ABSTRACT

Moment Representations of Exceptional Orthogonal Polynomials

John M. Osborn, Ph.D.

Chairperson: Constanze Liaw, Ph.D.

Exceptional orthogonal polynomials (XOPs) can be viewed as an extension of their classical orthogonal polynomial counterparts. They exclude polynomials of a certain order(s) from being eigenfunctions for their corresponding exceptional differential operator, while surprisingly still forming a complete set of eigenpolynomials for that operator. In Chapters Two and Three, we examine the (so-called) Type I X_1-Laguerre polynomial sequence, where the constant polynomial is omitted, and the Type I X_2-Laguerre polynomial sequence, where both the constant and linear polynomials are omitted.

In the literature, one method presented for obtaining XOPs has been to apply Gram–Schmidt orthogonalization to a so-called "flag". For our purposes, a flag is an infinite set of mutually linearly independent vectors which span the space of a particular XOP. Here, we use a known flag to generate the X_1 sequence, and find a new flag to generate the X_2 sequence. The particular canonical flag we pick keeps both the determinantal representation and the moment recursion manageable. We also develop a flag that could be extended to the general X_m exceptional Laguerre polynomials.

We derive two representations for the X_1 polynomials in terms of moments by using determinants. The first representation in terms of the canonical moments is rather cumbersome. We introduce adjusted moments and find a second, more elegant formula. We deduce a recursion formula for the moments and the adjusted ones. The adjusted moments are also expressed via a generating function. These representations are then extended to the X_2 case. We explore various complicating factors that arise in moving on to XOPs of a higher co-dimension.

Finally, by employing the Darboux transform, these representations in terms of adjusted moments are then generalized to any of the X_1 exceptional orthogonal polynomials, regardless of the underlying family (Jacobi or Laguerre). We include a recursion formula for the adjusted moments, and provide the initial adjusted moments for each system. Throughout we relate to the various examples of X_1 exceptional orthogonal polynomials, but we especially focus on and provide complete proofs for the Jacobi and the Type III Laguerre cases, as they are less prevalent in literature.

Moment Representations of Exceptional Orthogonal Polynomials

by

John M. Osborn, B.S.

A Dissertation

Approved by the Department of Mathematics

Lance L. Littlejohn, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

Approved by the Dissertation Committee

Constanze Liaw, Ph.D., Chairperson

Fritz Gesztesy, Ph.D.

Johnny Henderson, Ph.D.

Lance L. Littlejohn, Ph.D.

Lorin Matthews, Ph.D.

Accepted by the Graduate School
August 2018

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I am deeply indebted to my advisor, Constanze Liaw, for her unwavering support and encouragement during the several years of working on this dissertation. I am also grateful to the remaining dissertation committee members (Lance Littlejohn, Fritz Gesztesy, Johnny Henderson, and Lorin Matthews) for their work in reviewing my manuscript. Thanks also to all of my teachers during my six years at Baylor, each of whom helped me in my ongoing journey to become a professional mathematician.

Without the assistance and comradery I received from my cohort of fellow graduate students (Tiffany Jones, Josh Padgett, and Jordan Courtemanche), I am certain I would never have seen this challenging process of graduate school through to its conclusion.

To my parents, Phillip and Barbara Osborn,

my sister and brother-in-law, Linda and John Acock,

and my close friends of many years,

Mark Lynch and Brett Bachus

CHAPTER ONE

Exceptional Orthogonal Polynomials

1.1 Characteristics of Exceptional Orthogonal Polynomials

Exceptional orthogonal polynomial (XOP) systems are a generalization of the classical orthogonal polynomial systems. The classical polynomial systems of Laguerre, Jacobi, and Hermite were shown to be the only polynomial systems of their kind in Bochner's well-known classification theorem from 1929 [9, 10]. Ultimately, a new Bochner-like classification theorem for the exceptional systems has been proven [7, 11, 12, 14, 37].

In addition to satisfying the requirements to be a classical orthogonal polynomial system, XOP systems require that there be an associated finite set, $A \subset \mathbb{N}_0$, for which the associated second-order differential equation has no polynomial solution of degree n for $n \in A$. An XOP is designated as X_m where m is the cardinality of A; m is called the co-dimension of the XOP system. Each XOP system is complete in an appropriate Hilbert space, even though it lacks polynomials of a finite number of degrees. The elements of an XOP are the eigenfunctions of an associated differential expression, and satisfy a particular eigenvalue equation.

The fact that eigenfunctions of certain degrees are excluded from an XOP is caused by particular rational-function coefficients in the differential expression. When we apply the differential expression to eigenfunctions we must (at least) cancel any denominators introduced by these rational-function coefficients. This is because each element of an XOP is polynomial, and the result of the differential expression being applied to any of these eigenfunctions is also polynomial. This idea leads to what we call the *exceptional condition*, one of the central elements in this work.

From a purely mathematical perspective, these XOP families are of interest for their relationship to classical orthogonal polynomials and their associated properties, such as their spectral analysis, the asymptotic and interlacing properties and the location of their zeros, and recursion formulas [1, 5, 11–15, 17, 20–22, 27–29, 35, 37]. Up to a linear transformation of variable, the XOP families include: Types I, II, and III Laguerre; Types I and II Jacobi; and Hermite. The spectral analysis of the exceptional Laguerre polynomial systems, and a rigorous definition of a self-adjoint operator corresponding to each, has been presented in Atia–Littlejohn–Stewart [1] and Liaw–Littlejohn–Milson–Stewart [27].

In the literature we find several different representations of XOP sequences. For example, in [9] as well as Ho–Sasaki [22] the first several polynomials of the X_1-Laguerre XOP are given. The polynomials of an XOP can be expressed in terms of the associated classical Laguerre polynomials (see e.g. [27, Equation (3.2)]). Durán's work [5] contains representations of general exceptional orthogonal polynomials using determinants of classical polynomials. Other representations of exceptional orthogonal polynomial systems involve Wronskian, and sometimes pseudo-Wronskian, determinants of classical orthogonal polynomials, see e.g. [5, 8].

The classical orthogonal polynomials are obtained from the sequence $\{1, x, x^2, \ldots\}$ by applying Gram–Schmidt orthogonalization. This perspective is used to find the classical moment representation of Chihara, see [3, Chapter I.3]. An adaptation of these ideas leads to the adjusted moment determinantal representations given in this work.

Exceptional orthogonal polynomials were originally discovered as exact solutions to certain models in quantum mechanics. Those models include the supersymmetric setting, the Fokker–Planck, as well as the Dirac equations. See [6,18,20,35,41] for the connections to physics and mathematical physics. In recent years, the field has enjoyed much further attention from both the mathematics and the physics community; see e.g. [1, 5, 9–11, 17, 22, 27, 29, 36, 39] and the references therein.

The focused study of exceptional orthogonal polynomials began less than a decade ago. It arose as the result of extending exactly solvable and quasi-exactly solvable potentials in quantum mechanics beyond the Lie-algebraic setting [6, 18, 19, 39,40]. The Laguerre and Jacobi exceptional polynomial systems of codimension one were first introduced in 2009 as an extension of Bochner's classification theorem for the classical orthogonal polynomials [10]. At that time, the approach to exceptional orthogonal polynomial systems was as state-preserving solutions to second-order Sturm-Liouville-type problems.

1.2 Previous Representations of Orthogonal Polynomials

Although the literature has several examples of determinantal and/or moment representations of orthogonal polynomials, this dissertation arose as a natural extension of results by Ted Chihara from 1978. In his classic text *An Introduction to Orthogonal Polynomials* [3], Chihara presents the following result, which is left to the reader to verify. Namely, that $\{(\Delta_{n-1})^{-1}D_n(x)\}$ is the monic OPS for $\mathcal{L}$.

To interpret that result, we need to know that

$$\Delta_n = \det(\mu_{i+j})_{i,j=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix},$$

where

$$\mu_n = \int_a^b x^n w(x)\,dx, \qquad n = 0, 1, 2, ...$$

is the standard nth moment for a given weight function $w(x)$, that

$$D_n(x) = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ . & . & \dots & . \\ . & . & \dots & . \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix},$$

that an OPS is an orthogonal polynomial sequence, and that $\mathcal{L}$ denotes a quasi-definite moment functional with moment sequence μ_n. To prove the result above, it was necessary to first show that $\{D_n(x)\}$ is a (non-monic) OPS for $\mathcal{L}$. That intermediate result is actually the jumping off point for extending this determinantal moment representation to the realm of exceptional orthogonal polynomials.

Pandres (1960) [38] gives an interesting determinantal representation for the classical orthogonal polynomial (OPS) families. He makes use of the generalized Rodrigues formula, which gives the nth member of an OPS by the formula

$$P_N = \frac{1}{w}\frac{d^N}{dx^N}(wF^N),$$

where F is some polynomial, and w is the weight function, unique to each particular OPS.

He also uses an earlier result of his which showed that the nth derivative of a function $f(x)$ can be given by

$$\frac{d^N f(x)}{dx^N} = \Delta_N f(x),$$

where Δ_N denotes the determinant of an NxN matrix. This matrix is populated mostly by 0's above the main diagonal, and by D_1, ..., D_N on and below the main diagonal, where

$$D_K = \frac{1}{(K-1)!}\frac{d^K \log f(x)}{dx^K}.$$

Using this reformulation of the nth derivative, and substituting it into the Rodrigues formula, he finds that $P_N = F^N \Delta_N$, where the elements of Δ_N are now given by

$$D_K = \frac{1}{(K-1)!} \frac{d^K}{dx^K}[\log w + N \log F] \ .$$

Finally, using matrix properties and induction to pull the factor F^N inside the matrix, he achieves his main result that the nth member of an OPS has the determinantal representation $P_N = \Delta_N$, where the elements of Δ_N are now defined by

$$D_K = \frac{F^K}{(K-1)!} \frac{d^K}{dx^K}[\log w + N \log F] \ .$$

Gómez-Ullate, et al, [8] used Wronskian determinants of sequences of classical Hermite orthogonal polynomials to represent exceptional Hermite polynomials. Wronskians are the determinants of n x n matrices, where the first row is composed of a sequence of n functions, and each succeeding row contains progressively higher-order derivatives of those functions, starting with the first derivative on the 2nd row. The final row is composed of the $(n-1)$st derivatives of those n functions.

In their paper, the authors explain that if the Wronskian determinants of a sequence of classical Hermite polynomials are to form an orthogonal polynomial system with positive definite weight, then that sequence must be chosen using certain rules. They use variations on partitions of integer sequences to guide their selection of the sequence of Hermite polynomials which are the inputs to the Wronskian.

Durán's work [5] bears a close similarity to that of Gómez-Ullate, et al. He starts with Casorati determinants of Meixner polynomials. Casorati determinants are determinants of n x n matrices, where the first row (column) is composed of a sequence of n functions all evaluated at x, and each succeeding row (column) is that same set of functions evaluated at an argument that is progressively augmented by one. The last row (column) is that same set of functions, all evaluated at $(x+n-1)$. After passing to the limit as $a \to 1$, where a is a parameter appearing in the Casorati determinants, they become Wronskian determinants of classical Laguerre polynomials. Under certain specific conditions, these Wronskian determinants are shown to be the exceptional Laguerre polynomials.

While all of the results above are determinantal representations of orthogonal polynomials, Ismail and Stanton [23] develop moment representations for q-orthogonal polynomials. These polynomials are called q-orthogonal because they are given by q-integrals of the form $p_n(x) = \int_a^b y^n \, d\mu(y)$, so called because μ is a discrete measure whose masses are located at points of the form aq^n or bq^n. Notice that the integrals above are all moments of the independent variable y.

1.3 Attribution of Work

The majority of this dissertation has appeared in two peer-reviewed published papers, as well as one paper which is currently in submission. The attribution of the work appearing therein to the various authors is given below.

The contents of Chapter Two come primarily from a published paper titled "Moment representations of the exceptional X_1-Laguerre orthogonal polynomials" [32] which was jointly co-authored by my dissertation supervisor, Dr. Constanze Liaw, and me. Working on this paper constituted my introduction to the field of exceptional orthogonal polynomials. Thus, most of its results were initially developed by Dr. Liaw, then given to me to independently verify after being taught the general methods to use.

Chapter Three includes the contents of a paper titled "Moment representations of type I X_2 exceptional Laguerre polynomials" [30] which was jointly co-authored by me, Dr. Constanze Liaw, and Dr. Jessica Stewart-Kelly. This paper is currently in submission. My primary creative contributions to this dissertation occurred in the development of this paper. All of the major results of this paper were primarily my own results, which I was capable of producing after the close collaboration on the previous paper. This includes the creation of the doubly-subscripted adjusted moments, the choice of the flag elements, the exposition of the exceptional conditions for the co-dimension 2 case (as well as the general co-dimension m case) after receiving initial guidance from Dr. Liaw, the particular ansatz for representing the elements of the type I X_2 Laguerre XOP, the determination of the elements of the M matrix for the key linear system, the development of the suite of recursion-like formulas, and the calculation of the initial moments to start the recursion process. My co-authors' primary contributions on this paper consisted of taking primary responsibility for typing up various sections of this paper.

Finally, Chapter Four comes from a published paper titled "Moment representations of exceptional X_1 orthogonal polynomials" [31] which was jointly co-authored by Dr. Constanze Liaw, Dr. Jessica Stewart-Kelly, and me. Drs. Liaw and Stewart-Kelly produced all of the major results of this paper that were related to the use of the Darboux transform. My primary contributions to this paper were in the calculation of the initial moments for the exceptional Laguerre and Jacobi XOPs that were required to initiate the recursion formulas for the adjusted moments.

CHAPTER TWO

Representations of Type I Laguerre Exceptional Orthogonal Polynomials

2.1 Background Information

In this chapter, which comes from the author's work in [32], we examine moment representations for the Type I X_1-Laguerre exceptional orthogonal polynomials. We focus on these polynomials from the perspective of the seminal paper by Goméz-Ullate–Kamran–Milson [9]. This polynomial sequence, denoted by $\{L_n^\alpha(x)\}_{n\in\mathbb{N}}$, $\alpha > 0$, is orthogonal on $[0,\infty)$ with respect to the X_1-Laguerre weight

$$W^\alpha(x) = \frac{x^\alpha e^{-x}}{(x+\alpha)^2}.$$

The polynomials are complete in $L^2([0,\infty); W^\alpha)$ even though there is no degree 0 polynomial. Further, they are the eigenfunctions of the *exceptional* X_1*-Laguerre differential expression*

$$\ell^\alpha[y] = -xy'' + \left(\frac{x-\alpha}{x+\alpha}\right)[(x+\alpha+1)y' - y]\,. \tag{2.1.1}$$

The spectral analysis of this polynomial system and a rigorous definition of a self-adjoint operator corresponding to ℓ^α was presented in Atia–Littlejohn–Stewart [1]. The functions $y(x) = L_n^\alpha(x)$ satisfy the eigenvalue equation

$$\ell^\alpha[y] = (n-1)y \quad (0 < x < \infty).$$

Probably the most comprehensive study of the three types of exceptional Laguerre polynomials is contained in Liaw–Littlejohn–Milson–Stewart [27]. We refer the reader to Durán [5] for an interesting relation with so-called exceptional Meixner polynomials.

Literature reveals several representations of the X_1-Laguerre polynomial sequence:

For example, in [9] as well as Ho–Sasaki [22] the first couple of polynomials are listed

$$L_1^\alpha(x) = x + \alpha + 1, \tag{2.1.2}$$

$$L_2^\alpha(x) = x^2 - \alpha(\alpha+2), \tag{2.1.3}$$

$$L_3^\alpha(x) = \frac{1}{2}\left[-x^3 + (\alpha+3)x^2 + \alpha(\alpha+3)x - \alpha(\alpha^2+4\alpha+3)\right], \tag{2.1.4}$$

$$\vdots$$

Further, these polynomials can be expressed in terms of the classical Laguerre polynomials $\{p_n\}_{n\in\mathbb{N}_0}$ (here $p_{-1} \equiv 0$), see e.g. [27, Equation (3.2)]:

$$L_n^\alpha(x) = -(x+\alpha+1)p_{n-1}^{\alpha-1}(x) + (x+\alpha)p_{n-2}^\alpha(x) \qquad (n \in \mathbb{N}).$$

Durán's work [5] contains representations of general exceptional orthogonal polynomials using determinants of classical polynomials.

In [9] a three-term recurrence for the exceptional X_1-Laguerre polynomials was found

$$\begin{aligned} 0 = \quad & (n+1)[(x+\alpha)^2(n+\alpha) - \alpha]L_{n+2}^\alpha(x) \\ & + (n+\alpha)[n+\alpha)^2(x-2n-\alpha-1) + 2\alpha]L_{n+1}^\alpha(x) \\ & + (n+\alpha-1)[(x+\alpha)^2(n+\alpha+1) - \alpha]L_n^\alpha(x). \end{aligned}$$

We are most interested in the way the X_1-Laguerre polynomial sequence was introduced in the work of Goméz-Ullate–Kamran–Milson [15]. Namely, the sequence of polynomials $\{v_i(x)\}_{i=1}^\infty$ where

$$v_1(x) = x + \alpha + 1 \quad \text{and} \quad v_i(x) = (x+\alpha)^i \text{ for } i \geq 2, \tag{2.1.5}$$

spans the exceptional X_1-Laguerre polynomial flag. Via the Gram–Schmidt process, this sequence produces the sequence $\{L_n^\alpha\}_{n=1}^\infty$.

We note that the classical orthogonal polynomials are obtained from the sequence $\{1, x, x^2, \ldots\}$ also by applying Gram–Schmidt. This perspective is used to find the classical moment representation, see e.g. [3, Chapter I.3]. An adaptation of these ideas leads us to deduce our representations.

The proof of our recursion formula for the moments relies on an application of so-called symmetry factors for differential equations (see e.g. Cole [4, p. 66]) which was further developed from second to higher order differential equations by Littlejohn, see e.g. [33, 34]. In essence, every second order differential equation is symmetrizable, and the corresponding symmetry equation is solved by the weight (of orthogonality). In Krall's [25] well-known classification theorem (see e.g. Krall–Littlejohn [26, Theorem 1(ii)]) this theory was used to derive a moment equation for the classical moments. Here we carry out those techniques to obtain a recursive definition for the *(adjusted) moments*

$$\widetilde{\mu}_n = \widetilde{\mu}_n^\alpha := \int_0^\infty (x+\alpha)^n W^\alpha(x)dx.$$

A formula for the *(exceptional) moments*

$$\mu_n = \mu_n^\alpha := \int_0^\infty x^n W^\alpha(x)dx$$

is obtained.

Remark. We warn the reader that we often drop the subscript or superscript α to simplify notation.

2.1.1 Plan of the paper

In Section 2 we present a preliminary result: We characterize the subspace spanned by the first m exceptional X_1-Laguerre polynomials (Lemma 2.2.1), which yields what we call the exceptional condition satisfied by all exceptional X_1-Laguerre polynomials (Lemma 2.2.2).

A first representation (Theorem 2.3.1) of the exceptional X_1-Laguerre polynomials in terms of the exceptional moments is found in Section 3. The expression is rather cumbersome.

In Section 4 we focus on recursion formulae for the adjusted (Theorem 2.4.1) and the exceptional moments (Theorem 2.4.4). The adjusted moments are also expressed explicitly in two different ways (Corollary 2.4.2 and Theorem 2.4.3). One of the proofs involves a generating function. Throughout Section 4 we notice that the first two moments are different in nature from the others.

A more elegant representation (Theorem 2.5.1) of the exceptional X_1-Laguerre polynomials in terms of the adjusted moments is the topic of Section 5. In Remark 2.5.2 we determine the normalization constant so as to yield precisely those polynomials in the literature [9] and [27]. At the very end, we verify the representation formula for $n = 1$ and $n = 2$.

2.2 The Exceptional Condition

We characterize the span of the first m exceptional X_1-Laguerre polynomials as those polynomials of degree less than or equal to m for which the *exceptional condition*

$$p'(-\alpha) - p(-\alpha) = 0 \tag{2.2.1}$$

holds. This fact is well-known to specialists, but it is often written somewhat differently. We include the proof for the convenience of the reader. We mention aside that it is not hard to see that the polynomials given by (2.1.2) through (2.1.4), of course, satisfy the exceptional condition.

In order to formulate precisely and prove the above characterization result, we let $\mathcal{P}_m$ denote the set of polynomials of $\deg p \leq m$ and define the span of the first m exceptional X_1-Laguerre polynomials

$$\mathcal{L}_m := \operatorname{span}\{L_n^\alpha : n = 1, \ldots, m\}, \quad \text{and}$$

$$\mathcal{M}_m := \{p \in \mathcal{P}_m : p \text{ satisfies } (2.2.1)\}.$$

Lemma 2.2.1. *The sets $\mathcal{L}_m = \mathcal{M}_m$ for all $m \in \mathbb{N}$.*

Proof. To show $\mathcal{L}_m \subseteq \mathcal{M}_m$ take $p \in \mathcal{L}_m$. Then $\ell^\alpha[p] \in \mathcal{L}_m$, because $\mathcal{L}_m$ is a subspace of $L^2(\mu^\alpha)$ which is invariant under ℓ^α. In particular $\ell^\alpha[p](x)$ is a polynomial in x. In virtue of the definition of ℓ^α, some cancellation has to take place in the exceptional term

$$\left(\frac{x-\alpha}{x+\alpha}\right)\left[(x+\alpha+1)p'-p\right].$$

So the term in square brackets must equal zero when evaluated at $x=-\alpha$. This yields precisely the exceptional condition (2.2.1). And so we have $\mathcal{L}_m \subseteq \mathcal{M}_m$.

The opposite containment follows from a dimension argument. Namely, we have $\dim \mathcal{P}_m = m+1$. This implies $\dim \mathcal{M}_m = m$, due to the imposition of the one restriction (2.2.1). We also have $\dim \mathcal{L}_m = m$, because the set is spanned by m polynomials with mutually different orders (implying their linear independence). Finally, we recall that $\mathcal{L}_m$ and $\mathcal{M}_m$ are both subspaces of $\mathcal{P}_m$, and $\mathcal{L}_m \subseteq \mathcal{M}_m$. □

The exceptional condition (2.2.1) yields a condition on the coefficients of the polynomials L_n^α. To see this, we write

$$L_n^\alpha(x)=\sum_{k=0}^{n} c_{nk}x^k$$

and compute

$$\left(L_n^\alpha\right)'(x)=\sum_{k=1}^{n} k c_{nk}x^{k-1}.$$

Since $L_n^\alpha \in \mathcal{L}_n$, condition (2.2.1) applies and we have $0=\left(L_n^\alpha\right)'(-\alpha)-L_n^\alpha(-\alpha)$. We conclude:

Lemma 2.2.2. *The coefficients of the X_1-exceptional Laguerre polynomials $L_n^\alpha(x)=\sum_{k=0}^{n} c_{nk}x^k$ obey*

$$-c_{n0}+\sum_{k=1}^{n} c_{nk}\left[k(-\alpha)^{k-1}-(-\alpha)^k\right]=0. \tag{2.2.2}$$

2.3 First Representation of the XOP

Fix $n\in\mathbb{N}$. Recall that $L_n^\alpha=\sum_{k=0}^{n} c_{nk}x^k$. We determine c_{nk} for $k=0,1,\ldots,n$ by means of a linear system of $n+1$ equations $Ac=b$, where

$$c := \begin{bmatrix} c_{n0} \\ c_{n1} \\ \vdots \\ c_{nn} \end{bmatrix} \in \mathbb{R}^{n+1} \qquad \text{and} \qquad b := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ K_n \end{bmatrix} \in \mathbb{R}^{n+1}$$

with normalizing constant K_n and A is given below. In this section, we do not fix K_n, but merely assume $K_n \neq 0$. In Remark 2.5.2 below we determine the normalization constant such that we obtain exactly the polynomials described in [9] and [27].

