, once the rationality of a given knot has been determined, it is very simple (and quick) to put the tangle into standard form and to determine the corresponding fraction. At which point, the equivalence to a basic knot can be found instantaneously through finding the fraction in a $P(K_n)$ table.

The difficulty is actually finding the rationality of the knot. If one is given a random knot, it is difficult to tell where the original exterior arcs were and therefore may take multiple attempts to discover the knot's rationality. If a method could be discovered to ultimately determine the rationality of a knot without using the guess and check method I proposed, then this use of rational tangles to determine knot equivalence would be a very quick and practical method. Also, in the creation of the $P(K_n)$ tables, I was unable to find a comprehensive method that would find all of the fractions in the set. The method of using the external arcs outlined in Chapter Three was a method that would find a rational tangle in the knot, but may have failed to find all of the possible tangles.

We then left tangles and used a similar method using braids. Unlike with the tangles, there does exist an algorithm that will consistently and successfully produce a braid in the desired form. The issue then arises in Markov's Theorem, where a method for determining if two braids are Markov equivalent has yet to be discovered.

This makes the method of using braids to determine knot equivalence only practical when the number of Markov moves between the two braids is minimal and easily determined. With that being said, if such a method were to be discovered, this braid technique would be very practical in that it could be readily applied to any two knots and their equivalence could be quickly determined.

Regardless of the practicality of the use of the methods, these connections between knots, rational tangles, and braids are very insightful. The mere existence of conversions between the three is amazing and the fact that there are correlations that allow us to determine equivalence in the original knots is a wonderful result. It is the discovery of connections between knots and other branches of mathematics, such as braids and tangles, that will continue to aide in the discovery of a method that is both universal and practical in determining knot equivalence.

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