Theorem 2.3.1. *The exceptional X_1-Laguerre polynomials admit the representation*

$$L_n^\alpha(x) = \frac{1}{\det\ A} \sum_{k=0}^{n} (\det\ A_k) x^k = \frac{K_n}{\det\ A} \cdot \begin{vmatrix} (\textit{First } n \textit{ rows of } A) \\ 1 \quad x \quad x^2 \quad \ldots \quad x^n \end{vmatrix} \qquad (n \in \mathbb{N}),$$

where

$$A = \begin{bmatrix} -1 & 1(-\alpha)^0 - (-\alpha)^1 & \ldots & n(-\alpha)^{n-1} - (-\alpha)^n \\ \mu_1 + (\alpha+1)\mu_0 & \mu_2 + (\alpha+1)\mu_1 & \ldots & \mu_{n+1} + (\alpha+1)\mu_n \\ \sum_{m=0}^{2} \binom{2}{m} \mu_m\ \alpha^{2-m} & \sum_{m=0}^{2} \binom{2}{m} \mu_{m+1}\ \alpha^{2-m} & \ldots & \sum_{m=0}^{2} \binom{2}{m} \mu_{m+n}\ \alpha^{2-m} \\ \vdots & \vdots & & \vdots \\ \sum_{m=0}^{n} \binom{n}{m} \mu_m\ \alpha^{n-m} & \sum_{m=0}^{n} \binom{n}{m} \mu_{m+1}\ \alpha^{n-m} & \ldots & \sum_{m=0}^{n} \binom{n}{m} \mu_{m+n}\ \alpha^{n-m} \end{bmatrix},$$

the exceptional moments are given by $\mu_k = \mu_k^\alpha = \int_0^\infty x^k W^\alpha(x)dx$, and where the matrix A_k is obtained from A by replacing the $(k+1)$-st column with the vector b, as is done in Cramer's rule. (In Section 4 below we find a recursion formula for the moments μ_k.)

Remark. There is, indeed, no polynomial of order zero.

Remark. The matrix A is invertible, since the exceptional X_1-Laguerre polynomials are determined uniquely by exactly those conditions. Indeed, Lemma 2.2.1 ensures that the polynomial $\sum_{k=0}^{n} c_{nk}x^k$ belongs to the vector space $\mathcal{L}_n$, and the other conditions given by the rows of the linear system simply require that it is orthogonal to the subspace $\mathcal{L}_{n-1}$. Since $\dim(\mathcal{L}_n \setminus \mathcal{L}_{n-1}) = 1$, the polynomial L_n^α is uniquely (up to choosing the normalizing constant $K_n \neq 0$) defined by these conditions.

Proof. One condition on the coefficients c_{nk} was given by (2.2.2), which we rewrite in terms of the dot product

$$\begin{bmatrix} -1 & 1(-\alpha)^0 - (-\alpha)^1 & 2(-\alpha)^1 - (-\alpha)^2 & \ldots & n(-\alpha)^{n-1} - (-\alpha)^n \end{bmatrix} \cdot c = 0.$$

We use the row vector in square brackets as the first row of the matrix A.

The other n rows are obtained from orthogonality conditions: Recall that the polynomials L_n^α are obtained from the sequence v_i given by (2.1.5) via the Gram–Schmidt process. In particular, we have

$$\langle L_n^\alpha \, , \, v_k \rangle_{W^\alpha} = K_n \delta_{nk} \quad \text{for} \quad k = 1, \ldots, n,$$

where δ_{nk} is the Kronecker delta. We now find explicit expressions for these conditions for the different values of $k = 1, \ldots, n$.

For $k = 1$, we have

$$\begin{aligned} K_n \delta_{n1} &= \langle L_n^\alpha \, , \, v_1 \rangle_{W^\alpha} \\ &= \left\langle \sum_{k=0}^{n} c_{nk} x^k \, , \, x + \alpha + 1 \right\rangle_{W^\alpha} \\ &= \int_0^\infty \left(\sum_{k=0}^{n} c_{nk} x^{k+1} + (\alpha+1) \sum_{k=0}^{n} c_{nk} x^k \right) W^\alpha(x)\, dx \\ &= \sum_{k=0}^{n} c_{nk} (\mu_{k+1} + (\alpha+1)\mu_k). \end{aligned}$$

The second row of the matrix A is determined by the first factor of the dot product

$$\begin{bmatrix} \mu_1 + (\alpha+1)\mu_0 & \mu_2 + (\alpha+1)\mu_1 & \ldots & \mu_{n+1} + (\alpha+1)\mu_n \end{bmatrix} \cdot c = K_n \delta_{n1}.$$

When $n \geq 2$ we consider $2 \leq s \leq n$ and use the binomial theorem,

$$v_s = (x+\alpha)^s = \sum_{m=0}^{s} \binom{s}{m} x^m \, \alpha^{s-m}.$$

We compute

$$\begin{aligned}
K_n \delta_{ns} &= \langle L_n^\alpha \, , \, v_s \rangle_{W^\alpha} \\
&= \int_0^\infty \left(\sum_{k=0}^{n} c_{nk} \, x^k \right) \left(\sum_{m=0}^{s} \binom{s}{m} x^m \, \alpha^{s-m} \right) W^\alpha(x) \, dx \\
&= \int_0^\infty \sum_{k=0}^{n} c_{nk} \left(\sum_{m=0}^{s} \binom{s}{m} x^{m+k} \, \alpha^{s-m} \right) W^\alpha(x) \, dx \\
&= \sum_{k=0}^{n} c_{nk} \left(\sum_{m=0}^{s} \binom{s}{m} \int_0^\infty x^{m+k} \, \alpha^{s-m} \, W^\alpha(x) \, dx \right) \\
&= \sum_{k=0}^{n} c_{nk} \left(\sum_{m=0}^{s} \binom{s}{m} \mu_{m+k} \, \alpha^{s-m} \right).
\end{aligned}$$

Rewriting the summation as a dot product as before, we find that the $(s+1)$-st row of the matrix A is determined by

$$\left[\; \sum_{m=0}^{s} \binom{s}{m} \mu_m \, \alpha^{s-m} \quad \sum_{m=0}^{s} \binom{s}{m} \mu_{m+1} \, \alpha^{s-m} \quad \ldots \quad \sum_{m=0}^{s} \binom{s}{m} \mu_{m+n} \, \alpha^{s-m} \; \right] \cdot c = K_n \delta_{ns}.$$

We solve the linear system $Ac = b$ using Cramer's rule

$$c_{nk} = (\det A_k)/(\det A) \qquad (\text{for } k = 0, 1, \ldots, n).$$

It remains to build the polynomial from those coefficients c_{nk}, $k = 0, 1, \ldots, n$. With the definition of the vector b we easily obtain

$$\begin{vmatrix} \multicolumn{7}{c}{(\text{First } n \text{ rows of } A)} \\ 0 & \ldots & 0 & x^k & 0 & \ldots & 0 \end{vmatrix} = \frac{(\det \, A_k)}{K_n} \, x^k.$$

In the above determinant the entry x^k is in the $(k+1)$-st column of the matrix.

So the desired formula

$$\frac{1}{\det \, A} \sum_{k=0}^{n} (\det \, A_k) x^k = \frac{K_n}{\det \, A} \cdot \begin{vmatrix} \multicolumn{5}{c}{(\text{First } n \text{ rows of } A)} \\ 1 & x & x^2 & \ldots & x^n \end{vmatrix}$$

follows from expansion of the matrix by minors along the last row. □

2.4 Moment Formulas

Because our representation for $L_n^\alpha(x)$ effectively depends on the moments μ_k, we now develop a recursive formula for the moments. We do this in an indirect fashion. Namely, rather than working with the moments μ_k, we initially develop a recursion formula for the adjusted moments $\widetilde{\mu}_k$. We then apply the binomial formula to derive the expression for μ_k. We choose this indirect approach because it became evident during the course of our investigation that the moment calculations for the adjusted moments are more concise. (It is possible to do the computations for the exceptional moments μ_k directly, but such a direct computation turns out to be rather complicated.)

2.4.1 Adjusted Moments

Theorem 2.4.1. *The adjusted moments* $\widetilde{\mu}_k = \int_0^\infty (x+\alpha)^k \, W^\alpha(x)\, dx$

(a) satisfy the recursion formula

$$\widetilde{\mu}_{k+2} = (2\alpha + k)\widetilde{\mu}_{k+1} + \alpha(1-k)\widetilde{\mu}_k \qquad (k \in \mathbb{N}_0),$$

(b) and we can start the recursion with

$$\widetilde{\mu}_1^\alpha = e^\alpha \alpha^\alpha \Gamma(1+\alpha)\Gamma(-\alpha,\alpha), \text{ and} \tag{2.4.1}$$

$$\widetilde{\mu}_0^\alpha = \Gamma(\alpha) - 2e^\alpha \alpha^\alpha \Gamma(\alpha+1)\Gamma(-\alpha,\alpha),$$

where the Gamma function is given by $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$ *and the upper incomplete Gamma function by* $\Gamma(a,x) := \int_x^\infty t^{a-1} e^{-t} dt$ *for* $x > 0$.

We simplify notation by writing W for $W^\alpha(x)$.

Proof. We begin the proof of part (a) by observing two facts.

First, for functions f, g smooth on $[0,\infty)$ the moment functionals satisfy:

$$\langle W', f\rangle = -\langle W, f'\rangle \quad \text{and} \quad \langle gW, f\rangle = \langle W, fg\rangle$$

(here $\langle \cdot\,, \cdot \rangle$ denotes the inner product with respect to Lebesgue measure on $[0,\infty)$).

Second, we learn from [33] that for a linear operator of the form

$$\ell[y] = a_2 y'' + a_1 y' + a_0 y,$$

the related symmetry equation is given by

$$a_2 y' + (a_2' - a_1)y = 0,$$

and that it is solved by the weight function (with respect to which the eigenpolynomials are orthogonal).

That is

$$a_2 W' + (a_2' - a_1)W = 0.$$

And together with

$$\begin{aligned}\left\langle a_2 W',\ (x+\alpha)^k \right\rangle &= \left\langle W',\ a_2(x+\alpha)^k \right\rangle = -\left\langle W,\ (a_2(x+\alpha)^k)' \right\rangle \\ &= -k\left\langle W,\ a_2(x+\alpha)^{k-1} \right\rangle - \left\langle W,\ a_2'(x+\alpha)^k \right\rangle\end{aligned}$$

we find for $k \in \mathbb{N}$:

$$\begin{aligned}0 &= \left\langle a_2 W' + (a_2' - a_1)W,\ (x+\alpha)^k \right\rangle \\ &= \left\langle a_2 W',\ (x+\alpha)^k \right\rangle + \left\langle a_2' W,\ (x+\alpha)^k \right\rangle - \left\langle a_1 W,\ (x+\alpha)^k \right\rangle \\ &= -k\left\langle W,\ a_2(x+\alpha)^{k-1} \right\rangle - \left\langle a_1 W,\ (x+\alpha)^k \right\rangle. \end{aligned} \tag{2.4.2}$$

In our case, with the exceptional X_1-Laguerre expression (2.1.1) the coefficients are

$$a_2 = -x \quad \text{and} \quad a_1 = x - \alpha - 1 + \frac{2x}{x+\alpha},$$

or equivalently,

$$a_2 = -(x+\alpha) + \alpha \quad \text{and} \quad a_1 = (x+\alpha) - 2\alpha + 1 - \frac{2\alpha}{x+\alpha}.$$

We substitute these into (2.4.2), and collect the coefficients of terms of the form $\widetilde{\mu}_k = \left\langle W, (x+\alpha)^k \right\rangle$ for $k = k-1, k, k+1$ to obtain:

$$\begin{aligned}0 &= -k\left\langle W,\ (-(x+\alpha)+\alpha)(x+\alpha)^{k-1} \right\rangle \\ &\quad - \left\langle \left((x+\alpha) - 2\alpha + 1 - \frac{2\alpha}{x+\alpha}\right)W,\ (x+\alpha)^k \right\rangle \\ &= [-k\alpha + 2\alpha]\widetilde{\mu}_{k-1} + [k+2\alpha]\widetilde{\mu}_k - \widetilde{\mu}_{k+1}.\end{aligned}$$

We solve for $\widetilde{\mu}_{k+1}$

$$\widetilde{\mu}_{k+1} = (2\alpha + k - 1)\widetilde{\mu}_k + \alpha(2-k)\widetilde{\mu}_{k-1}.$$

Finally, shifting the index up by one, we see part (a) of the theorem.

We proceed to prove part (b). We recall the definition of $\Gamma(x)$ and $\Gamma(a,x)$, and let

$$E_a(x) = \int_1^\infty e^{-xt} t^{-a} dt, \quad x > 0$$

denote the exponential integral function. The two classes of functions are related by

$$E_a(x) = x^{a-1}\Gamma(1-a,x).$$

We also have the identity

$$(a-1)E_a(x) = e^{-x} - xE_{a-1}(x). \tag{2.4.3}$$

Our first claim is

$$\int_0^\infty \frac{e^{-x}x^\beta}{(x+\alpha)} dx = e^\alpha\, E_{1+\beta}(\alpha)\, \Gamma(1+\beta), \quad \alpha > 0, \beta > -1. \tag{2.4.4}$$

The claim is established by the following chain of manipulations:

$$\begin{aligned}
\int_0^\infty \frac{e^{-x}x^\beta}{(x+\alpha)} dx &= \int_1^\infty e^{\alpha(1-t)}(\alpha(t-1))^\beta t^{-1} dt \\
&= \int_\alpha^\infty e^{\alpha-s}(s-\alpha)^\beta s^{-1} ds \\
&= e^\alpha \int_\alpha^\infty \int_1^\infty e^{-st}(s-\alpha)^\beta dt ds \\
&= e^\alpha \int_1^\infty \int_0^\infty e^{-t\alpha-u}(u/t)^\beta t^{-1} du dt \\
&= e^\alpha \int_1^\infty e^{-t\alpha} t^{-\beta-1} dt \int_0^\infty e^{-u} u^\beta du.
\end{aligned}$$

As an immediate consequence, we obtain the following expressions for the first adjusted moment:

$$\begin{aligned}
\widetilde{\mu}_1 &= e^\alpha E_{1+\alpha}(\alpha)\Gamma(1+\alpha) \\
&= \Gamma(\alpha)(1-\alpha e^\alpha E_\alpha(\alpha)) \\
&= e^\alpha \alpha^\alpha \Gamma(1+\alpha)\Gamma(-\alpha,\alpha).
\end{aligned}$$

Further, notice that

$$\widetilde{\mu}_2 = \int_0^\infty (x+\alpha)^2 \frac{x^\alpha e^{-x}}{(x+\alpha)^2} dx = \int_0^\infty x^\alpha e^{-x} dx = \Gamma(\alpha+1). \tag{2.4.5}$$

With part (a) for $k=0$, we obtain $\Gamma(\alpha+1) = 2\alpha\widetilde{\mu}_1 + \alpha\widetilde{\mu}_0$ or equivalently

$$\widetilde{\mu}_0 = \Gamma(\alpha+1)/\alpha - 2\widetilde{\mu}_1 = \Gamma(\alpha) - 2e^{\alpha}\alpha^{\alpha}\Gamma(\alpha+1)\Gamma(-\alpha,\alpha).$$

The theorem is proved. □

Equation (2.4.5) is of interest by itself.

Remark. R. Milson contributed the evaluation of the moment $\widetilde{\mu}_1^{\alpha}$.

In the remainder of this subsection, we express the $(k+2)$-nd adjusted moment $\widetilde{\mu}_{k+2}$ "directly" in two ways. Define the matrix

$$B_n := \begin{bmatrix} 2\alpha+n & \alpha(1-n) \\ 1 & 0 \end{bmatrix}.$$

Corollary 2.4.2. *For $k \geq 2$*

$$\begin{bmatrix} \widetilde{\mu}_{k+2} \\ \widetilde{\mu}_{k+1} \end{bmatrix} = \Gamma(\alpha+1)\left(\prod_{n=2}^{k} B_n\right)\begin{bmatrix} 2\alpha+1 \\ 1 \end{bmatrix}.$$

Further, we have $\widetilde{\mu}_2 = \Gamma(\alpha+1)$ and $\widetilde{\mu}_3 = (2\alpha+1)\Gamma(\alpha+1)$.

(The moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ were given explicitly in Theorem 2.4.1.)

Remark. The first two moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ occur somewhat disconnected from the other moments $\widetilde{\mu}_k$ for $k \geq 2$, in that the forward recursion can be started from $k=2$, and the formulas for $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ contain the incomplete gamma function Γ. This observation also reflects the fact that the exceptional X_1-Laguerre polynomials do not have the degree zero polynomial.

Proof. By the recursion formula in Theorem 2.4.1 we can set up a discrete dynamical system to express

$$\begin{bmatrix} \widetilde{\mu}_{k+2} \\ \widetilde{\mu}_{k+1} \end{bmatrix} = B_k \begin{bmatrix} \widetilde{\mu}_{k+1} \\ \widetilde{\mu}_{k} \end{bmatrix}.$$

Repeated application yields

$$\begin{bmatrix} \widetilde{\mu}_{k+2} \\ \widetilde{\mu}_{k+1} \end{bmatrix} = \left(\prod_{n=2}^{k} B_n \right) \begin{bmatrix} \widetilde{\mu}_3 \\ \widetilde{\mu}_2 \end{bmatrix}. \tag{2.4.6}$$

We recall that $\widetilde{\mu}_2 = \Gamma(\alpha + 1)$ by equation (2.4.5). Further with the moment recursion formula in Theorem 2.4.1 for $k = 1$ we have

$$\widetilde{\mu}_3 = (2\alpha + 1)\widetilde{\mu}_2 = (2\alpha + 1)\Gamma(\alpha + 1). \tag{2.4.7}$$

Substitution into the vector on the right hand side of (2.4.6) yields the corollary. □

We use the standard technique of generating functions (see e.g. [24]) to find an explicit expression for the moments.

Theorem 2.4.3. *The adjusted moments* $\widetilde{\mu}_k = \int_0^\infty (x + \alpha)^k W^\alpha(x)\, dx$ *are given in hypergeometric notation by*

$$\widetilde{\mu}^\alpha_{k+2} = (-1)^k \Gamma(\alpha + 1)(-\alpha - k)_k \; {}_1F_1(-k, -\alpha - k; \alpha) \qquad (k \in \mathbb{N}_0),$$

where we use the Pochhammer symbol

$$(x)_n := \begin{cases} 1 & \text{for } n = 0 \\ x(x + 1) \cdot \ldots \cdot (x + n - 1) & \text{for } n > 0. \end{cases}$$

(Again, the moments $\widetilde{\mu}^\alpha_0$ *and* $\widetilde{\mu}^\alpha_1$ *cannot be obtained in this fashion, but their values are given in Theorem 2.4.1.)*

We check that indeed $\widetilde{\mu}^\alpha_2 = \Gamma(\alpha + 1)$ and $\widetilde{\mu}^\alpha_3 = (2\alpha + 1)\Gamma(\alpha + 1)$. The hypergeometric lay may consult equation (2.4.16) below to find an alternative expression for the adjusted moments without hypergeometric notation.

Proof. We begin by re-writing the recurrence relation from Theorem 2.4.1:

$$\widetilde{\mu}_{k+2} = (2\alpha + k)\widetilde{\mu}_{k+1} + \alpha(1 - k)\widetilde{\mu}_k \qquad (k \in \mathbb{N}_0) \tag{2.4.8}$$

to obtain

$$(k+1)\nu_{k+1} = (2\alpha + k + 1)\nu_k - \alpha\nu_{k-1} \qquad (k \in \mathbb{N}), \tag{2.4.9}$$

where

$$\widetilde{\mu}_{k+2} = k!\nu_k \qquad (k \in \mathbb{N}_0). \tag{2.4.10}$$

To see this we first replace $\tau_k := \widetilde{\mu}_{k+2}$ in (2.4.8) to see

$$\tau_k = (2\alpha + k)\tau_{k-1} + \alpha(1-k)\tau_{k-2} \qquad (k = 2, 3, 4, \ldots).$$

Shifting the index $k \mapsto k+1$ yields

$$\tau_{k+1} = (2\alpha + k + 1)\tau_k - \alpha k \tau_{k-1} \qquad (k \in \mathbb{N}).$$

We define the sequence $\{\nu_k\}_{k\in\mathbb{N}}$ by $\tau_k = k!\nu_k$ and so

$$(k+1)!\nu_{k+1} = (2\alpha + k + 1)k!\nu_k - \alpha k!\nu_{k-1} \qquad (k \in \mathbb{N}).$$

We divide the equation by $k!$ to arrive at the desired relation (2.4.9).

Our next goal is to write a related first order differential equation. In order to achieve this, we multiply relation (2.4.9) by the factor t^k and sum up for $k \in \mathbb{N}$:

$$\sum_{k=1}^{\infty}(k+1)\nu_{k+1}t^k = \sum_{k=1}^{\infty}(2\alpha + k + 1)\nu_k t^k - \sum_{k=1}^{\infty}\alpha\nu_{k-1}t^k. \tag{2.4.11}$$

In what follows, t is treated as the independent variable. We define the generating function

$$G(t) := \sum_{k=0}^{\infty}\nu_k t^k. \tag{2.4.12}$$

Note that the moments can be expressed by

$$\widetilde{\mu}^{\alpha}_{k+2} = k!\nu_k = G^{(k)}(0). \tag{2.4.13}$$

In order to write (2.4.11) using $G(t)$, we substitute

$$\sum_{k=1}^{\infty}(k+1)\nu_{k+1}t^k = G'(t) - \nu_1$$

on the left hand side, and

$$(2\alpha + 1)\sum_{k=1}^{\infty}\nu_k t^k + \sum_{k=1}^{\infty}k\nu_k t^k = (2\alpha+1)[G(t) - \nu_0] + tG'(t), \text{ as well as}$$

$$-\sum_{k=1}^{\infty}\alpha\nu_{k-1}t^k = -\alpha t\sum_{k=0}^{\infty}\nu_k t^k = -\alpha t G(t).$$

Apriori, we expect a first order *inhomogeneous* differential equation. However, when we collect the terms without the generating function (i.e. the inhomogeneity) we see

$$\nu_1 - (2\alpha+1)\nu_0 = \widetilde{\mu}_3 - (2\alpha+1)\widetilde{\mu}_2 = 0$$

by Equation (2.4.7). We obtain the first order *homogeneous* differential equation

$$(1-t)G'(t) + (\alpha t - 2\alpha - 1)G(t) = 0. \qquad (2.4.14)$$

The boundary condition

$$G(0) = \nu_0 = \widetilde{\mu}_2^\alpha = \Gamma(\alpha+1) \qquad (2.4.15)$$

follows from (2.4.12) for $t = 0$.

It is not hard to see that (near $t = 0$) the generating function

$$G(t) = \Gamma(\alpha+1)e^{\alpha t}(1-t)^{-(\alpha+1)}$$

solves this boundary value problem.

With the Leibniz rule and elementary simplifications we derive

$$\widetilde{\mu}_{k+2}^\alpha = \Gamma(\alpha+1)\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\alpha^m(-\alpha-k+m)_{k-m} \qquad (k \in \mathbb{N}_0). \qquad (2.4.16)$$

The theorem now follows from the standard hypergeometric notation by using the identity

$$(-\alpha-k+m)_{k-m} = \frac{(-\alpha-k)_k}{(-\alpha-k)_m}\,.$$

(We note that the numerator in the latter expression is independent of the summation variable m.) □

Remark. This method along with a first version of the first order differential equation was mentioned to us by M.E.H. Ismail.

2.4.2 Exceptional Moments

Starting from the recursion formula in Theorem 2.4.1 for the adjusted moments, we now derive a recursion formula for the exceptional moments by using the binomial formula.

Theorem 2.4.4. *The moments* $\mu_k = \int_0^\infty x^k\, W^\alpha(x)\, dx$

(a) satisfy the recursion formula

$$\mu_{k+2} = \sum_{m=0}^{k} \left[(2\alpha + k)\binom{k+1}{m} - \alpha\binom{k+2}{m} + (1-k)\binom{k}{m} \right] \alpha^{k+1-m}\, \mu_m$$
$$+ (1-\alpha)k\mu_{k+1}$$

for $k \in \mathbb{N}_0$. *Specifically, when* $k = 0$, *we find that* $\mu_2 = \alpha(\alpha+1)\mu_0$.

(b) with

$$\mu_0^\alpha = \Gamma(\alpha) - 2e^\alpha \alpha^\alpha \Gamma(\alpha+1)\Gamma(-\alpha, \alpha)$$
$$\mu_1^\alpha = -\Gamma(\alpha+1) + [2\alpha+1]e^\alpha \alpha^\alpha \Gamma(\alpha+1)\Gamma(-\alpha, \alpha).$$

Proof. To prove part (b) note that

$$\mu_0^\alpha = \int_0^\infty x^0 W^\alpha(x) dx = \int_0^\infty (x+\alpha)^0 W^\alpha(x) dx = \widetilde{\mu}_0^\alpha$$

and the first statement follows. To see the second statement

$$\widetilde{\mu}_1^\alpha = \int_0^\infty (x+\alpha) W^\alpha(x) dx = \int_0^\infty x^1 W^\alpha(x) dx + \alpha \int_0^\infty x^0 W^\alpha(x) dx = \mu_1^\alpha + \alpha\mu_0^\alpha.$$

So we have

$$\mu_1^\alpha = \widetilde{\mu}_1^\alpha - \alpha\mu_0^\alpha.$$

Substituting (2.4.1) as well as the formula for μ_0^α we obtain part (b).

Part (a) follows from lengthy but elementary computations, which we present for the convenience of the reader. The binomial theorem yields

$$\widetilde{\mu}_k = \int_0^\infty \left(\sum_{m=0}^{k} \binom{k}{m} x^m \alpha^{k-m} \right) W^\alpha(x)\, dx$$

$$= \sum_{m=0}^{k} \binom{k}{m} \alpha^{k-m} \int_0^\infty x^m W^\alpha(x)\, dx$$

$$\widetilde{\mu}_k = \sum_{m=0}^{k} \binom{k}{m} \mu_m\, \alpha^{k-m}. \tag{2.4.17}$$

By splitting off the last term we see that

$$\widetilde{\mu}_{k+2} = \sum_{m=0}^{k+2} \binom{k+2}{m} \mu_m\, \alpha^{k+2-m} = \sum_{m=0}^{k+1} \binom{k+2}{m} \mu_m\, \alpha^{k+2-m} + \mu_{k+2},$$

that is,

$$\mu_{k+2} = -\sum_{m=0}^{k+1} \binom{k+2}{m} \mu_m\, \alpha^{k+2-m} + \widetilde{\mu}_{k+2}.$$

Replacing $\widetilde{\mu}_{k+2}$ by the recursion formula for the adjusted moments in Theorem 2.4.1 and using (2.4.17) twice, we have

$$\mu_{k+2} = -\sum_{m=0}^{k+1} \binom{k+2}{m} \mu_m\, \alpha^{k+2-m} + (2\alpha+k)\widetilde{\mu}_{k+1} + \alpha(1-k)\widetilde{\mu}_k$$

$$\mu_{k+2} = -\sum_{m=0}^{k+1} \binom{k+2}{m} \mu_m\, \alpha^{k+2-m} \tag{2.4.18}$$

$$+ (2\alpha+k)\sum_{m=0}^{k+1} \binom{k+1}{m} \mu_m\, \alpha^{k+1-m} \tag{2.4.19}$$

$$+ \alpha(1-k)\sum_{m=0}^{k} \binom{k}{m} \mu_m\, \alpha^{k-m}. \tag{2.4.20}$$

Finally, we combine the sums. This is done by using

$$-\sum_{m=0}^{k+1} \binom{k+2}{m} \mu_m\, \alpha^{k+2-m} + 2\alpha \sum_{m=0}^{k+1} \binom{k+1}{m} \mu_m\, \alpha^{k+1-m}$$

$$=\alpha \left[-\sum_{m=0}^{k+1} \binom{k+2}{m} \mu_m\, \alpha^{k+1-m} + 2\sum_{m=0}^{k+1} \binom{k+1}{m} \mu_m\, \alpha^{k+1-m} \right]$$

$$=\alpha \sum_{m=0}^{k+1} \left[2\binom{k+1}{m} - \binom{k+2}{m} \right] \mu_m\, \alpha^{k+1-m}$$

$$=-k\alpha\mu_{k+1} + \alpha \sum_{m=0}^{k} \left[2\binom{k+1}{m} - \binom{k+2}{m} \right] \mu_m\, \alpha^{k+1-m}$$

— where the term in square brackets for $m = k+1$ evaluated as follows

$$\left[2\binom{k+1}{k+1} - \binom{k+2}{k+1}\right] = 2 - (k+2) = -k$$

— as well as

$$\begin{aligned}
&k\sum_{m=0}^{k+1}\binom{k+1}{m}\mu_m\,\alpha^{k+1-m} + \alpha(1-k)\sum_{m=0}^{k}\binom{k}{m}\mu_m\,\alpha^{k-m}\\
=&k\mu_{k+1} + l\sum_{m=0}^{k}\binom{k+1}{m}\mu_m\,\alpha^{k+1-m} + (1-k)\sum_{m=0}^{k}\binom{k}{m}\mu_m\,\alpha^{k+1-m}\\
=&k\mu_{k+1} + \sum_{m=0}^{k}\left[k\binom{k+1}{m} + (1-k)\binom{k}{m}\right]\mu_m\,\alpha^{k+1-m}.
\end{aligned}$$

Substitution into (2.4.18) through (2.4.20) yields

$$\begin{aligned}
\mu_{k+2} =& -k\alpha\mu_{k+1} + \alpha\sum_{m=0}^{k}\left[2\binom{k+1}{m} - \binom{k+2}{m}\right]\mu_m\,\alpha^{k+1-m}\\
&+ k\mu_{k+1} + \sum_{m=0}^{k}\left[k\binom{k+1}{m} + (1-l)\binom{k}{m}\right]\mu_m\,\alpha^{k+1-m} \qquad (2.4.21)
\end{aligned}$$

and we can easily verify the coefficients in the theorem. □

2.5 Alternative Representation of the XOP with Adjusted Moments

In Section 4 we saw that the adjusted moments can be computed much more easily. Here we obtain the more elegant representation for the exceptional X_1-Laguerre polynomials.

We begin by expressing the X_1-Laguerre orthogonal polynomials in terms of powers of $(x+\alpha)$. Consider

$$L_n^\alpha(x) = \sum_{k=0}^{n} a_{nk}(x+\alpha)^k.$$

As in Section 3, the idea is to express the coefficients a_{nk}, $k = 0, 1, \ldots, n$, via a system of $n+1$ linear equations $\widetilde{A}a = b$, where

$$a = \begin{bmatrix} a_{n0} \\ a_{n1} \\ \vdots \\ a_{nn} \end{bmatrix} \in \mathbb{R}^{n+1} \qquad \text{and} \qquad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \widetilde{K}_n \end{bmatrix} \in \mathbb{R}^{n+1},$$

with $\widetilde{K}_n \neq 0$ and where $\widetilde{A}$ is given in the next theorem:

Theorem 2.5.1. *The exceptional X_1-Laguerre polynomials admit the representation*

$$L_n^\alpha(x) = \frac{1}{\det \widetilde{A}} \sum_{k=0}^{n} \left(\det \widetilde{A}_k\right) (x+\alpha)^k = \frac{\widetilde{K}_n}{\det \widetilde{A}} \begin{vmatrix} \left(\textit{First } n \textit{ rows of the matrix } \widetilde{A}\right) \\ 1 \ (x+\alpha) \ (x+\alpha)^2 \ \ldots \ (x+\alpha)^n \end{vmatrix}$$

for $n \in \mathbb{N}$, where

$$\widetilde{A} = \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 \\ \widetilde{\mu}_0 + \widetilde{\mu}_1 & \widetilde{\mu}_1 + \widetilde{\mu}_2 & \ldots & \ldots & \widetilde{\mu}_n + \widetilde{\mu}_{n+1} \\ \widetilde{\mu}_2 & \widetilde{\mu}_3 & \ldots & \ldots & \widetilde{\mu}_{n+2} \\ \vdots & \vdots & & & \vdots \\ \widetilde{\mu}_n & \widetilde{\mu}_{n+1} & \ldots & \ldots & \widetilde{\mu}_{2n} \end{bmatrix},$$

the exceptional moments are given by $\widetilde{\mu}_k = \int_0^\infty (x+\alpha)^k W^\alpha(x)dx$, and where the matrix $\widetilde{A}_k$ is obtained from $\widetilde{A}$ by replacing the $(k+1)$-st column with the vector b; as is done in Cramer's rule. (In Section 4 we found a recursion formula for the moments $\widetilde{\mu}_k$.)

Proof. The idea of establishing matrix $\widetilde{A}$ is the same way as was for matrix A in Theorem 2.3.1. Since the moments are adjusted to better assimilate $v_k(x) = (x+\alpha)^k$, $k \geq 2$, from equations (2.1.5), the binomial formula is not required and $\widetilde{A}$ turns out to be simpler than A.

The first row of $\widetilde{A}$ also simplifies: As before, we use the exceptional condition (2.2.1). We obtain

$$\left(L_n^\alpha\right)'(x) = \sum_{k=1}^{n} k a_{nk} (x+\alpha)^{k-1}$$

and so

$$L_n^\alpha(-\alpha) = a_{n0} \quad \text{as well as} \quad \left(L_n^\alpha\right)'(-\alpha) = a_{n1}.$$

We have

$$0 = \left(L_n^\alpha\right)'(-\alpha) - L_n^\alpha(-\alpha) = a_{n1} - a_{n0}.$$

The entries in the first row of $\widetilde{A}$ follow.

The other n conditions are, again, obtained via orthogonality. So we have

$$\langle L_n^\alpha \ , \ v_k \rangle_{W^\alpha} = \widetilde{K}_n \delta_{nk} \quad (k = 1, \dots , n).$$

For $k = 1$, we observe as before

$$\begin{aligned}
\widetilde{K}_n \delta_{n1} &= \langle L_n^\alpha \ , \ v_1 \rangle_{W^\alpha} \\
&= \left\langle \sum_{k=0}^{n} a_{nk}(x+\alpha)^k \ , \ x + \alpha + 1 \right\rangle_{W^\alpha} \\
&= \int_0^\infty \left(\sum_{k=0}^{n} a_{nk}(x+\alpha)^k \right) (x + \alpha + 1) \, W^\alpha(x) \, dx \\
&= \sum_{k=0}^{n} a_{nk}(\widetilde{\mu}_k + \widetilde{\mu}_{k+1}).
\end{aligned}$$

So the second row of the matrix $\widetilde{A}$ is the first vector in the dot product

$$\begin{bmatrix} \widetilde{\mu}_0 + \widetilde{\mu}_1 & \widetilde{\mu}_1 + \widetilde{\mu}_2 & \dots & \widetilde{\mu}_n + \widetilde{\mu}_{n+1} \end{bmatrix} \cdot a = \widetilde{K}_n \delta_{n1}.$$

When $n \geq 2$ we consider $2 \leq s \leq n$. Again we compute

$$\begin{aligned}
\widetilde{K}_n \delta_{ns} &= \langle L_n^\alpha \ , \ v_s \rangle_{W^\alpha} \\
&= \left\langle \sum_{k=0}^{n} a_{nk}(x+\alpha)^k \ , \ (x+\alpha)^s \right\rangle_{W^\alpha} \\
&= \sum_{k=0}^{n} a_{nk} \ \widetilde{\mu}_{k+s}.
\end{aligned}$$

The corresponding coefficients fill the $(s+1)$-st row of the matrix $\widetilde{A}$.

Again, the matrix $\widetilde{A}$ is invertible and we apply Cramer's rule

$$a_{nk} = \left(\det \widetilde{A}_k\right) / \left(\det \widetilde{A}\right) \qquad (\text{for } k = 0, 1, \ldots, n).$$

With the definition of the vector b:

$$\begin{vmatrix} & & (\text{First } n \text{ rows of } A) & & \\ 0 \ \ldots \ 0 & & (x+\alpha)^k & & 0 \ \ldots \ 0 \end{vmatrix} = \frac{\left(\det \widetilde{A}_k\right)}{\widetilde{K}_n} (x+\alpha)^k,$$

and the desired formula follows from expansion of the determinant by minors along the last row. □

Remark 2.5.2. In order to embed this representation into the literature, we relate to the normalization used in [9] and [27]: The choice

$$\widetilde{K}_n = (-1)^n (\alpha + n) \Gamma(\alpha + n - 1)$$

yields the same normalization as in [9] and [27].

Indeed, there the leading coefficient of $L_n^\alpha(x)$ is given by $(-1)^n/(n-1)!$ and with this normalization they obtained

$$\|L_n^\alpha\|^2 = \frac{\Gamma(\alpha + n - 1)(\alpha + n)}{(n-1)!}.$$

And so by the Gram–Schmidt orthogonalization we have

$$\widetilde{K}_n = \langle L_n^\alpha(x), (x+\alpha)^n \rangle = (n-1)!/(-1)^n \|L_n^\alpha\|^2 = (-1)^n (\alpha + n)\Gamma(\alpha + n - 1).$$

This relation was pointed out to us by R. Milson.

Finally, we verify for $n = 1$ and $n = 2$ that the polynomials in Theorem 2.5.1 indeed agree with (2.1.2) and (2.1.3), respectively, as well as the normalization claimed in the latter remark.

Example. For $n = 1$ compute

$$\begin{vmatrix} -1 & 1 \\ 1 & x+\alpha \end{vmatrix} = -x - \alpha - 1,$$

which is a scalar multiple of (2.1.2). And, moreover, normalizing the formula in Theorem 2.5.1 according to Remark 2.5.2 we obtain

$$\frac{\widetilde{K}_n}{\det \widetilde{A}} \begin{vmatrix} -1 & 1 \\ 1 & x+\alpha \end{vmatrix} = \frac{-(\alpha+1)\Gamma(\alpha)}{(-\alpha-1)\Gamma(\alpha)}(-x-\alpha-1) = -x-\alpha-1,$$

the leading term of which is in agreement with the leading term in Remark 2.5.2, since for $n = 1$ we have $(-1)^n/(n-1)! = -1$.

Example. Take $n = 2$. By co-factor expansion along the last row we evaluate

$$L_2^\alpha(x) = \begin{vmatrix} -1 & 1 & 0 \\ \widetilde{\mu}_0 + \widetilde{\mu}_1 & \widetilde{\mu}_1 + \widetilde{\mu}_2 & \widetilde{\mu}_2 + \widetilde{\mu}_3 \\ 1 & x+\alpha & (x+\alpha)^2 \end{vmatrix}$$

$$= \widetilde{\mu}_2 + \widetilde{\mu}_3 + (x+\alpha)(\widetilde{\mu}_2 + \widetilde{\mu}_3) + (x+\alpha)^2[-\widetilde{\mu}_0 - 2\widetilde{\mu}_1 - \widetilde{\mu}_2].$$

For $k = 0$ in part (a) of Theorem 2.4.1 the recursion relation reduces to

$$\widetilde{\mu}_2 = 2\alpha\widetilde{\mu}_1 + \alpha\widetilde{\mu}_0,$$

so that

$$-\widetilde{\mu}_0 - 2\widetilde{\mu}_1 = -\widetilde{\mu}_2/\alpha.$$

And recall that the recursion relation for $k = 1$ reduces to

$$\widetilde{\mu}_3 = (2\alpha + 1)\widetilde{\mu}_2.$$

With this we have up to normalization

$$L_2^\alpha(x) = (2\alpha+2)\widetilde{\mu}_2 + (x+\alpha)(2\alpha+2)\widetilde{\mu}_2 - (x+\alpha)^2(1/\alpha+1)\widetilde{\mu}_2.$$

Factoring out

$$-\left(\frac{1}{\alpha}+1\right)\widetilde{\mu}_2 = -\frac{\alpha+1}{\alpha}\widetilde{\mu}_2$$

we obtain

$$L_2^\alpha(x) = -\frac{\alpha+1}{\alpha}\widetilde{\mu}_2\left[-2\alpha - 2\alpha(x+\alpha) + (x+\alpha)^2\right]$$

$$= -\frac{\alpha+1}{\alpha}\widetilde{\mu}_2\left[-2\alpha - 2\alpha x - 2\alpha^2 + x^2 + 2\alpha x + \alpha^2\right]$$

$$= -\frac{\alpha+1}{\alpha}\widetilde{\mu}_2\left[x^2-\alpha^2-2\alpha\right].$$

This is in agreement with the second degree polynomial given by (2.1.3), since they both span the same eigenspace.

Again, we compare this to the Remark 2.5.2 about normalization. It is tedious but elementary to compute

$$\frac{\widetilde{K}_2}{\det \widetilde{A}} = \frac{-(\alpha+2)}{\widetilde{\mu}_2(\alpha+3+2/\alpha)}$$

for $\widetilde{K}_2$ as in Remark 2.5.2. Now, the leading term of $L_2^\alpha(x)$ (with the normalization described in Theorem 2.5.1 and with $\widetilde{K}_2$ from Remark 2.5.2) is given by

$$\left(-\frac{\alpha+1}{\alpha}\widetilde{\mu}_2\right)\left(\frac{\widetilde{K}_n}{\det \widetilde{A}}\right) = \frac{(\alpha+1)(\alpha+2)}{\alpha^2+3\alpha+2} = 1.$$

And, again, this is in agreement with the predicted leading term coefficient in Remark 2.5.2, since for $n=2$ we have $(-1)^n/(n-1)! = 1$.

CHAPTER THREE

Type I Laguerre Exceptional Orthogonal Polynomials of Codimension Two

3.1 Background Information

In this chapter, which comes from the author's work in [30], we now examine moment representations for the Type I X_2-Laguerre exceptional orthogonal polynomial system, which has co-dimension two. We also explore the possibility of extending these methods to the general X_m-Laguerre XOP. Some results, such as Lemma 3.3.1, will be applicable for any co-dimension. As mentioned in [31], certain aspects of extending the determinantal results to higher co-dimension, that is, the $m \geq 2$ case, become more difficult. Complications arise as increasing the co-dimension m results in an increase in the number of choices one can make regarding the associated flag. These choices become more apparent in Section 3. However, the use of the adjusted moments as seen in [32] keeps this X_2 case manageable.

3.1.1 Outline

In Section 2 we introduce the X_m^I-Laguerre differential system including the polynomials and the weight function. We define and briefly discuss an alteration to the exceptional moments, which we call the *adjusted moments*. In Section 3, we explain how the exceptional conditions are responsible for excluding polynomials of degrees $0, \ldots, m-1$ from being eigenfunctions. In a sense, the main idea behind obtaining a determinantal representation for the X_2^I-Laguerre polynomials is to merge the exceptional conditions with the Gram–Schmidt algorithm. Namely, the exceptional conditions indicate which *flag* we will apply Gram–Schmidt to. The determinantal representations are the topic of Section 4. They provide a way to find a general X_2^I-Laguerre polynomial that does not explicitly go through the Darboux transform or classical orthogonal polynomials, but rather uses the exceptional conditions and the adjusted moments. Recursion formulas for computing the "matrix" of adjusted moments are presented (Section 5) along with expressions for the initial adjusted moments, that is, those needed to start the recursion (Section 6).

3.2 Differential Expression, Weight Function, and Adjusted Moments

The primary focus of our study will be on the X_m^I-Laguerre polynomials. We refer the reader to the literature for a more in-depth look at the Laguerre XOP families. In particular, a spectral study of the X_1-Laguerre polynomials [1]. The origination of the Type III Laguerre along with a comprehensive look at all three types of exceptional Laguerre OPS may be found in [27].

Remark 3.2.1. In order to distinguish between the classical Laguerre orthogonal polynomial system and the XOP system, we introduce the "hat" notation. The "hat" will appear on expressions, equations, polynomials, etc. which are associated with the exceptional case. For example, the classical Laguerre polynomial of degree n with parameter α is denoted by $L_n^\alpha(x)$; while our X_m^I-Laguerre polynomial of degree n with parameter α is denoted by $\widehat{L}_{m,n}^{I,\alpha}$.

The second-order differential expression associated with the X_m^I-Laguerre XOP system is given by

$$\begin{aligned}\widehat{\ell}_m^{I,\alpha}[y](x) := -xy''(x) + \left(x - \alpha - 1 + 2x\left(\log L_m^{\alpha-1}(-x)\right)'\right) y'(x) \\ + \left(2\alpha\left(\log L_m^{\alpha-1}(-x)\right)' - m\right) y(x)\,. \end{aligned} \tag{3.2.1}$$

We draw the reader's attention to the logarithmic derivative of the classical Laguerre polynomial, $L_m^{\alpha-1}(-x)$, which appears in the coefficients of the y' and y terms. Unlike its classical counterpart, XOP systems have expressions which involve non-polynomial coefficients. These coefficients will provide the requirements for what is referred to in Section 3 as the "exceptional condition."

The X_m^I-Laguerre polynomials $\left\{\widehat{L}_{m,n}^{I,\alpha}\right\}_{n=m}^{\infty}$ satisfy the Sturm-Liouville eigenvalue equation given by

$$\widehat{\ell}_m^{I,\alpha}[y](x) = (n-m)y(x) \quad (0 < x < \infty)\,. \tag{3.2.2}$$

The X_m^I-Laguerre XOP system includes polynomials of degree $n \geq m$.

Solutions to the eigenvalue equation (3.2.2) can be characterized in a variety of ways, see [27]. In particular, there exists the following relationship between the classical and exceptional X_m^I-Laguerre polynomials:

$$\widehat{L}_{m,n}^{I,\alpha}(x) = L_m^{\alpha}(-x)L_{n-m}^{\alpha-1}(x) + L_m^{\alpha-1}(-x)L_{n-m-1}^{\alpha}(x) \quad (n \geq m).$$

In addition, the exceptional X_m^I-Laguerre polynomials $\left\{\widehat{L}_{m,n}^{I,\alpha}\right\}_{n=m}^{\infty}$ satisfy an orthogonality condition on $(0,\infty)$ for $\alpha > 0$ with respect to the weight function

$$\widehat{W}_m^{I,\alpha} = \frac{x^{\alpha}e^{-x}}{(L_m^{\alpha-1}(-x))^2} \quad (0 < x < \infty). \tag{3.2.3}$$

Remarkably, for any finite co-dimension m, the X_m^I-Laguerre polynomials are complete in $L^2((0,\infty);\widehat{W}_m^{I,\alpha})$, see e.g. [15].

In order to simplify our calculations for the determinantal representations for the X_2-Laguerre XOP, we choose to use adjusted moments. We define these adjusted moments as

$$\widetilde{\mu}_{i,j} = \int_0^{\infty} (x-r)^i (x-s)^j \widehat{W}_m^{I,\alpha}(x)\,dx\,, \tag{3.2.4}$$

where r and s are the two distinct roots of $L_2^{\alpha-1}(-x)$ and $\widehat{W}(x)$ is an abbreviation for the X_2-Laguerre weight function. We describe the importance of these roots r and s in Section 3 below.

3.3 The Flag

In this section, we characterize the subspace spanned by the first n of the X_m^I-Laguerre polynomials as those which satisfy the following m exceptional conditions:

$$\xi_j y'(\xi_j) + \alpha y(\xi_j) = 0 \text{ for } j = 1, 2, \ldots, m, \tag{3.3.1}$$

and where $\{\xi_j\}_{j=1}^m$ denote the m real roots of the classical Laguerre polynomial $L_m^{\alpha-1}(-x)$. As the roots of $L_m^{\alpha-1}(-x)$ are simple, these m-exceptional conditions are not redundant. Recall that the exceptional polynomials, belonging to the X_m^I-Laguerre XOP sequence, will be of consecutive degrees beginning with degree m.

To tie these exceptional conditions in with the exceptional polynomials, two definitions are necessary: define the span of the first $k+1$ polynomials of the X_m^I-Type I Laguerre exceptional orthogonal polynomial system to be

$$\mathcal{L}_{m,k} := \text{span}\left\{ \widehat{L}_{m,j}^{I,\alpha} : j = m, m+1, \ldots, m+k \right\}.$$

Let $\mathcal{P}_k$ represent the set of all polynomials whose degree is at most k. Then define

$$\mathcal{M}_{m,k} := \{ p \in \mathcal{P}_{m+k} : p \text{ satisfies } (3.3.1) \}.$$

Lemma 3.3.1. *The sets $\mathcal{L}_{m,k} = \mathcal{M}_{m,k}$ for all $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$.*

Remark 3.3.2. Before proving Lemma 3.3.1, we note that the original dimension argument used in the proof to find the equality of these sets was first introduced in [27], while the entire proof is analogous to the proofs for [16, Proposition 5.3], [31, Lemma 4.1], and [32, Lemma 2.1]. As an adjustment is required to account for the m exceptional conditions, the proof is included.

Proof. Fix $m \in \mathbb{N}$. We will show the inclusion of subspaces $\mathcal{L}_{m,k} \leq \mathcal{M}_{m,k}$ for all $k \in \mathbb{N}_0$. Let $y \in \mathcal{L}_{m,k}$. Since $\mathcal{L}_{m,k}$ is invariant under $\widehat{\ell}_m^{I,\alpha}$, it follows that $\widehat{\ell}_m^{I,\alpha}[y] \in \mathcal{L}_{m,k}$. In particular, this requires that $\widehat{\ell}_m^{I,\alpha}[y]$ is polynomial—this will occur if and only if (3.3.1) is satisfied. Thus, $y \in \mathcal{M}_{m,k}$ and $\mathcal{L}_{m,k} \leq \mathcal{M}_{m,k}$.

On the other hand, a dimension argument is used to show that $\mathcal{M}_{m,k} \leq \mathcal{L}_{m,k}$. To find the dimension of $\mathcal{M}_{m,k}$, observe that P_{m+k} has dimension $m+k+1$ and that there are m exceptional conditions imposed on P_{m+k} in order to form $\mathcal{M}_{m,k}$. Therefore $\dim \mathcal{M}_{m,k} = k+1 < \infty$. Clearly, $\dim \mathcal{L}_{m,k} = k+1$ as it is spanned by $k+1$ linearly independent polynomials. As $\mathcal{L}_{m,k}$ and $\mathcal{M}_{m,k}$ are both subspaces of $\mathcal{P}_{m+k}$ and $\mathcal{M}_{m,k} \leq \mathcal{L}_{m,k}$, the result follows. □

Lemma 3.3.1 is applicable to the X_m^I-Laguerre orthogonal polynomial systems for all $m \in \mathbb{N}$. As discussed in [31, Lemma 4.1], the computational aspects related to exceptional orthogonal polynomial systems of higher order become difficult. With this in mind, we now restrict ourselves for the remainder of this chapter to $m = 2$. Let the two roots of $L_2^{\alpha-1}(-x)$ be denoted as r and s.

With the exceptional roots $r = -(\alpha+1) - \sqrt{\alpha+1}$ and $s = -(\alpha+1) + \sqrt{\alpha+1}$, we define degree k polynomials:

For $k = 2$,

$$\begin{aligned} v_2(x) &= L_2^{\alpha}(-x) \\ &= \frac{1}{2}(x-r)(x-s) + x - r - \sqrt{\alpha+1}\,, \end{aligned} \tag{3.3.2}$$

for $k = 3$,

$$v_3(x) = (x-r)^2(x-s+1)\,, \tag{3.3.3}$$

and for $k \geq 4$,

$$v_k(x) := (x-r)^{\overline{k}}(x-s)^{\underline{k}}\,, \tag{3.3.4}$$

where we use the floor and ceiling functions $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ (respectively) in the abbreviated notation $\overline{k} := \lceil k/2 \rceil$ and $\underline{k} := \lfloor k/2 \rfloor$.

Lemma 3.3.3. *The sequence of polynomials* $\{v_2, v_3, v_4, \ldots\}$ *forms a flag for* $\widehat{\ell}_2^{I,\alpha}$.

Remark 3.3.4. We recall that, informally, in the case of XOP systems, a *flag* is a sequence of polynomials whose span is preserved under an exceptional operator.

Proof. By definition of v_k, it is ensured that only polynomials of degree 0 and 1 will be excluded. As a result of Lemma 3.3.1, it is enough to show that the v_k satisfy the exceptional conditions outlined in (3.3.1). These exceptional conditions reduce in the X_2^I-Laguerre case to:

$$\begin{aligned} r y'(r) + \alpha y(r) &= 0 \\ s y'(s) + \alpha y(s) &= 0\,. \end{aligned} \tag{3.3.5}$$

Here v_2, the first element of the flag, is a classical Laguerre polynomial. We could use any polynomial for v_3 that satisfies the two exceptional conditions of

3.3.5. We found the current v_3 after using a previous version that was a much more complicated cubic polynomial. This v_3 seems to be the simplest possible, in terms of its representation using the two exceptional roots.

It is straightforward to check that v_2 and v_3 both satisfy the exceptional conditions. For $k \geq 4$, we have that both $v_k(r) = 0$ and $v_k(s) = 0$; therefore, it is enough to show that $v_k'(r) = 0$ and $v_k'(s) = 0$. This too follows easily as $\overline{k}, \underline{k} \geq 2$. □

3.4 Determinantal Representations

We use the adjusted moments defined in (3.2.4) to provide a determinantal representation formula for the X_2^I-Laguerre orthogonal polynomials.

This is done by expressing these polynomials in terms of powers of the terms $(x - r)$ and $(x - s)$, where r and s are the distinct roots of the generalized classical Laguerre polynomial $L_2^{\alpha-1}(-x)$. Fix $n \in \mathbb{N}$ and — as Ansatz — consider the expansion

$$\widehat{L}_{2,n}^{I,\alpha}(x) = \sum_{k=0}^{n} a_{nk}(x-r)^{\overline{k}}(x-s)^{\underline{k}}, \tag{3.4.1}$$

where $\overline{k}$ and $\underline{k}$ are as indicated previously in Section 3, that is, $\overline{k} = \lceil k/2 \rceil$ and $\underline{k} = \lfloor k/2 \rfloor$.

The main idea behind the determinantal representation is to determine the $n+1$ coefficients a_{nk} for $k = 0, \ldots, n$. This is accomplished through working with the linear system $Ma = b$ with $(n+1) \times (n+1)$ matrix M, as well as

$$a := \begin{pmatrix} a_{n0} \\ a_{n1} \\ \vdots \\ a_{nn} \end{pmatrix} \qquad \text{and} \qquad b := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ K_n \end{pmatrix}, \tag{3.4.2}$$

where $K_n = \|\widehat{L}_n^{I,\alpha}\|^2$ depends on the normalization convention. For us it merely matters that K_n is non-zero.

Theorem 3.4.1. *The X_2^I-Laguerre polynomials can be obtained through two representations*

$$\widehat{L}_{2,n}^{I,\alpha}(x) = (\det(M))^{-1} \sum_{k=0}^{n} \det(M_k)(x-r)^{\overline{k}}(x-s)^{\underline{k}} \text{ or, alternatively,} \tag{3.4.3}$$

$$\widehat{L}_{2,n}^{I,\alpha}(x) = \begin{vmatrix} & & & \textit{(First n rows of M)} & & & \\ 1 & x-r & (x-r)(x-s) & (x-r)^2(x-s) & (x-r)^2(x-s)^2 & \dots & (x-r)^{\overline{n}}(x-s)^{\underline{n}} \end{vmatrix}, \tag{3.4.4}$$

where the $(n+1) \times (n+1)$ matrix M is given by

$M =$

$$\begin{pmatrix} \alpha & r & r(r-s) & 0 & \dots & \dots & 0 \\ \alpha & s+\alpha(s-r) & s(s-r) & s(s-r)^2 & 0 & \dots & 0 \\ \frac{1}{2}\widetilde{\mu}_{1,1}+\widetilde{\mu}_{1,0}-\beta\widetilde{\mu}_{0,0} & \frac{1}{2}\widetilde{\mu}_{2,1}+\widetilde{\mu}_{2,0}-\beta\widetilde{\mu}_{1,0} & \frac{1}{2}\widetilde{\mu}_{2,2}+\widetilde{\mu}_{2,1}-\beta\widetilde{\mu}_{1,1} & \dots & \dots & \dots & \frac{1}{2}\widetilde{\mu}_{\overline{n}+1,\underline{n}+1}+\widetilde{\mu}_{\overline{n}+1,\underline{n}}-\beta\widetilde{\mu}_{\overline{n},\underline{n}} \\ \widetilde{\mu}_{2,1}+\widetilde{\mu}_{2,0} & \widetilde{\mu}_{3,1}+\widetilde{\mu}_{3,0} & \widetilde{\mu}_{3,2}+\widetilde{\mu}_{3,1} & \dots & \dots & \dots & \widetilde{\mu}_{\overline{n}+2,\underline{n}+1}+\widetilde{\mu}_{\overline{n}+2,\underline{n}} \\ \widetilde{\mu}_{2,2} & \widetilde{\mu}_{3,2} & \widetilde{\mu}_{3,3} & \dots & \dots & \dots & \widetilde{\mu}_{\overline{n}+2,\underline{n}+2} \\ \widetilde{\mu}_{3,2} & \widetilde{\mu}_{4,2} & \widetilde{\mu}_{4,3} & \dots & \dots & \dots & \widetilde{\mu}_{\overline{n}+3,\underline{n}+2} \\ \widetilde{\mu}_{3,3} & \widetilde{\mu}_{4,3} & \widetilde{\mu}_{4,4} & \dots & \dots & \dots & \widetilde{\mu}_{\overline{n}+3,\underline{n}+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\mu}_{\overline{n},\underline{n}} & \widetilde{\mu}_{\overline{n}+1,\underline{n}} & \widetilde{\mu}_{\overline{n}+1,\underline{n}+1} & \dots & \dots & \dots & \widetilde{\mu}_{2\overline{n},2\underline{n}} \end{pmatrix}$$

with $\beta = \sqrt{\alpha+1}$ and matrix M_k is obtained from M by replacing the $(k+1)$st column by the vector b.

(For the precise entries in rows 3 through n+1, please refer to equations (3.4.5) *and* (3.4.6) *in the proof.)*

Remark. Matrix M is invertible due to the fact that the polynomials are determined precisely by the conditions that give rise to the matrix.

Proof. First we focus on $\widehat{L}_{2,n}^{I,\alpha}(x) = (\det(M))^{-1} \sum_{k=0}^{n} \det(M_k)(x-r)^{\overline{k}}(x-s)^{\underline{k}}$. Since $Ma = b$ and by Cramer's rule we have

$$a_{nk} = \frac{\det(M_k)}{\det(M)}.$$

So, it suffices to show the entries of the matrix M. To that end we fill the first two rows of M with conditions that arise from the two exceptional conditions. Namely, given the two roots r and s of $L_2^{\alpha-1}(-x)$, every polynomial p that satisfies the eigenvalue equation for the exceptional differential expression necessarily satisfies the exceptional conditions

$$rp'(r) + \alpha p(r) = 0,$$
$$sp'(s) + \alpha p(s) = 0$$

(see Lemma 3.3.1).

Differentiating the Ansatz in (3.4.1) we see

$$\left(\widehat{L}^{I,\alpha}_{2,n}(x)\right)' = \sum_{k=1}^{n} \overline{k} a_{nk}(x-r)^{\overline{k}-1}(x-s)^{\underline{k}} + \sum_{k=2}^{n} \underline{k} a_{nk}(x-r)^{\overline{k}}(x-s)^{\underline{k}-1}.$$

And substituting this and (3.4.1) into the exceptional condition $rp'(r)+\alpha p(r) = 0$ yields

$$\begin{aligned} 0 &= r\left(\widehat{L}^{I,\alpha}_{2,n}(r)\right)' + \alpha \widehat{L}^{I,\alpha}_{2,n}(r) \\ &= r a_{n1} + r a_{n2}(r-s) + \alpha a_{n0} \\ &= \begin{bmatrix} \alpha & r & r(r-s) & 0 & \dots & 0 \end{bmatrix} a. \end{aligned}$$

We verified the first row of the matrix M.

For the second row, we proceed in analogy with the exceptional condition $sp'(s) + \alpha p(s) = 0$. Some more terms prevail:

$$\begin{aligned} 0 &= s\left(\widehat{L}^{I,\alpha}_{2,n}(s)\right)' + \alpha \widehat{L}^{I,\alpha}_{n}(s) \\ &= s a_{n1} + s a_{n2}(s-r) + s a_{n3}(s-r)^2 + \alpha a_{n0} + \alpha a_{n1}(s-r) \\ &= \begin{bmatrix} \alpha & s+\alpha(s-r) & s(s-r) & s(s-r)^2 & 0 & \dots & 0 \end{bmatrix} a. \end{aligned}$$

We verified the second row of the matrix M.

The remaining $n-1$ rows of M (rows 3 through $n+1$) are obtained from the orthogonality requirements

$$\langle \widehat{L}^{I,\alpha}_{2,n}, v_l \rangle_{W^\alpha} = K_n \delta_{nl} \qquad \text{for } l = 2, \dots, n.$$

Here we let δ_{nl} denote the Kronecker delta symbol. Since v_l are standard only for $l \geq 4$ we begin by treating the cases $l = 2$ and $l = 3$ separately. To avoid confusion later on we draw attention to the shift between l, and which row of M we are filling. That is, fixing some $l = 2, \ldots, n$, we will determine the entries in the $(l+1)$st row of M.

When $l = 2$ we have $v_2(x) = 1/2(x-r)(x-s) + x - r - \sqrt{\alpha+1}$ and so with the Ansatz (3.4.1) we see

$$\begin{aligned}
&\left\langle \widehat{L}_n^{I,\alpha}, v_2 \right\rangle_{W^\alpha} \\
&= \int_0^\infty \sum_{k=0}^n a_{nk}(x-r)^{\overline{k}}(x-s)^{\underline{k}} \left(\frac{1}{2}(x-r)(x-s) + x - r - \sqrt{\alpha+1} \right) W^\alpha(x)dx \\
&= \sum_{k=0}^n a_{nk} \int_0^\infty \left[\frac{1}{2}(x-r)^{\overline{k}+1}(x-s)^{\underline{k}+1} + (x-r)^{\overline{k}+1}(x-s)^{\underline{k}} \right. \\
&\qquad\qquad \left. - \sqrt{\alpha+1}(x-r)^{\overline{k}}(x-s)^{\underline{k}} \right] W^\alpha(x)dx \\
&= \sum_{k=0}^n a_{nk} \left[\frac{1}{2}\widetilde{\mu}_{\overline{k}+1,\underline{k}+1} + \widetilde{\mu}_{\overline{k}+1,\underline{k}} - \sqrt{\alpha+1}\widetilde{\mu}_{\overline{k},\underline{k}} \right]
\end{aligned}$$

by the definition of the adjusted moments in (3.2.4). As before (in virtue of linear algebra applied to the system $Ma = b$), the summands for the different values of k occupy the different entries of the 3rd row of M:

$$M_{3,k+1} = \frac{1}{2}\widetilde{\mu}_{\overline{k}+1,\underline{k}+1} + \widetilde{\mu}_{\overline{k}+1,\underline{k}} - \sqrt{\alpha+1}\widetilde{\mu}_{\overline{k},\underline{k}} \qquad \text{for } 0 \leq k \leq n. \tag{3.4.5}$$

Again, there is a shift between the value of k and the column of M. For example, for $k = 0$ we obtain the $(3,1)$ entry of M, $M_{3,1}$, to equal

$$M_{3,1} = \frac{1}{2}\widetilde{\mu}_{1,1} + \widetilde{\mu}_{1,0} - \sqrt{\alpha+1}\widetilde{\mu}_{0,0}.$$

For $k = 1$ and $k = 2$ we see

$$k = 1: \qquad M_{3,2} = \frac{1}{2}\widetilde{\mu}_{2,1} + \widetilde{\mu}_{2,0} - \sqrt{\alpha+1}\widetilde{\mu}_{1,0},$$

$$k = 2: \qquad M_{3,3} = \frac{1}{2}\widetilde{\mu}_{2,2} + \widetilde{\mu}_{2,1} - \sqrt{\alpha+1}\widetilde{\mu}_{1,1},$$

and so forth. We verified the third row of M.

Let us focus on the fourth row of M. To do so we take $l = 3$ and with our choice $v_3(x) = (x-r)^2(x-s) + (x-r)^2$ for the flag element of degree three we see

$$\begin{aligned}
&\left\langle \widehat{L}^{I,\alpha}_{2,n}, v_3 \right\rangle_{W^\alpha} \\
&= \int_0^\infty \sum_{k=0}^{n} a_{nk}(x-r)^{\overline{k}}(x-s)^{\underline{k}} \left[(x-r)^2(x-s) + (x-r)^2\right] W^\alpha(x)dx \\
&= \sum_{k=0}^{n} a_{nk} \int_0^\infty \left[(x-r)^{\overline{k}+2}(x-s)^{\underline{k}+1} + (x-r)^{\overline{k}+2}(x-s)^{\underline{k}}\right] W^\alpha(x)dx \\
&= \sum_{k=0}^{n} a_{nk} \left[\widetilde{\mu}_{\overline{k}+2,\underline{k}+1} + \widetilde{\mu}_{\overline{k}+2,\underline{k}}\right].
\end{aligned}$$

Again for $k = 0, 1, 2$ we obtain the matrix entries

$$\begin{aligned}
k = 0: &\qquad M_{4,1} = \widetilde{\mu}_{\overline{k}+2,\underline{k}+1} + \widetilde{\mu}_{\overline{k}+2,\underline{k}} \overset{k=0}{=} \widetilde{\mu}_{2,1} + \widetilde{\mu}_{2,0}, \\
k = 1: &\qquad M_{4,2} = \widetilde{\mu}_{\overline{k}+2,\underline{k}+1} + \widetilde{\mu}_{\overline{k}+2,\underline{k}} \overset{k=1}{=} \widetilde{\mu}_{3,1} + \widetilde{\mu}_{3,0}, \text{ and} \\
k = 2: &\qquad M_{4,3} = \widetilde{\mu}_{\overline{k}+2,\underline{k}+1} + \widetilde{\mu}_{\overline{k}+2,\underline{k}} \overset{k=2}{=} \widetilde{\mu}_{3,2} + \widetilde{\mu}_{3,1}.
\end{aligned}$$

This shows the entries in the fourth row of M.

When the degree of the exceptional polynomial is $n \geq 4$, the rows five through $n+1$ are obtained from the standard flag elements

$$v_l = (x-r)^{\overline{l}}(x-s)^{\underline{l}},$$

for $l = 4, \ldots, n$. And so for those values of l we have

$$\begin{aligned}
\left\langle \widehat{L}^{I,\alpha}_{2,n}, v_l \right\rangle_{W^\alpha} &= \int_0^\infty \sum_{k=0}^{n} a_{nk}(x-r)^{\overline{k}}(x-s)^{\underline{k}} \left[(x-r)^{\overline{l}}(x-s)^{\underline{l}}\right] W^\alpha(x)dx \\
&= \sum_{k=0}^{n} a_{nk} \int_0^\infty (x-r)^{\overline{k}+\overline{l}}(x-s)^{\underline{k}+\underline{l}} W^\alpha(x)dx = \sum_{k=0}^{n} a_{nk}\widetilde{\mu}_{\overline{k}+\overline{l},\underline{k}+\underline{l}}.
\end{aligned}$$

This results in the $(l+1, k+1)$ matrix entry

$$M_{l+1,k+1} = \widetilde{\mu}_{\overline{k}+\overline{l},\underline{k}+\underline{l}} \qquad \text{for } 3 \leq l \leq n, 0 \leq k \leq n. \tag{3.4.6}$$

This concludes the proof for the entries of the matrix M, and thereby the proof of equation (3.4.3).

To obtain the second representation, (3.4.4), we notice that by the definition of the vector b in (3.4.2) we have

$$\begin{vmatrix} & & & \text{(First } n \text{ rows of } M) & & & \\ 0 & \dots & 0 & (x-r)^{\overline{k}}(x-s)^{\underline{k}} & 0 & \dots & 0 \end{vmatrix} = \frac{\det(M_k)}{K_n}\,(x-r)^{\overline{k}}(x-s)^{\underline{k}}$$

for $k = 0, 1, \dots, n$. In the last row of the matrix, the entry $(x-r)^{\overline{k}}(x-s)^{\underline{k}}$ is in the $(n+1, k)$ position. Formula (3.4.4) now follows simply by co-factor expansion of the determinant in (3.4.4) along the last row. □

3.5 Moment Recursion Formulas

Because the matrix M used in the determinantal representation of the polynomials $\widehat{L}^{I,\alpha}_{2,n}(x)$ is largely populated by various adjusted moments, we need to develop a suite of recursion-like formulas which will allow us to compute all entries in the 2-dimensional array of these moments.

Theorem 3.5.1. *The adjusted moments* $\widetilde{\mu}_{i,j} = \int_0^\infty (x-r)^i (x-s)^j\, W^\alpha(x)\, dx$

(a) satisfy the 3-term recursion-like formula

$$\widetilde{\mu}_{i+1,j} = \widetilde{\mu}_{i,j+1} + 2\sqrt{\alpha+1}\,\widetilde{\mu}_{i,j} \qquad (i, j \in \mathbb{N}_0), \tag{3.5.1}$$

as well as

(b) the 4-term recursion-like formula

$$\begin{aligned} \widetilde{\mu}_{i+1,j+1} = {} & [i + j - 1 + 2\alpha + \sqrt{\alpha+1}]\,\widetilde{\mu}_{i+1,j} \\ & + [(1-i-j)(\alpha+1) + (3 - 3i - j - 4\alpha)\sqrt{\alpha+1}]\,\widetilde{\mu}_{i,j} \\ & + [(2i-4)(\alpha+1)(\sqrt{\alpha+1}+1)]\,\widetilde{\mu}_{i-1,j} \qquad (i, j \in \mathbb{N}_0). \end{aligned} \tag{3.5.2}$$

Remark. The following result will be key to our proof of part (b) below. (Here, we simplify notation by writing W for $\widehat{W}^{I,\alpha}_2$.)

To obtain this result, we make use of several facts. First, for functions f, g which are smooth on $[0, \infty)$ the moment functionals satisfy:

$$\langle W', f\rangle = -\langle W, f'\rangle \quad \text{and} \quad \langle gW, f\rangle = \langle W, fg\rangle ,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product with respect to Lebesgue measure on $[0,\infty)$.

Next, for a linear operator of the form

$$\ell[y] = a_2 y'' + a_1 y' + a_0 y,$$

the related symmetry equation is given by

$$a_2 y' + (a_2' - a_1)y = 0,$$

and it is solved by the weight function (with respect to which the eigen-polynomials are orthogonal) [33]. That is,

$$a_2 W' + (a_2' - a_1)W = 0.$$

And together with

$$\begin{aligned}\left\langle a_2 W',\ (x-r)^i (x-s)^j \right\rangle = -\left\langle W,\ a_2'(x-r)^i(x-s)^j\right\rangle - i\left\langle W,\ a_2(x-r)^{i-1}(x-s)^j\right\rangle \\ - j\left\langle W,\ a_2(x-r)^i(x-s)^{j-1}\right\rangle ,\end{aligned}$$

where the goal is to convert W' into W, we find for $i,j\in\mathbb{N}$:

$$\begin{aligned}0 &= \left\langle a_2W' + (a_2'-a_1)W,\ (x-r)^i(x-s)^j\right\rangle \\ &= i\left\langle W,\ a_2(x-r)^{i-1}(x-s)^j\right\rangle + j\left\langle W,\ a_2(x-r)^i(x-s)^{j-1}\right\rangle \\ &\quad + \left\langle W,\ a_1(x-r)^i(x-s)^j\right\rangle .\end{aligned} \tag{3.5.3}$$

Proof. The proof of part (a) will simply rely on the relation $r = s - 2\sqrt{\alpha+1}$, where r and s are the exceptional roots first mentioned in Section 3. We obtain

$$\begin{aligned}\widetilde{\mu}_{i+1,j} &= \int_0^\infty (x-r)^{i+1}(x-s)^j\, W(x)\, dx \\ &= \int_0^\infty (x-r)^i(x-s+2\sqrt{\alpha+1})(x-s)^j\, W(x)\, dx \\ &= \int_0^\infty (x-r)^i(x-s)^{j+1}\, W(x)\, dx + 2\sqrt{\alpha+1}\int_0^\infty (x-r)^i(x-s)^j\, W(x)\, dx \\ &= \widetilde{\mu}_{i,j+1} + 2\sqrt{\alpha+1}\widetilde{\mu}_{i,j}.\end{aligned}$$

To start the proof of part (b), we note that the differential expression for the X_2^I-Laguerre XOP system is

$$\widehat{\ell}_2^{I,\alpha}[y](x) = -xy'' + \left(x - \alpha - 1 + 2x\frac{(L_2^{\alpha-1}(-x))'}{L_2^{\alpha-1}(-x)}\right)y' + \left(\frac{2\alpha(L_2^{\alpha-1}(-x))'}{L_2^{\alpha-1}(-x)} - 2\right)y,$$

where the classical Laguerre polynomial $L_2^{\alpha-1}(-x) = \frac{1}{2}x^2 + (\alpha+1)x + \frac{\alpha(\alpha+1)}{2}$, and $(L_2^{\alpha-1}(-x))' = x + \alpha + 1$.

For this differential expression, the coefficient $a_2(x) = -x$ and, after a slight simplification,

$$a_1(x) = x - \alpha - 1 + \frac{4x(x+\alpha+1)}{x^2 + 2(\alpha+1)x + \alpha(\alpha+1)}.$$

In order to have adjusted moments appear from these calculations, we re-write both a_1 and a_2 so that x only appears in powers of the linear factors $(x-r)$ and $(x-s)$.

Accordingly, $a_2(x)$ can be represented as $-(x-r)-r$, and we find that

$$a_1(x) = (x-r) + A - \frac{B}{(x-r)} + \frac{C}{(x-r)(x-s)},$$

where $A = -2\alpha + 2 - \sqrt{\alpha+1}$, $B = 4(\alpha+1)$, and $C = (\alpha+1)(1-\sqrt{\alpha+1})$, and where we have arbitrarily given priority to the factor $(x-r)$ in these representations.

Substituting a_1 and a_2 into (3.5.3), we get

$$\begin{aligned}0 &= i\left\langle W,\ [-(x-r)-r](x-r)^{i-1}(x-s)^j\right\rangle + j\left\langle W,\ [-(x-r)-r](x-r)^i(x-s)^{j-1}\right\rangle \\ &\quad + \left\langle W,\ \left[(x-r) + A - \frac{B}{(x-r)} + \frac{C}{(x-r)(x-s)}\right](x-r)^i(x-s)^j\right\rangle \\ &= -i\widetilde{\mu}_{i,j} - ir\widetilde{\mu}_{i-1,j} - j\widetilde{\mu}_{i+1,j-1} - jr\widetilde{\mu}_{i,j-1} + \widetilde{\mu}_{i+1,j} + A\widetilde{\mu}_{i,j} - B\widetilde{\mu}_{i-1,j} + C\widetilde{\mu}_{i-1,j-1}.\end{aligned}$$

Thus,

$$\widetilde{\mu}_{i+1,j} = j\widetilde{\mu}_{i+1,j-1} + (i-A)\widetilde{\mu}_{i,j} + (ir+B)\widetilde{\mu}_{i-1,j} + jr\widetilde{\mu}_{i,j-1} - C\widetilde{\mu}_{i-1,j-1}.$$

After shifting the index $j \mapsto j+1$, we have

$$\begin{aligned}\widetilde{\mu}_{i+1,j+1} &= (j+1)\widetilde{\mu}_{i+1,j} + (i-A)\widetilde{\mu}_{i,j+1} + (ir+B)\widetilde{\mu}_{i-1,j+1} \\ &\quad + (j+1)r\widetilde{\mu}_{i,j} - C\widetilde{\mu}_{i-1,j}.\end{aligned} \tag{3.5.4}$$

Using the identity (3.5.1) of part (a) for $\widetilde{\mu}_{i,j+1}$ and $\widetilde{\mu}_{i-1,j+1}$, (3.5.4) may be reduced from 5 to 3 terms on the RHS. After some simplifying, we have

$$\begin{aligned}\widetilde{\mu}_{i+1,j+1} &= (i+j+1-A)\,\widetilde{\mu}_{i+1,j} + [(i-A)(-2\sqrt{\alpha+1}) + (ir+B) + (j+1)r]\,\widetilde{\mu}_{i,j} \\ &\quad + [(ir+B)(-2\sqrt{\alpha+1}) - C]\,\widetilde{\mu}_{i-1,j}.\end{aligned}$$

Finally, after substituting in the values for A, B, C, and r, we arrive at the result of part (b). □

Remark. By switching the priority from the linear factor $(x-r)$ to $(x-s)$, but otherwise exactly following the steps of the proof of part (b) of Theorem 3.5.1, we obtain a second 4-term recursion-like formula

$$\begin{aligned}\widetilde{\mu}_{i+1,j+1} &= [i+j-1+2\alpha-\sqrt{\alpha+1}]\,\widetilde{\mu}_{i,j+1} \\ &\quad + [(1-i-j)(\alpha+1) + (-3+i+3j+4\alpha)\sqrt{\alpha+1}]\,\widetilde{\mu}_{i,j} \\ &\quad + [(-2j+4)(\alpha+1)(\sqrt{\alpha+1}-1)]\,\widetilde{\mu}_{i,j-1} \qquad (i,j\in\mathbb{N}_0).\end{aligned} \tag{3.5.5}$$

We have little doubt that numerous recursion-like relationships exist for the adjusted moments. However, we will see in the next section that the three formulas given above in 3.5.1, 3.5.2, and 3.5.5, in conjunction with three initial values, $\widetilde{\mu}_{2,2}$, $\widetilde{\mu}_{1,2}$, and $\widetilde{\mu}_{2,1}$, will suffice to compute all of the adjusted moments.

3.6 Initial Moments

Theorem 3.6.1. *The adjusted moments for the X_2^I-Laguerre XOP are completely determined by the recursion-like formulas of Theorem 3.5.1, and the three initial values*

$$\widetilde{\mu}_{2,2} = 4\Gamma(1+\alpha),$$

$$\widetilde{\mu}_{1,2} = 4e^{-r}(-r)^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,-r), \text{ and}$$

$$\widetilde{\mu}_{2,1} = 4e^{-s}(-s)^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,-s),$$

where the Gamma function is given by $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$, and the incomplete Gamma function by $\Gamma(x,a) := \int_a^\infty t^{x-1}e^{-t}dt$ for $a > 0$.

Proof. Recall that $L_2^{\alpha-1}(-x) = \frac{1}{2}(x-r)(x-s)$. The first adjusted moment given above, $\widetilde{\mu}_{2,2}$, is the simplest to compute because of the complete cancellation of the linear factors $(x-r)$ and $(x-s)$ that occurs. We have

$$\widetilde{\mu}_{2,2} = \int_0^\infty (x-r)^2(x-s)^2\, W^\alpha(x)\, dx = \int_0^\infty (x-r)^2(x-s)^2\, \frac{x^\alpha e^{-x}}{(L_2^{\alpha-1}(-x))^2}\, dx$$
$$= 4\int_0^\infty x^\alpha e^{-x}\, dx = 4\Gamma(1+\alpha).$$

For $\widetilde{\mu}_{1,2}$ and $\widetilde{\mu}_{2,1}$, we rely on the result of Theorem 4.1 in [32], where it is established that

$$\int_0^\infty \frac{e^{-x}x^\beta}{(x+\alpha)} dx = e^\alpha\, E_{1+\beta}(\alpha)\, \Gamma(1+\beta), \quad \alpha > 0, \beta > -1, \quad \text{where}$$
$$E_a(x) = \int_1^\infty e^{-xt}t^{-a}dt = x^{a-1}\Gamma(1-a,x), \quad x > 0.$$

This result allows us to compute both

$$\widetilde{\mu}_{1,2} = \int_0^\infty (x-r)(x-s)^2\, W^\alpha(x)\, dx = 4\int_0^\infty \frac{x^\alpha e^{-x}}{(x-r)}\, dx$$
$$= 4e^{-r} E_{1+\alpha}(-r)\, \Gamma(1+\alpha) = 4e^{-r}(-r)^\alpha\Gamma(1+\alpha)\Gamma(-\alpha,-r),$$

and similarly, by symmetry,

$$\widetilde{\mu}_{2,1} = 4e^{-s}(-s)^\alpha\Gamma(1+\alpha)\Gamma(-\alpha,-s).$$

To see that these three adjusted moments are sufficient to start a recursion-like process that can be used to calculate all other moments, consider the four diagrams in Figures 3.1 and 3.2.

Figure 3.1 (a) represents the behavior of the 3-term recursion-like formula given in (3.5.1). Namely, if we know the values of any two of the three adjusted moments marked by the symbol *, the formula allows us to compute the third moment. (In this and the subsequent figures, the first subscript of the adjusted moments is indicated going down the left side of the figure, while the second subscript is given along the top.)

Figure 3.1 (b) represents the behavior of the 4-term recursion-like formula given in (3.5.5). That is, if we know the values of any three of the four adjusted moments marked by the symbol *, the formula allows us to compute the fourth moment.

And Figure 3.1 (c) represents the behavior of the 4-term recursion-like formula given in (3.5.2). As before, if we know the values of any three of the four adjusted moments marked by the symbol *, the formula allows us to compute the fourth moment.

Finally, Figure 3.2 demonstrates the first few steps in starting to compute all entries of the 2-D array of adjusted moments. The cells marked "A" represent the 3 adjusted moments $\widetilde{\mu}_{2,2}$, $\widetilde{\mu}_{1,2}$, and $\widetilde{\mu}_{2,1}$ calculated at the start of this theorem. Next, $\widetilde{\mu}_{1,1}$, marked with a "B", can be computed using the 3-term formula represented in Figure 3.1 (a), along with the known values of $\widetilde{\mu}_{1,2}$ and $\widetilde{\mu}_{2,1}$.

Then we find one of the two adjusted moments marked with a "C" by using the 4-term formula displayed in Figure 3.1 (b), and we get the other moment marked "C" from the other 4-term formula, as demonstrated in Figure 3.1 (c). To compute the adjusted moment marked "D", we once again rely on the 3-term formula, using the now-known values of $\widetilde{\mu}_{0,1}$, and $\widetilde{\mu}_{1,0}$.

Finally, the values of $\widetilde{\mu}_{0,2}$, and $\widetilde{\mu}_{2,0}$ are computed making use of the two 4-term formulas – one for each. Once the 3-by-3 sub-array at the upper left corner of the infinite array of adjusted moments has been filled out, it is clear that the remaining entries can all be computed using the trio of recursion-like formulas – (3.5.1), (3.5.2), and (3.5.5).

$\square$

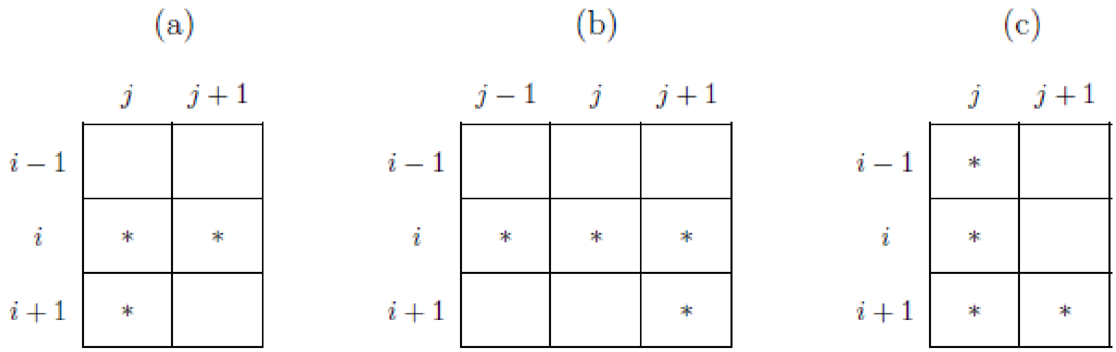

FIGURE 3.1. Behavior of recursion-like formulas

$i \backslash j$	0	1	2	$\cdots$
0	D	C	$\cdots$	$\cdots$
1	C	B	A	$\cdots$
2	$\vdots$	A	A	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

FIGURE 3.2. Filling out array of adjusted moments

CHAPTER FOUR

Exceptional Orthogonal Polynomials of Codimension One

4.1 Background Information

In this chapter, which comes from the author's work in [31], we use the Darboux transform and its relation to the exceptional Laguerre (Types I, II, and III) and Jacobi (Types I and II) polynomial systems to obtain the corresponding exceptional polynomials via the Gram-Schmidt method. In the case of classical orthogonal polynomials it is a well-established perspective to view the polynomials as the result of applying Gram-Schmidt to the *flag sequence* $\{1, x, x^2, \ldots\}$, as is outlined in [3, Chapter 1.3]. The author, in collaboration [32], used these ideas to derive two representations for the Type I X_1-Laguerre polynomials in terms of their moments by using determinants. An adaptation and generalization of this method leads us to our main result, which relies solely on the moment functions and modified weights of the exceptional orthogonal polynomial systems. Our results are universal in the sense that they can be applied to X_1 orthogonal polynomial systems (independent of being "Laguerre" or "Jacobi" specific). As we are restricting ourselves to the X_1 systems, we refer to both the Type I and II Jacobi systems as "Jacobi" because in the codimension one case, they are equivalent. Due to structural reasons, there are no X_1-Hermite polynomials [8].

Other representations of exceptional orthogonal polynomial systems involve Wronskian, and sometimes pseudo-Wronskian, determinants of classical orthogonal polynomials, see e.g. [5, 8].

The idea of this paper hinges on the following simple observations regarding the essential characteristics of exceptional orthogonal polynomials. First of all, the exclusion of eigenfunctions with certain degrees is caused by particular rational-function coefficients in the differential expression. When we apply the differential expression to eigenfunctions we must (at least) cancel any denominators introduced by these rational-function coefficients. This idea leads to what we call the *exceptional condition*, one of the key factors in the theory. In this paper we primarily consider X_1 polynomials, so that this denominator consists of a linear polynomial. We call the root of this polynomial the *exceptional root*, and denote it by ξ. Our work is suspended on the ansatz that writing the Taylor expansion of the exceptional polynomials around ξ inherits beneficial properties, see [32, Section 5]. There it was also noticed that adjusting moments by replacing the integrand x^l by $(x-\xi)^l$ drastically simplifies matters.

We first gather some preliminary information about the Darboux transform and its relations to exceptional orthogonal polynomials (Section 2). For subsequent comparison, we introduce the Jacobi and the Type III Laguerre cases in more detail. In Section 3, we focus on a universal expression for the exceptional condition as it was obtained recently in [7]. We write their condition in terms of the basic functions of the theory, and relate the condition to our examples. The flag sequence, which after the Gram-Schmidt algorithm returns the exceptional orthogonal polynomials, is the topic of Section 4. We state the determinantal representation in Theorem 4.5.1 of Section 5. Like in [32], we give the degree n exceptional polynomial in terms of a determinant of an $(n+1)\times(n+1)$-matrix. The first row of the matrix comes from the exceptional condition, while the second through last row contains certain adjusted moments. We then, in Section 6, present a recursion formula to compute these adjusted moments. We observe a curious fact: the moment representations for the Type I and II Laguerre polynomials only differ in the exceptional condition. The computation of the initial adjusted moments is addressed in Section 7. Then, in the final chapter (Chapter 5), we include some preliminary observations on the

Table 4.1. Other notation.

Symbol:	Definition:
$\eta(x)$	Polynomial function that occurs in the natural operator
$s(x)$	Linear function that occurs in the natural operator
ξ	The exceptional root; that is the root of $b(x)$ or, equivalently, the root of $\phi(x)$ or that of $\eta(x)$ after the α-shift
$\mathcal{E}_n$	Span of the first n exceptional orthogonal polynomials
$\mathcal{F}_n$	Set of polynomials of degree $\leq n$ satisfying the exceptional condition
v_k	Flag elements $v_k(x) = (x-\xi)^k$
$c_{n,i}$	Coefficients of the Taylor expansion $\widehat{y}_n(x) = \sum_{i=0}^{n} c_{n,i}(x-\xi)^i$ of the exceptional polynomial around the exceptional root
$\widetilde{\mu}_m$	Adjusted moments $\widetilde{\mu}_m = \int_I (x-\xi)^m \widehat{W}(x)dx$

flags and recursive moment formulas for X_m orthogonal polynomials when $m > 1$, mainly so as to indicate some difficulties we expect to encounter when extending determinantal representations to the higher co-dimension setting. For example, flags have been explored for $m = 1$ (see [15]), whereas the Darboux transform is used when $m > 1$.

4.1.1 Notation

Most of our notation leans on the standards used in the field of exceptional orthogonal polynomials. The general trend is that the classical objects are denoted by letters without "$\widehat{\ }$". For example, T^α represents the classical Laguerre differential expression/operator; $L_n^\alpha(x)$, the classical Laguerre polynomial of degree n; and $W^{(\alpha,\beta)}(x)$, the classical Jacobi weight. Their exceptional counterparts are denoted by the respective letters but with the "$\widehat{\ }$". Here a subscript is used to refer to the codimension and superscripts I through III specify with which type of exceptional

Laguerre we are dealing, e.g. $\widehat{T}_m^{I,\alpha}$ stands for the Type I X_m-Laguerre differential operator. When we talk more generally about a weight, differential expression or a general polynomial (and not specifically about either Laguerre or Jacobi) we do not include the parameters α and/or β. For example we use W, T or p_n for the classical objects, and $\widehat{W}$, $\widehat{T}$ or $\widehat{p}_n$ when we consider the exceptional counterparts. For the reader's convenience we include the other notation in Table 4.1.

4.2 The Darboux Transform

We first recall how the Darboux transform may be used to generate exceptional orthogonal polynomial systems. For further reading on the relationship of the Darboux transform and exceptional orthogonal polynomial systems, see [7,11,12,14], upon which the following exposition is based.

Suppose $T[y]$ is a second-order differential operator with rational coefficients; that is

$$T[y] = p(x)y''(x) + q(x)y'(x) + r(x)y(x)\,. \tag{4.2.1}$$

Define the following quasi-rational functions

$$P(x) = \exp\left(\int \frac{q(x)}{p(x)}\,dx\right)\,, \tag{4.2.2}$$

$$W(x) = \frac{P(x)}{p(x)}\,, \tag{4.2.3}$$

$$R(x) = r(x)W(x)\,. \tag{4.2.4}$$

By multiplying the eigenvalue equation $T[y] = \lambda y$ by $W(x)$, the Sturm-Liouville type equation

$$(Py')' + Ry = \lambda W y$$

is formed. Thus, we refer to $W(x)$ as the weight function associated with T.

For T and a quasi-rational function $\phi(x)$, $\phi(x)$ is called a *quasi-rational eigenfunction* if

$$T[\phi] = \lambda\phi \quad \lambda \in \mathbb{C}\,. \tag{4.2.5}$$

In order to create the operator associated with the exceptional orthogonal polynomials, we first use the following decomposition proposition to rewrite T as a composition of two first-order operators.

Proposition 4.2.1. *[7, Proposition 3.5] For a second-order differential operator $T[y]$ having rational coefficients, let $\phi(x)$ be a quasi-rational eigenfunction of T with eigenvalue λ, and let $b(x)$ be an arbitrary, non-zero rational function. Define rational functions*

$$w = \frac{\phi'}{\phi}, \tag{4.2.6}$$

$$\widehat{b} = \frac{p}{b}, \tag{4.2.7}$$

$$\widehat{w} = -w - \frac{q}{p} + \frac{b'}{b} \tag{4.2.8}$$

and first-order operators A and B by

$$A[y] = b(y' - wy) \quad \text{and} \quad B = \widehat{b}(y' - \widehat{w}y)\,. \tag{4.2.9}$$

With A and B as above, T has a rational factorization of the form $T = BA + \lambda$.

Using the factorization of T and the definitions of A and B given in Proposition 4.2.1, we will define a new operator, called the *partner operator*, to be $\widehat{T} := AB + \lambda$. The rational Darboux transformation maps T to $\widehat{T}$. Operator $\widehat{T}$ will also be a second-order differential operator with rational coefficients; that is

$$\widehat{T}[y] = p(x)y''(x) + \widehat{q}(x)y'(x) + \widehat{r}(x)y(x)\,. \tag{4.2.10}$$

Note that the coefficient of the second-order term is the same for both the original and partner operators. Additionally, another Sturm-Liouville type equation is induced.

Proposition 4.2.2. *[7, Proposition 3.6] Suppose that T and $\widehat{T}$ are second-order differential operators with rational coefficients which are related via a rational Darboux transformation. Then T and $\widehat{T}$ have the same second-order coefficients, while first-*

and zero-order coefficients $q, \widehat{q}, r, \widehat{r} \in \mathbb{Q}$*, and the quasi-rational weights* $W(x)$ *and* $\widehat{W}(x)$ *are related by*

$$\widehat{q} = q + p' - \frac{2pb'}{b}\,, \tag{4.2.11}$$

$$\widehat{r} = -p(\widehat{w}' + \widehat{w}^2) - \widehat{q}\widehat{w} \tag{4.2.12}$$

$$= r + q' + wp' - \frac{b'}{b}(q + p') + \left(2\left(\frac{b'}{b}\right)^2 - \frac{b''}{b} + 2w'\right)p\,, \text{ and} \tag{4.2.13}$$

$$\widehat{W} = \frac{pW}{b^2} = \frac{P(x)}{b^2}\,. \tag{4.2.14}$$

We are interested in the Darboux transform applied to classical Bochner systems; in particular, the systems of Laguerre and Jacobi. When T is defined to be one of these classical systems and ϕ is carefully chosen, the partner operator will be an exceptional polynomial system operator.

4.2.1 General X_m Expression

The moment representations will be approached in the general case (without reliance on the specifics of the Laguerre or Jacobi setting) and will rely solely on the information provided via the Darboux transform and Bochner systems. Therefore, the differential expression for an X_m orthogonal polynomial system will be defined by $\widehat{T}[\widehat{y}]$, with the weight function given by equation (4.2.14). Recall that ξ is the exceptional root. (That is, ξ is the root of the denominator function $b(x)$ of the weight $\widehat{W}(x)$.) Later, when we restrict ourselves to $m = 1$, $b(x)$ will be of degree two and have one repeated root.

Since we assume that T is a classical Bochner operator and ϕ has been carefully chosen to produce an X_m orthogonal polynomial operator, the differential equation $\widehat{\ell}[\widehat{y}] = \widehat{\lambda}\widehat{y}$ will be satisfied by a sequence of polynomials $\widehat{y} = \{\widehat{y}_n\}_{n \in \mathbb{N}_0 \setminus A}$, where $\widehat{y}_n$ is of degree n and A is a finite set of dimension m, and corresponding sequence of eigenvalues $\widehat{\lambda} = \{\widehat{\lambda}_n\}_{n \in \mathbb{N}_0 \setminus A}$. We mention that the eigenvalues for the classical orthogonal systems and exceptional orthogonal systems are *not* necessarily equal as shifting may occur under the Darboux transform. On an open interval, $I = (a, b)$, this sequence of polynomials will satisfy the orthogonality relation

$$\langle \widehat{y}_n, \widehat{y}_k \rangle_{\widehat{W}} = \int_I \widehat{y}_n \widehat{y}_k \widehat{W}\, dx = K_n \delta_{n,k} \tag{4.2.15}$$

where $\delta_{n,k}$ is the Kronecker-δ symbol (equal to 1 when $n = k$ and 0 when $n \neq k$). Of course, the interval $I = (0, \infty)$ for Laguerre and $I = (-1, 1)$ when we work with Jacobi systems.

4.2.2 X_m-Laguerre Expression

Here we provide a brief insight into the Type I, II, and III exceptional Laguerre orthogonal polynomial systems. A more rigorous look at the Darboux transform applied to the classical Laguerre expression may be found in [11]. Recall the classical Laguerre differential operator

$$T^{\alpha}[y] = xy'' + (-x + \alpha + 1)y', \tag{4.2.16}$$

and corresponding weight function

$$W^{\alpha}(x) = x^{\alpha} e^{-x}.$$

The classical Laguerre polynomials are shape-invariant under the factorizations:

$$\begin{aligned} T^{\alpha} &= B^{\alpha} A^{\alpha} \\ \widehat{T}^{\alpha+1} &= A^{\alpha} B^{\alpha} + 1, \end{aligned}$$

where

$$\begin{aligned} A^{\alpha}(y) &= y' \quad \text{and} \\ B^{\alpha}(y) &= xy' + (\alpha + 1 - x)y. \end{aligned}$$

The quasi-rational eigenfunctions and eigenvalues of $T^{\alpha}[y]$ are:

$$\begin{aligned} \phi_1(x) &= L_m^{\alpha}(x), & \lambda &= -m, \\ \phi_2(x) &= x^{-\alpha} L_m^{-\alpha}(x), & \lambda &= \alpha - m, \\ \phi_3(x) &= e^x L_m^{\alpha}(-x), & \lambda &= \alpha + 1 + m, \text{ and} \\ \phi_4(x) &= x^{-\alpha} e^x L_m^{\alpha}(-x), & \lambda &= m + 1, \end{aligned}$$

where $m \in \mathbb{N}_0$. The factorizations corresponding to each of these eigenfunctions have been studied, see [11, 17]. It has been shown that ϕ_1 with $m = 0$ corresponds to a state-deleting transformation and corresponds to the classical Laguerre polynomials. For $m > 0$, the eigenfunctions corresponding to ϕ_1 yield singular operators, which means that no new families of orthogonal polynomials arise. The family associated with ϕ_4 is state-adding and therefore, the resulting orthogonal polynomials are not of codimension m. The factorizations of ϕ_2 and ϕ_3 result in new orthogonal polynomials—in fact, these factorizations respectively produce the Type I and Type II exceptional orthogonal polynomials of codimension m. For further reading regarding the properties of the Type I and Type II exceptional Laguerre polynomial systems, see [27].

The Type III exceptional Laguerre polynomials $\{\widehat{L}^{III,\alpha}_{m,n}(x)\}_{n=0 \text{ or } n>m}$ are a class of Laguerre-type orthogonal polynomials which were extensively studied in [27]. We will focus on the Type III exceptional operator for several of our examples in this paper.

There is another rational factorization of the classical Laguerre expression $T^\alpha[\cdot]$, which yields the Type III second-order expression. Let

$$A^{III,\alpha}_m(y) = xL^{-\alpha}_m(-x)y' - (m+1)L^{-\alpha-1}_{m+1}(-x)y\,, \text{ and}$$
$$B^{III,\alpha}_m(y) = \frac{y'}{L^{-\alpha}_m(-x)}\,.$$

The classical Laguerre operator may be written as

$$T^\alpha = B^{III,\alpha}_m A^{III,\alpha}_m + m + 1\,,$$

and the Darboux transformation associated with the above factorizations yields the Type III operator

$$\widehat{T}^{III,\alpha}_m = A^{III,\alpha+1}_m B^{III,\alpha+1}_m + m - \alpha.$$

That is, explicitly, we have

$$\widehat{T}^{III,\alpha}_m[y] = -xy'' + \left(-1 + \alpha + x + 2x\frac{(L^{-\alpha-1}_m(-x))'}{L^{-\alpha-1}_m(-x)}\right)y' - \alpha y\,. \tag{4.2.17}$$

The corresponding weight for the Type III case is

$$\widehat{W}_m^{III,\alpha}(x) = \frac{x^\alpha e^{-x}}{\left(L_m^{-\alpha-1}(-x)\right)^2}\,. \tag{4.2.18}$$

The Type III eigenvalue equation will have orthogonal polynomial solutions on $(0,\infty)$ if and only if $-1 < \alpha < 0$. In fact, the associated Type III differential equation will have a polynomial solution $y(x) = \widehat{L}_{m,n}^{III,\alpha}(x)$ of degree n for $n = 0$ and for $n \geq m+1$; that is, solutions of degrees $\{1, 2, \ldots, m\}$ are missing.

We may write the Type III exceptional Laguerre polynomials using the Darboux transformation

$$\widehat{L}_{m,n}^{III,\alpha}(x) = \begin{cases} -A_m^{III,\alpha+1}[L_{n-m-1}^{\alpha+1}(x)], & n = m+1, m+2, \ldots \\ 1 & n = 0\,. \end{cases} \tag{4.2.19}$$

4.2.3 X_m-Jacobi Expression

We provide a brief outline of the Type I exceptional Jacobi orthogonal polynomial systems. A more rigorous look at the Darboux transform applied to the classical Jacobi expression may be found in [17]. For $\alpha, \beta > -1$,

$$T^{\alpha,\beta}[y] = (1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' \tag{4.2.20}$$

is the classical Jacobi differential operator. For

$$\alpha, \beta > -1, \quad \alpha + 1 - m - \beta \notin \{0, 1, \ldots, m-1\}\,, \quad \text{and} \quad \operatorname{sgn}(\alpha + 1 - m) = \operatorname{sgn}(\beta),$$

define

$$A_m^{\alpha,\beta}[y] = (1-x)P_m^{(-\alpha,\beta)}y' + (m-\alpha)P_m^{(-\alpha,\beta)}y\,,$$

$$B_m^{\alpha,\beta}[y] = \frac{(1+x)y' + (1+\beta)y}{P_m^{(-\alpha,\beta)}}\,,$$

where $P_m^{(\alpha,\beta)}$ is the classical Jacobi polynomial of degree m. It follows that

$$T^{\alpha,\beta}[y] = B_m^{\alpha,\beta}A_m^{\alpha,\beta}y - (m-\alpha)(m+\beta+1)y$$

and the exceptional Jacobi operator is defined as

$$\begin{aligned}\widehat{T}_m^{\alpha,\beta}[y] &= A_m^{\alpha+1,\beta-1} B_m^{\alpha+1,\beta-1} y - (m-\alpha-1)(m+\beta) y \\ &= T^{\alpha,\beta}[y] + (\alpha-\beta-m+1) m y - (\log P_m^{(-\alpha-1,\beta-1)})' \left(\beta(1-x)y + (1-x^2)y'\right)\end{aligned} \tag{4.2.21}$$

For $\alpha, \beta > -1$ and $n \geq m$, the X_m-Jacobi polynomial of degree n can be written as

$$\widehat{P}_{m,n}^{(\alpha,\beta)}(x) = \frac{(-1)^{m+1}}{\alpha+1+j} A_m^{\alpha+1,\beta-1} \left[P_j^{(\alpha+1,\beta-1)}(x)\right], \quad j = n-m \geq 0\,.$$

Note that the exceptional operator extends the classical Jacobi operator; that is

$$\widehat{T}_0^{\alpha,\beta}[y] = T^{\alpha,\beta}[y]\,.$$

The X_m-Jacobi polynomials $\{\widehat{P}_{m,n}^{(\alpha,\beta)}\}_{n\geq m}$ are orthogonal on (-1,1) with respect to the weight function

$$\widehat{W}_m^{\alpha,\beta}(x) = \frac{(1-x)^\alpha(1+x)^\beta}{\left(P_m^{(-\alpha-1,\beta-1)}(x)\right)^2}\,.$$

4.3 Exceptional Condition

The most noticeable difference between a classical orthogonal polynomial expression, as classified by Bochner, and the exceptional orthogonal polynomial expressions, is that the coefficient functions for the first- and zero-order terms are no longer polynomial. From Proposition 4.2.2, we see that the coefficient functions for any second-order partner operator formed via a rational Darboux transformation will have denominators containing powers of $b(x)$ and $\phi(x)$.

We turn our attention to our particular case, where we are working with $\widehat{T}$, an exceptional polynomial operator. By [7, Definition 7.1, ii-c], $f(x)p(x)\widehat{W}(x) \to 0$ at the endpoints of (a, b) for every polynomial $f(x)$. Consequently, the operator $\widehat{T}$ will be polynomially regular and thus, semi-simple [7, Remark 4.10]. In other words, since we seek polynomial solutions to the associated eigenvalue problem, these non-polynomial coefficients require a specific structure condition for the remaining polynomials; that is, "cancellation" must occur in order to have polynomial eigenfunctions. The coefficient functions which are non-polynomial form the *exceptional term*, and the specific structure induced on solutions is referred to as the *exceptional condition*. Polynomials of every degree cannot satisfy both the exceptional condition and form a maximal invariant subspace under $\widehat{T}$. Therefore, we do not have a full sequence of polynomial eigenfunctions for the exceptional operators—that is A is ensured to be non-empty.

In order to find the exceptional condition which characterizes the exceptional polynomial systems we introduce an additional setting in which we can consider the exceptional orthogonal polynomial operators. We say that for any two second-order operators having rational coefficients, T and $\widehat{T}$ are *gauge-equivalent* if there exists a rational function σ such that

$$\sigma T = \widehat{T}\sigma\,.$$

By [7, Proposition 2.5, Theorem 5.4], every exceptional operator will be gauge equivalent to a natural operator. A natural operator is a second-order operator of the form

$$py'' + \left(\frac{p'}{2} + s - \frac{2p\eta'}{\eta}\right)y' + \left(\frac{p\eta''}{\eta} + \left(\frac{p'}{2} - s\right)\frac{\eta'}{\eta}\right)y\,, \tag{4.3.1}$$

where p is a second-degree polynomial, η is a polynomial, and s is a linear function. It is the case that the polynomial p is the same regardless of whether $\widehat{T}$ is written in standard form (4.2.10) or in natural gauge form (4.3.1). Obviously, if the standard form of $\widehat{T}$ is known (that is, b is known), by setting the coefficients to be equal, one can find the polynomial s. Rather, we would like to approach the task of finding s without knowing b. To do this, we will utilize [7, Corollary 5.25] which states (in paraphrased form) that the exceptional term for $\widehat{T}$ in the natural gauge is given by

$$\frac{2p\eta' y' - \left(p\eta'' + \dfrac{p'\eta'}{2} - s\eta'\right)y}{\eta}\,. \tag{4.3.2}$$

Finding the linear polynomial s given the exceptional terms from the two representations of $\widehat{T}$ will be the focus of the following subsection.

4.3.1 Finding the Linear Polynomial s

We now turn our attention to finding the linear polynomial s, which is found in (4.3.1).

While it is clear that η should be the polynomial part of the quasi-rational eigenfunction ϕ (that initiates the Darboux transform), it seems less obvious what the function $s \in \mathcal{P}_1$ would be. Here $\mathcal{P}_1$ denotes the polynomials of degree less than or equal to one. Of course, one could extract s by comparing the coefficient of y' in $\widehat{T}$ to $\widehat{q}$ in (4.2.11) after having computed everything. But we found it beneficial to express s directly from η and the factorization gauge.

Lemma 4.3.1. *The linear polynomial* $s = q + \dfrac{p'}{2} + 2p\left(\dfrac{\eta'}{\eta} - \dfrac{b'}{b}\right)$.

From this formula, it is not immediately clear that $s \in \mathcal{P}_1$ in general. However, this is exactly the topic of [7, Section 5]. Below we include the values for s for the Type I, II, and III Laguerre and for the Jacobi case.

Proof. We set the coefficient of y' in (4.3.1) equal to $\widehat{q}$ in Proposition 4.2.2

$$-\frac{2\eta' p}{\eta} + \frac{p'}{2} + s = q + p' - \frac{2b'p}{b}.$$

Solving for s yields the lemma. □

Table 4.2 includes the choices of η and s for the three types of Laguerre systems. We took the liberty of already including all the shifts. As required, we observe that $s \in \mathcal{P}_1$ in all cases.

We exemplify how these expressions play out in the case of the Type III Laguerre system.

Table 4.2. Choices of η and s for the natural operator in equation (4.3.1).

XOP Type:	$\phi(x)$:	$\eta(x)$:	$b(x)$:	Shift:	$s(x)$:
Type I Lag.	$e^x L_m^\alpha(-x)$	$L_m^\alpha(-x)$	$L_m^\alpha(-x)$	$\alpha \mapsto \alpha - 1$	$x - \alpha - 1/2$
Type II Lag.	$x^{-\alpha} L_m^{-\alpha}(x)$	$L_m^{-\alpha}(x)$	$x L_m^{-\alpha}(x)$	$\alpha \mapsto \alpha + 1$	$x - \alpha - 1/2$
Type III Lag.	$x^{-\alpha} e^x L_m^{-\alpha}(-x)$	$L_m^{-\alpha}(-x)$	$x L_m^{-\alpha}(-x)$	$\alpha \mapsto \alpha + 1$	$x - \alpha - 1/2$
Jacobi	$(1-x)^{-\alpha} \cdot P_m^{(-\alpha,\beta)}(x)$	$P_m^{(-\alpha,\beta)}(x)$	$(1-x) \cdot P_m^{(-\alpha,\beta)}(x)$	$\alpha \mapsto \alpha + 1$ $\beta \mapsto \beta - 1$	$\beta - \alpha - (\alpha + \beta + 1)x$

Example. First recall that the coefficients of the classical Laguerre expression (4.2.16) are $p(x) = -x$, $q(x) = x - \alpha - 1$ and $r(x) \equiv 0$. The Type III Laguerre orthogonal polynomial system is derived from the classical Laguerre system by using the quasi-rational eigenfunction $\phi_4(x) = x^{-\alpha} e^x L_m^{-\alpha}(-x)$, so that $\widetilde{\eta}(x) := L_m^{-\alpha}(-x)$. In [27] we used the factorization gauge $b(x) = x L_m^{-\alpha}(-x)$. We also use the shift $\alpha \mapsto \alpha + 1$. This shift can be seen from the $\alpha + 1$ superscript in the partner operator

$$-\widehat{T}_m^{III,\alpha} = A_m^{III,\alpha+1} \circ B_m^{III,\alpha+1} + m - \alpha.$$

We compute

$$\frac{\widetilde{\eta}'(x)}{\widetilde{\eta}(x)} = \frac{(L_m^{-\alpha}(-x))'}{L_m^{-\alpha}(-x)} \qquad \text{and} \qquad \frac{b'(x)}{b(x)} = \frac{1}{x} + \frac{(L_m^{-\alpha}(-x))'}{L_m^{-\alpha}(-x)},$$

so that the expression in Lemma 4.3.1

$$q + \frac{p'}{2} + 2p\left(\frac{\widetilde{\eta}'}{\widetilde{\eta}} - \frac{b'}{b}\right) = x - \alpha - 1 - \frac{1}{2} - 2x\left(-\frac{1}{x}\right) = x - \alpha + \frac{1}{2}.$$

And after the shift $\alpha \mapsto \alpha + 1$ we obtain

$$\eta(x) = L_m^{-\alpha-1}(-x) \qquad \text{and} \qquad s(x) = x - \alpha - \frac{1}{2}.$$

We verify that the coefficient of y' is correct by comparing

$$\left(\frac{p'}{2} + s - \frac{2p\eta'}{\eta}\right) y' = \left(-\frac{1}{2} + x - \alpha - \frac{1}{2} - 2x\frac{(L_m^{-\alpha-1}(-x))'}{L_m^{-\alpha-1}(-x)}\right) y'$$

$$= \left(x - \alpha - 1 - 2x\frac{(L_m^{-\alpha-1}(-x))'}{L_m^{-\alpha-1}(-x)}\right) y'$$

with the X_m expression (4.2.17) for Type III Laguerre.

We also compute the coefficient of the zero-order term to ensure that this choice for s produces an equivalent expression in standard form. Using $\eta = L_m^{-\alpha-1}(-x)$ and the coefficient for y from (4.3.1), we have the following calculation, which relies on the fact that $\eta = L_m^{-\alpha-1}(-x)$ is a solution to the classical Laguerre differential equation $T[\eta] = m\eta$, where T is defined in (4.2.16):

$$\begin{aligned}\frac{\left(p\eta'' + \frac{p'\eta'}{2} - s\eta'\right)}{\eta} &= \frac{-x\left(L_m^{-\alpha-1}(-x)\right)'' + \left(-\frac{1}{2} - x + \alpha + \frac{1}{2}\right)\left(L_m^{-\alpha-1}(-x)\right)'}{L_m^{-\alpha-1}(-x)} \\ &= \frac{-x\left(L_m^{-\alpha-1}(-x)\right)'' + (-x+\alpha)\left(L_m^{-\alpha-1}(-x)\right)'}{L_m^{-\alpha-1}(-x)} \\ &= \frac{mL_m^{-\alpha-1}(-x)}{L_m^{-\alpha-1}(-x)} \\ &= m.\end{aligned}$$

Note that the coefficient of the zero-order term we found using the natural operator and s differs from the coefficient in the standard form given in (4.2.17) by a constant. This is a result of the shifting of eigenvalues which occurs when the Darboux transformation is applied. We are not concerned by this discrepancy as the structure of the operator remains the same.

Example. The Jacobi expression, $\phi(x) = (1-x)^{-\alpha}P_m^{(-\alpha,\beta)}(x)$, is the quasi-rational eigenfunction. Using Lemma 4.3.1 together with the function $\eta(x) = P_m^{(-\alpha,\beta)}(x)$, the gauge $b(x) = (1-x)P_m^{(-\alpha,\beta)}(x)$ and the shifts $\alpha \mapsto \alpha+1$ and $\beta \mapsto \beta - 1$ yields

$$s(x) = \beta - \alpha - (\alpha+\beta+1)x.$$

We verify that the coefficient of y' is correct by computing

$$\begin{aligned}\left(\frac{p'}{2} + s - \frac{2p\eta'}{\eta}\right)y' &= \left(-x + \beta - \alpha - (\alpha+\beta+1)x - 2(1-x^2)\frac{\left(P_m^{(-\alpha-1,\beta-1)}(x)\right)'}{P_m^{(-\alpha-1,\beta-1)}(x)}\right)y' \\ &= \left(\beta - \alpha - (\alpha+\beta+2)x - 2x\frac{\left(P_m^{(-\alpha-1,\beta-1)}(x)\right)'}{P_m^{(-\alpha-1,\beta-1)}(x)}\right)y'\end{aligned}$$

with the X_m expression for Jacobi given by (4.2.20) and (4.2.21).

4.4 *The Flag*

In this section, we characterize the subspace spanned by the first n of the X_1-exceptional orthogonal polynomials as those polynomials satisfying the exceptional condition

$$\left[2p\eta' y' - \left(p\eta'' + \frac{p'\eta'}{2} - s\eta'\right) y\right]\Big|_{x=\xi} = 0\,. \tag{4.4.1}$$

Recall that for the exceptional orthogonal systems of codimension one, ξ represents the root of η (that is, the exceptional root). The first m X_1-exceptional orthogonal polynomials will be those of degree less than or equal to m excluding degree 0 in the case of the Type I and Type II X_1-Laguerre or X_1-Jacobi polynomial systems. For the Type III X_1-Laguerre polynomials system, the first m polynomials will be those of degree 0 or between 2 and m, inclusively.

In preparation for Lemma 4.4.1 below, we present two definitions. The definitions, as given, are directly applicable for the Type I and II Laguerre and Jacobi systems. We abuse notation slightly for the Type III Laguerre case by using $\widehat{y}_1$ to represent $\widehat{y}_0$. Recall that $\widehat{y}_n$ is defined to have degree n and for the Type III Laguerre polynomials, we have a polynomial $\widehat{y}_0$, but not $\widehat{y}_1$. Let $\mathcal{P}_n$ denote the set of polynomials of degree less than or equal to n and define the span of the first n exceptional orthogonal polynomials

$$\mathcal{E}_n := \operatorname{span}\{\widehat{y}_j : j = 1, \dots, n\}\,,$$

where $\widehat{y}_j$ is defined as in Table 4.3.

Further we define

$$\mathcal{F}_n := \{p \in \mathcal{P}_n : p \text{ satisfies } (4.4.1)\}\,.$$

Lemma 4.4.1. *The sets* $\mathcal{E}_n = \mathcal{F}_n$ *for all* $n \in \mathbb{N}$*.*

The proof is analogous to the proof for [32, Lemma 2.1], but as in [16, Proposition 5.3], the exceptional condition is now replaced by a universal one. We choose to include the argument for the convenience of the reader.

Table 4.3. In the definition of $\mathcal{E}_n = \text{span}\,\{\widehat{y}_j : j = 1, \dots, n\}$ we have $\widehat{y}_j$.

XOP Family:	Type:	$\widehat{y}_j$:
Exceptional Laguerre	I	$\widehat{L}^{I,\alpha}_{1,j}$
	II	$\widehat{L}^{II,\alpha}_{1,j}$
	III	$\begin{cases} \widehat{L}^{III,\alpha}_{1,j} & j \geq 2 \\ 1 & j = 1 \end{cases}$
Exceptional Jacobi		$\widehat{P}^{(\alpha,\beta)}_{1,j}$

Proof. Since $\mathcal{E}_n$ and $\mathcal{F}_n$ are clearly vector spaces of equal dimension n, it suffices to show that $\mathcal{E}_n \subseteq \mathcal{F}_n$.

To see this, take $f \in \mathcal{E}_n$. Then f is a linear combination of the first n exceptional polynomials. So $f \in \mathcal{P}_n$. Further, $\widehat{T}[f]$ is polynomial, and so is the expression (4.4.1) for $y = f$. It follows that $f \in \mathcal{F}_n$. □

With the exceptional root ξ, we define degree k polynomials

$$v_k(x) := \begin{cases} \widehat{y}_1(x) & k = 1 \\ (x-\xi)^k & k \geq 2 \end{cases} \tag{4.4.2}$$

where $\widehat{y}_1(x)$ is given in Table 4.3.

Lemma 4.4.2. *The sequence of polynomials* $\{v_1, v_2, v_3, \dots\}$ *forms a flag for* $\widehat{T}$.

Proof. Our definition of v_k ensures that the polynomials have the appropriate degrees. That is, for Types I and II X_1-Laguerre and X_1-Jacobi systems, the constant polynomial is missing, and for Type III Laguerre the linear polynomial is excluded.

In virtue of the previous lemma, it suffices to prove that all v_k satisfy the exceptional condition (4.4.1). This follows immediately for the first flag element, as $\widehat{y}_1 = v_1$ is the first exceptional polynomial and hence an eigenfunction of $\widehat{T}$.

For $k \geq 2$, we recall that $v_k(x) := (x-\xi)^k$. In particular, we have $v_k(\xi) = 0$ so that the second term of (4.4.1) vanishes. On the left-hand side of (4.4.1) we are left with $2p(\xi)\eta'(\xi)v_k'(\xi)$. But we also have $v_k'(\xi) = k(x-\xi)^{k-1}\big|_{x=\xi} = 0$, because $k \geq 2$. So the exceptional condition (4.4.1) is satisfied and $\mathcal{F}_n = \text{span}\{v_1, v_2, v_3, \ldots, v_n\}$. $\square$

4.5 Determinantal Representations

In this section, we provide the details for finding the determinantal representations for the Type II and III X_1-Laguerre and X_1-Jacobi orthogonal polynomial systems. In the case of the Type I X_1-Laguerre polynomials, we do confirm that our results agree with [32] and refer the reader to [32] for the details of that particular case. It is the case that η will have one and only one exceptional root, ξ. The exceptional condition as discussed in Sections 3 and 4 and in [7, Corollary 5.25] reduces to (4.4.1).

To find the first row of entries in the determinantal representation, we follow the methods outlined in [32] and use the ansatz for degree n, $n \geq 2$, to write the exceptional polynomial

$$\widehat{y}_n(x) := \sum_{i=0}^{n} c_{n,i}(x-\xi)^i. \tag{4.5.1}$$

Note that $\widehat{y}_n(x)$ may be any X_1-Laguerre-type or X_1-Jacobi polynomial of degree n. We fill in specific details for each case below.

Then

$$\widehat{y}_n{}'(x) := \sum_{i=1}^{n} i c_{n,i}(x-\xi)^{i-1}. \tag{4.5.2}$$

and $\widehat{y}_n(\xi) = c_{n,0}$ and $\widehat{y}_n{}'(\xi) = c_{n,1}$.

In order to obtain the determinantal representation we notice that the coefficients $c_{n,i}$, $i = 0, 1, \ldots, n$, are given as the unique solution of a system of $n+1$ linear equations with matrix form $Ac = b$. The objects A, c and b are given below.

Table 4.4. Square of the norms of the X_1 polynomials.

XOP Family:	Type:	K_n :
Exceptional Laguerre	I	$\frac{(\alpha+n)\Gamma(\alpha+n-1)}{(n-1)!}$ for $n \geq 1$
	II	$\frac{(\alpha+n-1)\Gamma(\alpha+n+1)}{(n-1)!}$ for $n \geq 1$
	III	$\begin{cases} \frac{n\Gamma(n+\alpha)}{(n-2)!} & n \geq 2 \\ \frac{\Gamma(\alpha+1)\Gamma(-\alpha)}{\Gamma(1-\alpha)} & n = 0 \end{cases}$
Exceptional Jacobi		$\frac{2^{\alpha+\beta+1}(\alpha+n)(\beta+n)\Gamma(\alpha+n)\Gamma(\beta+n)}{4(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n-1)\Gamma(n)\Gamma(\alpha+\beta+n)}$

To this end we define the *adjusted moments*

$$\widetilde{\mu}_m := \int_I (x-\xi)^m \widehat{W}(x)dx \tag{4.5.3}$$

and the vectors

$$c := (c_{n,0}, c_{n,1}, \ldots, c_{n,n})^\top \in \mathbb{R}^{n+1} \quad \text{and} \quad b := (0, \ldots, 0, K_n)^\top \in \mathbb{R}^{n+1}.$$

The constant $K_n := \langle \widehat{p}_n, \widehat{p}_n \rangle$ determines the normalization of the exceptional polynomials.

Remark. For the reader's reference we also include the square, K_n, of the norms for each of the exceptional sequences in Table 4.4. The norms for the X_m-Laguerre polynomials are found in [27]; for Jacobi, see [10].

Theorem 4.5.1. *The X_1 orthogonal polynomials have the determinantal representation formula*

$$\widehat{y}_n(x) = \frac{1}{\det A} \sum_{i=0}^{n} (\det A_i)(x-\xi)^i \tag{4.5.4}$$

$$= \frac{K_n}{\det A} \begin{vmatrix} & & \textit{First } n \textit{ rows of the matrix } A & & \\ 1 & (x-\xi) & (x-\xi)^2 & \ldots & (x-\xi)^n \end{vmatrix} \tag{4.5.5}$$

where the $(n+1)\times(n+1)$*-matrix* A *is given by*

$$A = \begin{bmatrix} p(\xi)\eta''(\xi) + \frac{p'(\xi)\eta'(\xi)}{2} - s(\xi)\eta'(\xi) & 2p(\xi)\eta'(\xi) & 0 & \dots & 0 \\ c_{1,0}\widetilde{\mu}_0 + c_{1,1}\widetilde{\mu}_1 & c_{1,0}\widetilde{\mu}_1 + c_{1,1}\widetilde{\mu}_2 & c_{1,0}\widetilde{\mu}_2 + c_{1,1}\widetilde{\mu}_3 & \dots & c_{1,0}\widetilde{\mu}_n + c_{1,1}\widetilde{\mu}_{n+1} \\ \widetilde{\mu}_2 & \widetilde{\mu}_3 & \widetilde{\mu}_4 & \dots & \widetilde{\mu}_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \widetilde{\mu}_n & \widetilde{\mu}_{n+1} & \widetilde{\mu}_{n+2} & \dots & \widetilde{\mu}_{2n} \end{bmatrix},$$

the adjusted moments are defined to be

$$\widetilde{\mu}_m = \int_I (x-\xi)^m \widehat{W}(x)\,dx,$$

and where the matrix A_k *is obtained from* A *by replacing the* $(k+1)$*-st column with the vector* b*.*

In Section 6, we work out recursion formulas for the adjusted moments, $\widetilde{\mu}_k$, of each X_1 family. In addition, we compute $L_{1,2}^{III,\alpha}(x)$ as an example.

In Table 4.5 we present the specifics for the matrix corresponding to each of the exceptional cases, noting that the constants $c_{1,0}$ and $c_{1,1}$ are easily obtained by comparing the formula (4.5.1) with the table of $v_1 = \widehat{y}_1$ from Section 4.

Table 4.5: The values of the constants $c_{1,0}$ and $c_{1,1}$ in the matrix A. At the same time $c_{1,0}$ and $c_{1,1}$ are also the coefficients of $\widehat{y}_1(x)$, which is defined in (4.5.1).

XOP Family:	Type:	$c_{1,0}$:	$c_{1,1}$:
Exceptional Laguerre	I	1	1
	II	1	1
	III	1	0
Exceptional Jacobi		$\frac{\alpha+\beta}{\beta-\alpha}$	$\frac{1}{2}$

Proof of Theorem 4.5.1. We fill the matrix A row-wise.

Substituting the ansatz (4.5.1) and its derivative (4.5.2) into the exceptional equation (4.4.1), leads to

$$\left[2p\eta' c_{n,1} - \left(p\eta'' + \frac{p'\eta'}{2} - s\eta'\right) c_{n,0}\right]\Bigg|_{x=\xi} = 0\,.$$

We collect this information in the first row of the determinantal representation. Since our matrix equation has the form $Ac = b$, the first row of the matrix A now reads

$$\begin{bmatrix} p(\xi)\eta''(\xi) + \dfrac{p'(\xi)\eta'(\xi)}{2} - s(\xi)\eta'(\xi) & 2p(\xi)\eta'(\xi) & 0 & \dots & 0 \end{bmatrix}.$$

The other n rows come from the orthogonality relations (4.2.15). These conditions inform us that not only $\widehat{y}_n \perp \widehat{y}_k$ for $n \neq k$, but also that $\widehat{y}_n \perp \mathcal{E}_m$ for $m < n$. Since $v_m \in \mathcal{E}_m$, the orthogonality conditions imply

$$\begin{aligned} \langle \widehat{y}_n, v_k \rangle_{\widehat{W}} &= 0 \quad \text{for } 1 \leq k < n, \text{ and} \\ \langle \widehat{y}_n, \widehat{y}_n \rangle_{\widehat{W}} &= K_n. \end{aligned}$$

The second row of the matrix A is obtained by substitution of

$$v_1(x) = \widehat{y}_1 = c_{1,0} + c_{1,1}(x - \xi)$$

and the ansatz (4.5.1) into this orthogonality relation. Using v_1 and the adjusted moments (4.5.3), we compute

$$\begin{aligned} 0 = \langle \widehat{y}_n, v_1 \rangle_{\widehat{W}} &= \sum_{i=0}^{n} c_{n,i} \langle (x-\xi)^i, v_1 \rangle \\ &= \sum_{i=0}^{n} c_{n,i} \left[\langle (x-\xi)^i, c_{1,0} \rangle + c_{1,1} \langle (x-\xi)^i, (x-\xi) \rangle \right] \\ &= \sum_{i=0}^{n} c_{n,i} \left[c_{1,0} \widetilde{\mu}_i + c_{1,1} \widetilde{\mu}_{i+1} \right]\,. \end{aligned}$$

Thus, the second row of the determinantal representation is

$$\begin{bmatrix} c_{1,0}\widetilde{\mu}_0 + c_{1,1}\widetilde{\mu}_1 & c_{1,0}\widetilde{\mu}_1 + c_{1,1}\widetilde{\mu}_2 & \dots & c_{1,0}\widetilde{\mu}_n + c_{1,1}\widetilde{\mu}_{n+1} \end{bmatrix}.$$

For $2 \leq k \leq n$ the $(k+1)$-st row of matrix A is found by analogy. Namely, recalling the definition of $v_k = (x-\xi)^k$ for $k \geq 2$, we compute

$$\begin{aligned} 0 = \langle \widehat{y}_n, v_k \rangle_{\widehat{W}} &= \sum_{i=0}^{n} c_{n,i} \langle (x-\xi)^i, v_k \rangle_{\widehat{W}} \\ &= \sum_{i=0}^{n} c_{n,i} \widetilde{\mu}_{i+k} \,. \end{aligned}$$

Thus, row $l = k+1$, $3 \leq l \leq n+1$, is given by

$$[\widetilde{\mu}_k \quad \widetilde{\mu}_{k+1} \quad \dots \quad \widetilde{\mu}_{n+k}] \qquad \text{or}$$

$$[\widetilde{\mu}_{l-1} \quad \widetilde{\mu}_{l} \quad \dots \quad \widetilde{\mu}_{n+l-1}] \,.$$

Equation (4.5.4) now follows from Cramer's rule, and (4.5.5) comes about from the co-factor definition of determinants, upon expanding (4.5.5) along the last row. □

4.6 Recursion Relations for the Adjusted Moments

The exceptional polynomials $\widehat{y}_n$ may be represented using the adjusted moments $\widetilde{\mu}_k$. Thus, we can develop a recursive formula for the moments via the three-term recursion relation associated with the polynomial sequence. It is remarkable that these adjusted moments follow three-term recurrence relations as the exceptional polynomials themselves follow a five-term recurrence relation at best. In Table 4.6 we present the recursive formulas for all X_1 orthogonal polynomial systems. We include the proofs for both the Type III Laguerre and Jacobi moments. The proofs follow in analogy to the recursive formula of the Type I moments found in [32].

Prior to stating Theorem 4.6.1 we note that to simplify notation, we allow $\widehat{W}$ and I to respectively represent the appropriate weight function and interval of orthogonality pertaining to each exceptional system.

Table 4.6. Recursion formulas for X_1 moments.

XOP Family:	Type:	Recursion Formula:
Exceptional Laguerre	I	$\widetilde{\mu}_{k+2} = (2\alpha + k)\widetilde{\mu}_{k+1} + \alpha(1-k)\widetilde{\mu}_k$
	II	$\widetilde{\mu}_{k+2} = (2\alpha + k)\widetilde{\mu}_{k+1} + \alpha(1-k)\widetilde{\mu}_k$
	III	$\widetilde{\mu}_{k+2} = k\widetilde{\mu}_{k+1} - \alpha(1-k)\widetilde{\mu}_k$
Exceptional Jacobi		$\widetilde{\mu}_{k+2} = \left[\frac{(2-\alpha-\beta-2k)\xi+\beta-\alpha}{\alpha+\beta+k}\right]\widetilde{\mu}_{k+1} + \left[\frac{(k-2)(1-\xi^2)}{\alpha+\beta+k}\right]\widetilde{\mu}_k$

Theorem 4.6.1. *For an X_1 orthogonal polynomial sequence satisfying*

$$a_2y'' + a_1y' + a_0y = \lambda y$$

the adjusted moments $\widetilde{\mu}_k = \int_I (x-\xi)^k \widehat{W}(x)\,dx$ satisfy the recursion formulas for $k \in \mathbb{N}_0$:

$$\widetilde{\mu}_{k+2} = -((k+1)r_2 + s_1)^{-1}\left[((k+1)r_0 + s_{-1})\widetilde{\mu}_k + ((k+1)r_1 + s_0)\widetilde{\mu}_{k+1}\right],$$

where $a_2(x) = \sum_{\ell=0}^{2} r_\ell\,(x-\xi)^\ell$ and $a_1(x) = \sum_{m=-1}^{1} s_m\,(x-\xi)^m$.

Specifically, for each of the exceptional Laguerre and Jacobi families, the recursion formulas are provided in Table 4.6. The initial moments with which to begin the recursion are provided in Section 7.

Remark. The details of the recursion formula of Theorem 4.6.1 are provided below and the details of the initial moments are given in Section 7. Before the proof, we provide two critical observations. First, note that for functions f and g, which are smooth on I, the associated moment functionals satisfy:

$$\left\langle \widehat{W}', f \right\rangle = -\left\langle \widehat{W}, f' \right\rangle \text{ and } \left\langle g\widehat{W}, f \right\rangle = \left\langle \widehat{W}, fg \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ represents the inner product with respect to the Lebesgue measure on I.

Second, the second order linear operator given by

$$\ell[y] = a_2y'' + a_1y' + a_0y$$

may be written as a symmetry equation

$$a_2 y' + (a_2' - a_1)y = 0$$

that is solvable by the weight function $\widehat{W}$ to which the associated eigenfunctions are orthogonal.

Combining these two notes, we observe that

$$\begin{aligned}\left\langle a_2\widehat{W}', (x-\xi)^k\right\rangle = \left\langle \widehat{W}', a_2(x-\xi)^k\right\rangle &= -\left\langle \widehat{W}, \left(a_2(x-\xi)^k\right)'\right\rangle \\ &= -k\left\langle \widehat{W}, a_2(x-\xi)^{k-1}\right\rangle - \left\langle \widehat{W}, a_2'(x-\xi)^k\right\rangle .\end{aligned}$$

Therefore, for $k \in \mathbb{N}$,

$$\begin{aligned}0 &= \left\langle a_2\widehat{W}' + (a_2' - a_1)\widehat{W}, (x-\xi)^k\right\rangle \\ &= \left\langle a_2\widehat{W}', (x-\xi)^k\right\rangle + \left\langle a_2'\widehat{W}, (x-\xi)^k\right\rangle - \left\langle a_1\widehat{W}, (x-\xi)^k\right\rangle \\ &= -k\left\langle \widehat{W}, a_2(x-\xi)^{k-1}\right\rangle - \left\langle a_1\widehat{W}, (x-\xi)^k\right\rangle . \qquad (4.6.1)\end{aligned}$$

Proof. Using equation (4.6.1), we aim to prove a general recursion formula. First rewrite the coefficients a_2 and a_1 from the differential expression to be in terms of powers of $(x-\xi)$. In the case of the X_1-Laguerre families, a_2 is of degree 1; for the X_1-Jacobi family, a_2 is of degree 2. Therefore, we have

$$a_2(x) = \sum_{\ell=0}^{2} r_\ell (x-\xi)^\ell,$$

where the coefficients r_ℓ are appropriately chosen as indicated below:

$$a_2(x) = \begin{cases} -(x-\xi) - \xi & \text{Types I, II, III } X_1\text{-Laguerre} \\ -(x-\xi)^2 - 2\xi(x-\xi) + (1-\xi^2) & X_1\text{-Jacobi}\,. \end{cases} \qquad (4.6.2)$$

Since a_1 may be written as $p' + q + \frac{2b'p}{b}$, using a degree argument, a_1 may be written as

$$a_1(x) = \sum_{m=-1}^{1} s_m (x-\xi)^m,$$

where the coefficients s_m are appropriately chosen as indicated below:

$$a_1(x) = \begin{cases} (x-\xi) + (1+2\xi) + 2\xi(x-\xi)^{-1} & \text{Type I } X_1\text{-Lag.} \\ (x-\xi) - (3+2\xi) + 2\xi(x-\xi)^{-1} & \text{Type II } X_1\text{-Lag.} \\ (x-\xi) + 2\xi(x-\xi)^{-1} & \text{Type III } X_1\text{-Lag.} \\ -(\alpha+\beta)(x-\xi) + (2-\alpha-\beta)\xi + \beta\alpha + 2(\xi^2-1)(x-\xi)^{-1} & X_1\text{-Jacobi.} \end{cases} \tag{4.6.3}$$

Substituting a_2 and a_1 into (4.6.1), rearranging, and collecting coefficients produces

$$0 = -(kr_0 + s_{-1})\widetilde{\mu}_{k-1} - (kr_1 + s_0)\widetilde{\mu}_k - (kr_2 + s_1)\widetilde{\mu}_{k+1} \quad (\text{for } k \in \mathbb{N})\,.$$

In other words, for $k \in \mathbb{N}$,

$$\widetilde{\mu}_{k+1} = -(kr_2 + s_1)^{-1}\left((kr_0 + s_{-1})\widetilde{\mu}_{k-1} + (kr_1 + s_0)\widetilde{\mu}_k\right)\,. \tag{4.6.4}$$

Shifting $k \mapsto k+1$, we obtain the result. □

Example. It is a short exercise, using $\xi = \alpha$ and equations (4.6.2) and (4.6.3), for the reader to see that (4.6.4) simplifies to the formulas given in Table 4.6.

We will verify for $n = 2$, that the polynomials in Theorem 4.5.1 indeed agree with (4.2.19).

For $n = 2$, recall that $c_{1,0} = 1$ and $c_{1,1} = 0$ as in Table 5. Therefore,

$$L_{1,2}^{III,\alpha}(x) = \begin{vmatrix} 0 & 2\alpha & 0 \\ \widetilde{\mu}_0 & \widetilde{\mu}_1 & \widetilde{\mu}_2 \\ 1 & (x-\alpha) & (x-\alpha)^2 \end{vmatrix}$$

$$= 2\alpha\widetilde{\mu}_2 - 2\alpha\widetilde{\mu}_0(x-\alpha)^2.$$

Using the recursion formula in part (a) of Theorem 4.6.1 with $k = 0$,

$$\widetilde{\mu}_0 = \frac{\widetilde{\mu}_2}{-\alpha}\,,$$

we have, up to normalization,

$$L_{1,2}^{III,\alpha}(x) = 2\widetilde{\mu}_2\left(x^2 - 2\alpha x + \alpha(\alpha+1)\right).$$

This is in agreement with the polynomial given in (4.2.19) since they both span the same eigenspace.

Remark 4.6.2. Although different in many ways, the moment representations for the Type I and II Laguerre polynomials *only* differ in the exceptional condition (up to normalization, see Table 4.4). All of the rows in the matrix of Theorem 4.5.1 except the first, the moment recursion formulas in Table 4.6, and the initial moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ in Table **??** turned out completely identical for the Type I and II Laguerre polynomials.

Remark 4.6.3. There are additional ways to compute the moments of the X_1 polynomial families. In particular, generating functions may be used. This method may be seen for the Type I Laguerre family in [32].

4.7 Initial Moments

Lemma 4.7.1. *The initial moments for the X_1-exceptional orthogonal polynomial systems of Laguerre and Jacobi are given in Table* **??***, where the Gamma function is given by*

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$$

and the incomplete Gamma function by

$$\Gamma(a, x) := \int_x^\infty t^{a-1} e^{-t}\, dt \text{ for } x > 0\,.$$

Further we use the notation

$$J_1 = \frac{-1}{(\alpha+1)(\alpha+\beta)}\, F_1\left(1, -\beta, 1, \alpha+2; -1, \frac{\beta-\alpha}{\alpha+\beta}\right)$$

as well as

$$J_2 = \frac{-1}{(\beta+1)(\alpha+\beta)}\, F_1\left(1, -\alpha, 1, \beta+2; -1, \frac{\alpha-\beta}{\alpha+\beta}\right),$$

where $F_1(\cdot)$ denotes the first Appell hypergeometric series.

In the case of the Type I Laguerre polynomial family, the proof has been published in [32, Theorem 4.1]. We will prove that the moments given in Table **??** are correct for the Type III X_1-Laguerre and X_1-Jacobi polynomial families. The

Table 4.7. The values of the initial moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$.

XOP Family:	Type:	$\widetilde{\mu}_0$:	$\widetilde{\mu}_1$:
Exceptional Laguerre	I	$\Gamma(\alpha) - 2e^{\alpha}\alpha^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,\alpha)$	$e^{\alpha}\alpha^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,\alpha)$
	II	$\Gamma(\alpha) - 2e^{\alpha}\alpha^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,\alpha)$	$e^{\alpha}\alpha^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,\alpha)$
	III	$\frac{-\Gamma(\alpha+1)}{\alpha}$	$e^{-\alpha}(-\alpha)^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,-\alpha)$
Exceptional Jacobi		$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha+\beta)}{2\alpha\beta\Gamma(\alpha+\beta+2)} + \frac{(J_1+J_2)(2\alpha\beta-\alpha-\beta)}{\alpha\beta}$	$\frac{4}{\beta-\alpha}(J_1+J_2)$

proof for the Type II X_1-Laguerre family follows in analogy to the other Laguerre types.

Type III X_1-Laguerre Proof. We prove that the expressions given in Table 4.7 are the moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ associated with the Type III X_1-Laguerre expression. Set

$$E_a(x) = \int_1^{\infty} e^{-xt}t^{-a}\,dt\,, \qquad \text{where} \qquad x > 0\,.$$

Recalling the definitions of $\Gamma(x)$ and $\Gamma(a,x)$, these functions and the exponential integral function, E_a, are related via

$$E_a(x) = x^{a-1}\Gamma(1-a,x)\,.$$

In addition,

$$(a-1)E_a(x) = e^{-x} - xE_{a-1}(x)\,.$$

Following a chain of manipulations, which may be found in [32], we note the following relation:

$$\int_0^{\infty} \frac{e^{-x}x^{\beta}}{x-\alpha}dx = e^{-\alpha}E_{1+\beta}(-\alpha)\Gamma(1+\beta)\,, \qquad \text{for } \alpha > 0\,,\ \beta > -1\,.$$

As a consequence, we obtain the following expression for the first adjusted moment, $\widetilde{\mu}_1 = \int_0^{\infty} \frac{x^{\alpha}e^{-x}}{x-\alpha}\,dx$:

$$\begin{aligned}\widetilde{\mu}_1 &= e^{-\alpha}E_{1+\alpha}(-\alpha)\Gamma(1+\alpha)\\ &= e^{-\alpha}(-\alpha)^{\alpha}\Gamma(-\alpha,-\alpha)\Gamma(1+\alpha)\,.\end{aligned}$$

Notice that

$$\widetilde{\mu}_2 = \int_0^{\infty}(x-\alpha)^2\frac{x^{\alpha}e^{-x}}{(x-\alpha)^2}\,dx = \int_0^{\infty} x^{\alpha}e^{-x}\,dx = \Gamma(\alpha+1)\,.$$

To finish the proof, we set $k = 0$, use the recursion formula in Table 4.6, and solve for $\widetilde{\mu}_0$ to show $\widetilde{\mu}_0 = \frac{-\Gamma(\alpha+1)}{\alpha}$. □

X_1-Jacobi Proof. We prove that the expressions given in Table 4.7 are the moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ associated with the X_1-Jacobi expression. Per (4.5.3),

$$\widetilde{\mu}_1 := \int_I (x - \xi)\widehat{W}(x)dx.$$

For the X_1-Jacobi system, $\xi = \frac{\alpha+\beta}{\beta-\alpha}$, and

$$\widehat{W}_1^{\alpha,\beta}(x) = \frac{(1-x)^\alpha(1+x)^\beta}{\left(P_1^{(-\alpha-1,\beta-1)}(x)\right)^2}.$$

Substituting these into the equation for $\widetilde{\mu}_1$, along with the classical Jacobi polynomial in the denominator of the weight function, we find that

$$\widetilde{\mu}_1 = \frac{4}{\beta-\alpha}\int_{-1}^{1} \frac{(1-x)^\alpha(1+x)^\beta}{(\beta-\alpha)x-\alpha-\beta}dx.$$

We recast this integral as the sum of two integrals, so that now

$$\widetilde{\mu}_1 = \frac{4}{\beta-\alpha}[J_1 + J_2],$$

where

$$J_1 = \int_0^1 \frac{(1-x)^\alpha(1+x)^\beta}{(\beta-\alpha)x-\alpha-\beta}dx,$$

and

$$J_2 = \int_0^1 \frac{(1-x)^\beta(1+x)^\alpha}{(\alpha-\beta)x-\alpha-\beta}dx.$$

To obtain the value of these integrals, we note that the first Appell hypergeometric series has an integral representation if two of its parameters meet certain restrictions. Namely, we find

$$F_1(a,b_1,b_2,c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}dt$$

for Re $c >$ Re $a > 0$, see [2, Chapter 9].

Observe that setting $a = 1, b_1 = -\beta, b_2 = 1, c = \alpha+2, x = -1$, and $y = \frac{\beta-\alpha}{\alpha+\beta}$ allows us to equate J_1 with the Appell series. These values also satisfy the restrictions for a and c. Similarly, we set $a = 1, b_1 = -\alpha, b_2 = 1, c = \beta+2, x = -1$, and $y = \frac{\alpha-\beta}{\alpha+\beta}$ to equate J_2 with the Appell series.

This results in the following values:

$$J_1 = \frac{-1}{(\alpha+1)(\alpha+\beta)} F_1\left(1, -\beta, 1, \alpha+2; -1, \frac{\beta-\alpha}{\alpha+\beta}\right),$$

and

$$J_2 = \frac{-1}{(\beta+1)(\alpha+\beta)} F_1\left(1, -\alpha, 1, \beta+2; -1, \frac{\alpha-\beta}{\alpha+\beta}\right).$$

As mentioned above, $\widetilde{\mu}_1 = \frac{4}{\beta-\alpha}[J_1 + J_2]$.

We calculate $\widetilde{\mu}_0$ indirectly by first finding $\widetilde{\mu}_2$, which is an easier computation. Again, per equation (4.5.3),

$$\widetilde{\mu}_2 := \int_I (x-\xi)^2 \widehat{W}(x)dx.$$

Substituting the expressions for ξ and the exceptional weight, we find that

$$\widetilde{\mu}_2 = \frac{4}{(\beta-\alpha)^2} \int_{-1}^1 (1-x)^\alpha (1+x)^\beta dx.$$

A standard table of integrals informs us that for $m, n > -1$, and $b > a$,

$$\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}.$$

Setting $a = -1, b = 1, m = \beta$, and $n = \alpha$ allows us to compute the value of $\widetilde{\mu}_2$. The restriction on m and n matches those for the Jacobi parameters α and β:

$$\widetilde{\mu}_2 = \frac{4\,\Gamma(\alpha+1)\Gamma(\beta+1)}{(\beta-\alpha)^2\,\Gamma(\alpha+\beta+2)}.$$

Finally, using the recursion formula given in Table 4.6 for the X_1-Jacobi system, and setting $k = 0$, we calculate $\widetilde{\mu}_0$. After substituting in the expression for ξ in terms of α and β, and further simplifications, we obtain the following value:

$$\widetilde{\mu}_0 = \frac{(\alpha+\beta)\Gamma(\alpha+1)\Gamma(\beta+1)}{2\alpha\beta\Gamma(\alpha+\beta+2)} + \frac{(2\alpha\beta-\alpha-\beta)(J_1+J_2)}{\alpha\beta}.$$

This concludes the proof of Lemma 4.7.1. □

CHAPTER FIVE

Some Observations Regarding Higher Co-Dimensions

The introduction of the Darboux approach to the field [39, 40] allowed the community to study X_m orthogonal polynomials when $m > 1$. The approach towards Sections 2 and 3 was taken from the perspective of allowing higher co-dimensional sequences. Results regarding the codimension two case can be found in [30], but as can be seen in the discussion below, generalized and higher order results should not be expected for $m > 1$.

Further, it is not hard to see that Lemma 4.4.1 can be generalized to the case of $m > 1$ by simply replacing the exceptional condition by a set of m exceptional conditions. We must assume that condition (4.4.1) holds for $x = \xi_i$ where ξ_i ($i = 1, \ldots, m$) denote the m roots of the function η. Namely, we just replace (4.4.1) by the m exceptional conditions

$$\left[2p\eta' y' - \left(p\eta'' + \frac{p'\eta'}{2} - s\eta'\right) y\right]\Bigg|_{x=\xi_i} = 0 \qquad \text{for } i = 1, \ldots, m. \tag{5.0.1}$$

5.1 Possible Flags

As the codimension increases, there become increasingly more possibilities for the organization of the flag; that is, there are more choices in which degrees are removed from the sequence. Certain choices simplify subsequent computations like moment recursion formulas and the set-up of matrix A in Theorem 4.5.1. The importance of making "good" choices that simplify the computations has already become clear in [32]. Here we do not go into the details of how to simplify the computations, but rather provide some preliminary discussions on what kind of choices are allowed in the case of the Type I X_2-Laguerre polynomials.

For example, consider the case when $m = 2$. Of course, we must take the first flag element to be

$$v_2(x) = L_2^\alpha(-x).$$

After some consideration, it appears that for $m = 2$ the choice

$$v_3(x) = (x - \xi_1)^2(x - \xi_2 + 1)$$

is appropriate. To prove that a set $\{v_2, v_3, \ldots\}$ indeed forms a flag, it suffices to take polynomials that satisfy the exceptional conditions (5.0.1) and are of the appropriate degree. As a result, there are now several choices for higher degree flag elements. For $n \geq 4$ we set

$$v_n(x) = (x - \xi_1)^k(x - \xi_2)^{n-k} \qquad \text{where we choose } 2 \leq k \leq n - 2.$$

In general, for $m \geq 2$, we can take

$$v_m(x) = L_m^\alpha(-x),$$

$$v_n(x) = \prod_{i=1}^{m}(x - \xi_i)^{k_i} \qquad \text{for } n \geq 2m, k_i \geq 2 \text{ and } \sum k_i = n.$$

We expect that the flag elements of degree $m+1, \ldots, 2m-1$ are more complicated. But in principle the only requirements are that they have the correct degree and that they satisfy the exceptional conditions (5.0.1).

5.2 Recursion Type Formulas

The general method of finding recursion type formulas for moments of the exceptional weights applies to the higher co-dimension setting. In [32], using an adjusted moment greatly simplifies the computations. As in the discussion of possible flags above, there is again much more freedom, and it is not expected that all choices will yield favorable results.

Currently, it is clear that a “good” adjustment for the moments is given by

$$\widetilde{\mu}_{(l_1,\ldots,l_m)} := \int_I \prod_{i=1}^{m}(x - \xi_i)^{l_i}\widehat{W}(x)dx \qquad \text{where } l_i \in \mathbb{N}_0.$$

For example for $m = 2$, the adjusted moments of interest will take the form

$$\widetilde{\mu}_{(l_1,l_2)} := \int_I (x-\xi_1)^{l_1}(x-\xi_2)^{l_2}\widehat{W}(x)dx,$$

so that recursion formulas will also need to generate $\widetilde{\mu}_{(l_1,l_2)}$ for all $(l_1, l_2) \in \mathbb{N}_0 \times \mathbb{N}_0$.

The choices made for the flag and for the ansatz used to generalize (4.5.1) will determine which moments will occur in the generalization of matrix A. This in turn tells us which of the adjusted moments we need to generate.

BIBLIOGRAPHY

[1] M. Atia, L. L. Littlejohn, and J. Stewart, *The Spectral Theory of the X_1-Laguerre Polynomials*, Adv. Dyn. Syst. Appl., 8(2) (2013) 81-92.

[2] W.N. Bailey, *Appell's Hypergeometric Functions of Two Variables*, Cambridge, England: Cambridge University Press, 1935.

[3] T. Chihara, *An Introduction to Orthogonal Polynomials,* Gordon and Breach, Science Publisher Inc., New York, 1978. Unabridged republication by Dover Publications, 2011.

[4] R.H. Cole, *Theory of Ordinary Differential Equations*, Appleton-Century-Crofts, NewYork, 1968.

[5] A.J. Duran, *Exceptional Meixner and Laguerre orthogonal polynomials*, J. Approx. Theory 184, (2014) 176-208.

[6] D. Dutta and P. Roy, *Conditionally exactly solvable potentials and exceptional orthogonal polynomials,* J. Math. Phys. 51 (2010) 042101.

[7] M. Ángeles García–Ferrero, D. Gómez-Ullate, and R. Milson. *A Bochner type classification theorem for exceptional orthogonal polynomials.* ArXiv e-prints, March 2016.

[8] D. Gómez-Ullate, Y. Grandati, R. Milson, *Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials*, J. Phys. A: Math. Theor. 47 (2014) 1-27.

[9] D. Gómez-Ullate, N. Kamran, and R. Milson, *An extended class of orthogonal polynomials defined by a Sturm-Liouville problem*, J. Math. Anal. Appl. 359 (2009), no. 1, 352–367.

[10] D. Gómez-Ullate, N. Kamran, and R. Milson, *An extension of Bochner's problem: exceptional invariant subspaces*, J. Approx. Theory 162 (2010) 987-1006.

[11] D. Gómez-Ullate, N. Kamran, and R. Milson, *Exceptional orthogonal polynomials and the Darboux transformation*, J. Phys. A: Math. Theor. 43 (2010) 434016.

[12] D. Gómez-Ullate, N. Kamran, and R. Milson, *Two-step Darboux transformations and exceptional Laguerre polynomials*, J. Math. Anal. Appl., 387 (2012) 410–418.

[13] D. Gómez–Ullate, N. Kamran, and R. Milson, *Structure theorems for linear and non-linear differential operators admitting invariant polynomial subspaces*, Discrete Contin. Dyn. Syst. 18 (2007) 85–106.

[14] D. Gómez-Ullate, N. Kamran, and R. Milson, *A conjecture on exceptional orthogonal polynomials*, Found. Comput. Math. 13 (2013) 615–666.

[15] D. Gómez-Ullate, N. Kamran, and R. Milson, *On orthogonal polynomials spanning a non-standard flag,* Contemp. Math., 563 (2012) 51–71.

[16] D. Gómez-Ullate, A. Kasman, A.B.J. Kuijlaars, R. Milson, *Recurrence relations for exceptional Hermite polynomials*, J. Approx. Theory, 204 (2016) 1–16.

[17] D. Gómez-Ullate, F. Marcellán, and R. Milson, *Asymptotic and interlacing properties of zeros of exceptional Jacobi and Laguerre polynomials*, J. Math. Anal. Appl. 399 (2013) 480–495.

[18] Y. Grandati, *Solvable rational extensions of the isotonic oscillator*, Ann. Phys. 326, (2011) 2074-2090.

[19] Y. Grandati, *Rational extensions of solvable potentials and exceptional orthogonal polynomials*, J. Physics, 343, (2012) 012041.

[20] C. L. Ho, *Dirac(-Pauli), Fokker-Planck equations and exceptional Laguerre polynomials,* Ann. Phys. 326 (2011) 2074-2090.

[21] C. L. Ho, S. Odake, and R. Sasaki, *Properties of the exceptional (X_ℓ) Laguerre and Jacobi polynomials,* SIGMA, vol. 7, article 107, (2011) 24 pages

[22] C. L. Ho and R. Sasaki, *Zeros of the Exceptional Laguerre and Jacobi Polynomials*, ISRN Mathematical Physics, Article ID 920475 (2012) 27 pages.

[23] M.E.H. Ismail, *q-Integral and Moment Representations for q-Orthogonal Polynomials*, Canad. J. Math. 54 (2002), 709–735.

[24] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Encyclopedia of Mathematics and its Applications 98, Cambridge, UK, 2005. Paperback 1st edition reprint by Cambridge University Press, 2009.

[25] H.L. Krall, *Certain differential equations for Tchebycheff polynomials,* Duke. Math. J., 4 (1938), 705–718.

[26] H.L. Krall, L.L. Littlejohn, *On the classification of Differential Equations Having Orthogonal Polynomials Solutions - II,* Annali di Matematica pura ed applicata (IV), vol. CIL (1987), 77–102.

[27] C. Liaw, L. Littlejohn, R. Milson, and J. Stewart, *The spectral analysis of three families of exceptional Laguerre polynomials,* J. Approx. Theory 202 (2016) 5-41.

[28] C. Liaw, L. Littlejohn, and J. S. Kelly, *Spectral analysis for the exceptional X_m-Jacobi equation,* Electron. J. Differential Equations 194 (2015) 1-10.

[29] C. Liaw, L. Littlejohn, J. Stewart, and Q. Wicks *A spectral study of the second-order exceptional X_1-Jacobi differential expression and a related non-classical Jacobi differential expression,* J. Math. Anal. Appl. 422 (2015) no. 1, 212-239.

[30] J. S. Kelly, C. Liaw, and J. Osborn *Moment representations of the Type I X_2 Exceptional Laguerre Polynomials,* submitted.

[31] J.S. Kelly, C. Liaw and J. Osborn *Moment representations of exceptional X_1 orthogonal polynomials,* J. Math. Anal. Appl. 455 (2017) 1848–1869.

[32] C. Liaw and J. Osborn *Moment representations of the exceptional X_1-Laguerre orthogonal polynomials,* Mathematische Nachrichten 290 (2017) 1716–1731.

[33] L.L. Littlejohn, *On the classification of Differential Equations Having Orthogonal Polynomials Solutions,* Annali di Matematica pura ed applicata (IV), vol. CXXXVIII (1984), 35–53.

[34] L.L. Littlejohn, *Symmetry Factors for Differential Equations,* Amer. Math. Monthly, vol. 90, no. 7 (1983), 462–464.

[35] B. Midya and B. Roy, *Exceptional orthogonal polynomials and exactly solvable potentials in position dependent mass Schrödinger Hamiltonians,* Phys. Lett. A 373 (45) (2009) 4117-4122.

[36] S. Odake and R. Sasaki, *Infinitely many shape invariant potentials and new orthogonal polynomials*, Physics Letters B 679 (2010), 173–176.

[37] S. Odake and R. Sasaki, *Another set of infinitely many exceptional* (X_ℓ) *Laguerre polynomials*, Physics Letters B 684 (2009) 414-417.

[38] D. Pandres, Jr., *A determinant representation for the classical orthogonal polynomials*, Amer. Math. Monthly, vol. 67, no. 7 (1960), 658–659.

[39] C. Quesne, *Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry,* J. Phys. A: Math. Theor. 41 (2008) 392001.

[40] C. Quesne, *Solvable rational potentials and exceptional orthogonal polynomials in supersymmetric quantum mechanics,* SIGMA, vol. 5, article 084, (2009) 24 pages.

[41] T. Tanaka, *N-fold supersymmetry and quasi-solvability associated with X_2-Laguerre polynomials,* J. Math. Phys. 51 (2010) 032101.